Thm. (Bubley - Dyer 1997) ①

For a probability measure $\mu$ on $V$, we put

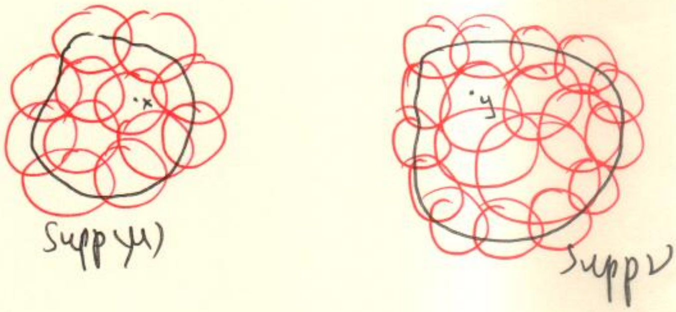
$$\mu P(\cdot) := \sum_x \mu(x) m_x^P(\cdot).$$

Suppose $W_1(m_x^P, m_y^P) \leq (1-k)\, d(x,y)$ for all $x \sim y$.

Then we have for any two probability measures $\mu$ and $\nu$ on $V$ that

$$W_1(\mu P, \nu P) \leq (1-k)\, W_1(\mu, \nu).$$

Proof: Intuition:




Recall from last time, by triangle ineq, we have

$$W_1(m_x^P, m_y^P) \leq (1-k)\, d(x,y), \text{ for all } x, y.$$

(not necessarily adjacent).

Let $\eta : V \times V \to \mathbb{R}$ be an optimal coupling of $\mu$ and $\nu$, that is

$$W_1(\mu, \nu) = \sum_{x,y \in V} \eta(x,y)\, d(x,y).$$

For any $x, y \in V$, let $\xi^{x,y} : V \times V \to \mathbb{R}$ be an optimal coupling of $m_x^P(\cdot)$ and $m_y^P(\cdot)$, that is

$$W_1(m_x^P, m_y^P) = \sum_{u,v \in V} \xi^{x,y}(u,v)\, d(u,v) \leq (1-k)\, d(x,y).$$

Define a map $\xi : \cancel{\phantom{x}}\ V \times V \to \mathbb{R}$ s.t.

$$\xi(u,v) = \sum_{x,y \in V} \eta(x,y)\, \xi^{x,y}(u,v)$$

We check

$$\sum_{u \in V} \xi(u,v) = \sum_{x,y \in V} \eta(x,y) \sum_{u \in V} \xi^{x,y}(u,v) = \sum_{x,y \in V} \eta(x,y)\, m_y(v)$$

$$= \sum_{y \in V} \nu(y)\, m_y(v) = \nu P(v).$$

and
$$\sum_{v \in V} \xi(u,v) = \sum_{x,y \in V} \eta(x,y) \sum_{v \in V} \xi^{x,y}(u,v) = \sum_{x,y \in V} \eta(x,y)\, m_x(u)$$

$$= \sum_{x \in V} \mu(x)\, m_x(u) = \mu P(u).$$

That is, $\xi$ is a coupling of $\mu P$ and $\nu P$. By definition we have

$$W_1(\mu P, \nu P) \le \sum_{u,v \in V} \xi(u,v)\, d(u,v)$$

$$= \sum_{u,v \in V} \sum_{x,y \in V} \eta(x,y)\, \xi^{x,y}(u,v)\, d(u,v)$$

$$= \sum_{x,y \in V} \eta(x,y) \underbrace{\sum_{u,v \in V} \xi^{x,y}(u,v)\, d(u,v)}$$

$$\le (1-k) \sum_{x,y \in V} \eta(x,y)\, d(x,y)$$

$$= (1-k)\, W_1(\mu(x), \mu(y)). \qquad \square.$$

<u>Discrete-time heat equation</u>:

~~Let P~~ Consider

$$(*) \quad \begin{cases} f(x,t+1) - f(x,t) = \Delta f(x,t), & t = 0, 1, 2, \dots \\ f(x,0) = f_0(x). & \forall x \end{cases}$$

Observe that $\Delta f(x,t) = \frac{1}{\cdot} \sum (f(y,t) - f(x,t)) = \sum f(y,t)\, m_x(y) - f(x,t)$

<u>Define</u>: $Pf(x) := \sum_y f(y) m_x(y) = \sum_y f(y)\, \delta_x P(y)$.

The solution to $\circledast$ is

$$f(x,t) = P^t f_0(x) = \sum_y f(y)\, \delta_x P^t(y)$$

where $\delta_x P^t(\cdot) = \sum_{x_1, \dots, x_{t-1}} m_x(x_1)\, m_{x_1}(x_2) \cdots m_{x_{t-1}}(\cdot)$

Applying Bubley-Dyer, the curvature condition $\kappa(x,y) \geq k$

implies $W_1(\delta_x P^t, \delta_y P^t) \leq (1-k)^{t-1} W_1(m_x, m_y)$
$\leq (1-k)^t d(x,y)$

<u>Thm</u>: On a graph $G$. TFAE.

(i) $\kappa(x,y) \geq k$, $\forall x \sim y$.

(ii) $\cancel{RC^t}$ $|\underset{P^t f_0(x)}{f(x,t)} - \underset{P^t f_0(y)}{f(y,t)}| \leq (1-k)^t \mathrm{Lip}(f_0)$, $\forall f_0 : V \to \mathbb{R}$,
$\forall x \sim y$, $\forall t \in \mathbb{N}$.

<u>Proof</u>: (i) $\Rightarrow$ (ii)

$$|f(x,t) - f(y,t)| = |P^t f_0(x) - P^t f_0(y)|$$

$$= \Big| \sum_u f_0(u)\, \delta_x P^t(u) - \sum_v f_0(v)\, \delta_y P^t(v) \Big|$$

$$= \Big| \sum_{u,v} (f_0(u) - f_0(v))\, \xi^{x,y}(u,v) \Big|,$$ $\xi^{x,y}$ is the optimal coupling of $\delta_x P^t$ & $\delta_y P^t$.

$$\leq \mathrm{Lip}(f_0) \sum_{u,v} d(u,v)\, \xi^{x,y}(u,v)$$

$$= \mathrm{Lip}(f_0)\, W_1(\delta_x P^t, \delta_y P^t) \leq (1-k)^t \mathrm{Lip}(f_0)\, d(x,y).$$

(ii) $\Rightarrow$ (i). By Kantorovich duality. $\square$

Extension to $m_x^p$:

Consider

$$\begin{cases} f(x,t+1) - f(x,t) = \sum_y m_x^p(y) f(y,t) - f(x,t) \\ f(x,0) = f_0(x), \ \forall x \end{cases}$$

Observing $m_x^p(\cdot) = p\delta_x + (1-p) m_x(\cdot)$

$$\Rightarrow \sum_y m_x^p(y) f(y,t) - f(x,t) = (p-1) f(x,t) + (1-p) \sum_y f(y,t) m_x(y)$$

$$= (1-p) \Delta f(x,t).$$

That is, the heat equation reads

$$\begin{cases} f(x,t+1) - f(x,t) = (1-p) \Delta f(x,t) \\ f(x,0) = f_0(x). \end{cases}$$

Denote by $(P_p)^t f(x) := \sum_y f(y) m_x^p(y)$

$$= \sum_y f(y) \delta_x P_p(y)$$

The solution

$$f(x,t) = (P_p)^t f_0(x) = \sum_y f(y) \delta_x (P_p)^t(y)$$

where $\delta_x (P_p)^t(y) := \sum_{x_1,\dots,x_{t-1}} m_x(x_1) \cdots m_{x_{t-1}}(y)$

Similarly, we have.

Thm. On a graph $G$. TFAE:

(i) $\kappa_p(x,y) \geq k$, $\forall x \sim y$.

(ii) $|(P_p)^t f_0(x) - (P_p)^t f_0(y)| \leq (1-k)^t \operatorname{Lip}(f_0)$.

$\forall f_0 : V \to \mathbb{R}$, $\forall x \sim y$, $\forall t \in \mathbb{N}$.

Application: Eigenvalue estimate.

Let $\lambda_0$ be an nonzero eigenvalue of $\Delta$ with eigenfct $f$. Recall $\langle f, 1 \rangle = 0$.

Applying discrete heat equ. with the eigenfct as the initial data, we derive from $\kappa_p \geq k$, $\forall\, x \sim y$ that

$$|(P_p)^t f(x) - (P_p)^t f(y)| \leq (1-k)^t \, Lip(f) \, d(x,y).$$

Since $\Delta f = -\lambda f$. $\Rightarrow$ $\underset{\displaystyle (P_p) f - I \cdot f}{\underset{\shortparallel}{(1-p)\,\Delta f}} = -(1-p)\lambda f$.

$$\Rightarrow P_p(f) = (1-(1-p)\lambda) f$$

$$\Rightarrow |1-(1-p)\lambda|^t \, |f(x) - f(y)| \leq (1-k)^t \, Lip(f) \, d(x,y).$$

Since $f$ is not constant, $\exists\, x_0, y_0$ s.t. $|f(x_0) - f(y_0)| = \varepsilon_0 > 0$.

Then $|1-(1-p)\lambda|^t \varepsilon_0 \leq (1-k)^t \, Lip(f) \, d(x_0, y_0)$.

For $\lambda \neq \frac{1}{1-p}$, we have

$$\varepsilon_0 \leq \left( \frac{1-k}{|1-(1-p)\lambda|} \right)^t Lip(f) \, d(x_0, y_0)$$

Letting $t \to \infty$, we have $\frac{1-k}{|1-(1-p)\lambda|} \geq 1$

$$\Rightarrow \frac{k_p}{1-p} \leq \lambda \leq \frac{2-k_p}{1-p}$$

In conclusion, we obtain:

<u>Thm</u>: Let $G=(V,E)$ be a finite graph satisfying

$\kappa_p(x,y) \geq k > 0, \forall x \sim y$, ~~p where p~~.

Then for any nonzero graph Laplacian eigenvalue $\lambda \neq \frac{1}{1-p}$, we have

$$\frac{k}{1-p} \leq \lambda \leq \frac{2-k}{1-p}.$$

In particular,

(1) If $p > \frac{1}{2}$, any nonzero eigenvalue $\lambda$ satisfies

$$\frac{k}{1-p} \leq \lambda \boxed{\leq \frac{2-k}{1-p}}$$

trivial since $\frac{2-k}{1-p} > 2(2-k) \geq 2$ since $k \leq 1$.

(2) If $p=0$, any nonzero eigenvalue $\lambda$ satisfies

$$k \leq \lambda \leq 2-k.$$

After Lin-Lu-Yau's modification. That is, they show

$p \longmapsto \frac{\kappa_p(x,y)}{1-p}$ is non-decreasing (by showing concavity of $\kappa_p$).

and $\kappa_{LLY}(x,y) := \lim_{p \to 1} \frac{\kappa_p(x,y)}{1-p}$.

Assume $\kappa_{LLY}(x,y) \geq k > 0 \Rightarrow \forall \varepsilon > 0, \exists\, p_0 \in (0,1)$, s.t.

for any $p \in (p_0, 1]$, that $\frac{\kappa_p(x,y)}{1-p} \geq (1-\varepsilon)k$

$\Rightarrow \kappa_p(x,y) \geq (1-p)(1-\varepsilon)k$.

Applying Thm (1), we have $\lambda \geq \frac{(1-p)(1-\varepsilon)k}{1-p}$.

Letting $\varepsilon \to 0$ yields $\lambda \geq k$. $\square$