Ricci curvature of Cartesian product graphs.

Recall $G=(V(G),E(G))$, $H=(V(H),E(H))$.

The Cartesian product $G\times H$ is defined as

$$V(G\times H) = V(G)\times V(H).$$

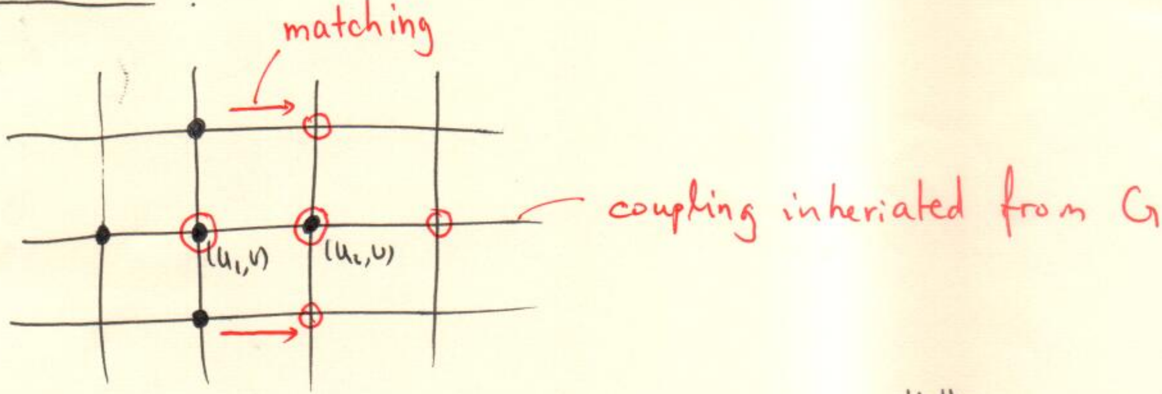
$(u_1,v_1)\sim_{G\times H}(u_2,v_2)$ iff either $u_1\sim_G u_2$, $v_1=v_2\in V(H)$
or $u_1=u_2\in V(G)$, $v_1\sim_H v_2$

Assume G is d_G-regular, H is d_H-regular.

~~Thm~~ Lemma 1. For $p\in[0,1]$ we have

$$\kappa_p^{G\times H}\big((u_1,v),(u_2,v)\big)\geq \frac{d_G}{d_G+d_H}\kappa_p^G(u_1,u_2)$$
$$u_1\sim_G u_2$$

Intuition :



$\bullet\ m^\alpha_{(u_1,v)}$ $\quad \circ\ m^\alpha_{(u_2,v)}$

$A^{u_1,u_2}: V(G)\times V(G)\to\mathbb{R}$ is the coupling s.t.
$$W_1^G(m^\alpha_{u_1},m^\alpha_{u_2}) = \sum_{x_1,x_2\in V(G)} A^{u_1,u_2}(x_1,x_2)\,d(x_1,x_2)$$

(u_1,v) $\qquad\qquad$ (u_2,v)

$\{(x_1,v): x_1\sim_G u_1\}$ $\xrightarrow{A^{u_1,u_2}}$ $\{(x_2,v): x_2\sim_G u_2\}$

$\{(u_1,y_1): y_1\sim_H v\}$ $\xrightarrow[\text{matching}]{}$ $\{(u_2,y_2): y_2\sim_H v\}$

Coupling : $B: V(G\times H)\times V(G\times H)\to\mathbb{R}$

$$B\big((u_1,y_1),(u_2,y_1)\big) = \frac{d_H}{d_G+d_H}m^\alpha_v(y_1),\quad y_1\neq v$$

$$B\big((x_1,v),(x_2,v)\big) = \frac{d_G}{d_G+d_H}A^{u_1,u_2}(x_1,x_2).\quad (x_1,x_2)\neq(u_1,u_2)$$

$$B\big((u_1,v),\ (u_2,v)\big) = \frac{d_G}{d_G+d_H} A^{u_1,u_2}(u_1,u_2) + \frac{d_H}{d_G+d_H}\cdot\alpha .$$

上述两种情形迭加

One can check B is a coupling of $m^{\alpha}_{(u_1,v)}$ & $m^{\alpha}_{(u_2,v)}$ and calculate the corresponding transport cost.

Alternatively, we formulate the proof a bit "abstractly".

Proof: First, we observe W_1 satisfies a convexity property: For any $\lambda \in [0,1]$, ~~μ, ν be two prob~~ μ_i, ν_i, $i=1,2$ be probability measures. Then

$$W_1(\lambda\mu_1 + (1-\lambda)\mu_2,\ \lambda\nu_1 + (1-\lambda)\nu_2)$$
$$\leq \lambda W_1(\mu_1,\nu_1) + (1-\lambda) W_1(\mu_2,\nu_2). \qquad (*)$$

(This has been shown last time when we prove the concavity of $K_{\cancel{p}}(x,y)$. Indeed, let ξ_i, $i=1,2$ be the optimal coupling of μ_1 & ν_1, μ_2 & ν_2 respectively. Then $\lambda\xi_1 + \mu\xi_2$ is a coupling of $\lambda\mu_1 + (1-\lambda)\mu_2$ & $\lambda\nu_1 + (1-\lambda)\nu_2$. The convexity $(*)$ follows from a direct calculation)

Observe that

$$m^{\alpha}_{(u_1,v)} = \frac{d_G}{d_G+d_H} m^{\alpha}_{u_1} \otimes \delta_v + \frac{d_H}{d_G+d_H} \delta_{u_1} \otimes m^{\alpha}_v$$

where the tensor~~ization~~ product is defined as

$$\left(m^{\alpha}_{u_1} \otimes \delta_v\right)(x,y) := m^{\alpha}_{u_1}(x)\cdot\delta_v(y).$$

Therefore, we have

$$W_1\left(m^\alpha_{(u_1,v)},\ m^\alpha_{(u_2,v)}\right)$$

$$= W_1\left(\lambda\, m^\alpha_{u_1}\otimes\delta_v + (1-\lambda)\,\delta_{u_1}\otimes m^\alpha_v,\ \lambda m^\alpha_{u_2}\otimes\delta_v + (1-\lambda)\,\delta_{u_2}\otimes m^\alpha_v\right)$$

with $\lambda = \dfrac{d_G}{d_G+d_H}$.

By convexity, we derive

$$W_1\left(m^\alpha_{(u_1,v)},\ m^\alpha_{(u_2,v)}\right)$$

$$\leq \lambda\, \underbrace{W_1\left(m^\alpha_{u_1}\otimes\delta_v,\ m^\alpha_{u_2}\otimes\delta_v\right)}_{=W_1^G(m^\alpha_{u_1},\, m^\alpha_{u_2})} + (1-\lambda)\, \underbrace{W_1\left(\delta_{u_1}\otimes m^\alpha_v,\ \delta_{u_2}\otimes m^\alpha_v\right)}_{\leqslant d(u_1,u_2)=1}$$

Since there is a perfect matching
$(u_1, y) \to (u_2, y)$
for any $y\in \mathrm{supp}(m^\alpha_v)$

$$\leqslant \lambda W_1^G(m^\alpha_{u_1}, m^\alpha_{u_2}) + (1-\lambda)$$

Consequently, we obtain

$$K_\alpha^{G\times H}\left((u_1,v),(u_2,v)\right) = 1 - W_1\left(m^\alpha_{(u_1,v)},\ m^\alpha_{(u_2,v)}\right)$$

$$\geqslant 1-\lambda W_1^G(m^\alpha_{u_1}, m^\alpha_{u_2}) - (1-\lambda)$$

$$= \lambda\left(1 - W_1^G(m^\alpha_{u_1}, m^\alpha_{u_2})\right) = \lambda\, K_\alpha^G(u_1,u_2).$$ $\square$

In particular, the positivity of K_p is preserved under taking Cartesian product.

<u>Lemma 2</u>: $K_\alpha^{G\times H}\left((u_1,v),(u_2,v)\right) \leqslant \dfrac{d_G+\alpha d_H}{d_G+d_H}\, K^G_{\alpha'}(u_1,u_2)$

for $\alpha\in[0,1]$, $u_1\sim_G u_2$, $\alpha' = \alpha\cdot\dfrac{d_G+d_H}{d_G+\alpha d_H}$

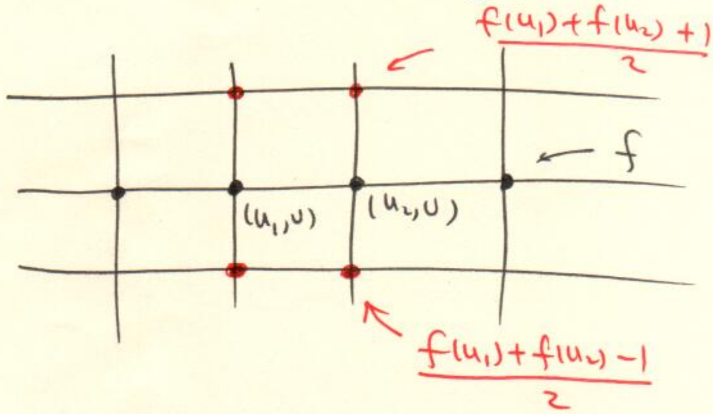
Proof: We do it by duality formula.

Let $f: V(G) \to \mathbb{R}$ be a 1-Lipschitz function

s.t. $W_1^G(m_{u_1}^{\alpha'}, m_{u_2}^{\alpha'}) = \sum_{x \in V(G)} f(x)\left(m_{u_1}^{\alpha'}(x) - m_{u_2}^{\alpha'}(x)\right)$

Construct $F : V(G \times H) \longrightarrow \mathbb{R}$ such that.

$$F(x,y) = \begin{cases} f(x), & \text{if } y = v \\ \dfrac{f(u_1)+f(u_2)+1}{2} & \text{if } x = u_1,\ y \sim v,\ y \neq v \\ \dfrac{f(u_1)+f(u_2)-1}{2} & \text{if } x = u_2,\ y \sim v,\ y \neq v \end{cases}$$



$W_1(m_{(u_1,v)}^{\alpha}, m_{(u_2,v)}^{\alpha})$

$$\geq \sum_{(x,y)} F(x,y)\left(m_{(u_1,v)}^{\alpha}(x,y) - m_{(u_2,v)}^{\alpha}(x,y)\right)$$

$$= \lambda \sum_{(x,y)} F(x,y)\left(m_{u_1}^{\alpha} \otimes \delta_v(x,y) - m_{u_2}^{\alpha} \otimes \delta_v(x,y)\right)$$

$$+ (1-\lambda) \sum_{(x,y)} F(x,y)\left(\delta_{u_1} \otimes m_v^{\alpha}(x,y) - \delta_{u_2} \otimes m_v^{\alpha}(x,y)\right)$$

$$= \lambda \sum_x f(x)\left(m_{u_1}^{\alpha}(x) - m_{u_2}^{\alpha}(x)\right)$$

$$+ (1-\lambda)\frac{f(u_1)+f(u_2)+1}{2} \sum_{y \neq v} m_v^{\alpha}(y) - (1-\lambda)\frac{f(u_1)+f(u_2)-1}{2} \sum_{y \neq v} m_v^{\alpha}(y)$$

$$+ (1-\lambda) f(u_1)\alpha - f(u_2)\alpha.(1-\lambda)$$

$$= (1-\alpha)(1-\lambda) + \lambda \sum_{x \neq u_1} f(x)\, m^{\alpha}_{u_1}(x) - \lambda \sum_{x \neq u_2} f(x)\, m^{\alpha}_{u_2}(x)$$

$$+ \lambda f(u_1)\cdot\alpha - \lambda f(u_2)\alpha$$

$$+ (1-\lambda) f(u_1)\alpha - (1-\lambda) f(u_2)\alpha$$

$$= (1-\alpha)(1-\lambda) + \cancel{\lambda}\Big(\sum_{x\neq u_1} f(x)\lambda m^{\alpha}_{u_1}(x) + f(u_1)\alpha\Big) - \cancel{\lambda}\Big(\sum_{x\neq u_2} f(x)\lambda m^{\alpha}_{u_2}(x) + \cancel{\lambda} f(u_2)\alpha\Big)$$

The measure

$$p_{u_1}(x) = \begin{cases} \lambda m^{\alpha}_{u_1}(x) = \lambda \dfrac{1-\alpha}{d_G}, & x \sim u_1 \\ \alpha & x = u_1 \end{cases}$$

satisfies $\sum_x p_{u_1}(x) = (1-\alpha)\lambda + \alpha$

Similarly, for $p_{u_2}(x) := \begin{cases} \lambda m^{\alpha}_{u_2}(x) = \lambda \dfrac{1-\alpha}{d_G}, & x \sim u_2 \\ \alpha & x = u_2 \end{cases}$

has $\sum_x p_{u_2}(x) = (1-\alpha)\lambda + \alpha$.

Scaling them to be probability measures:

$$\frac{1}{(1-\alpha)\lambda+\alpha}\, p_{u_1}(\cdot) \quad \text{and} \quad \frac{1}{(1-\alpha)\lambda+\alpha}\, p_{u_2}(\cdot)$$

$\Rightarrow \cancel{W_1(m^{\alpha}_{(u_1,\cdot)}, m^{\alpha}_{(u_2,\cdot)}) = (1-\alpha)(1-\lambda) +}$ Observe that

$$\frac{1}{(1-\alpha)\lambda+\alpha}\, p_{u_1}(x) = \begin{cases} \dfrac{(1-\alpha)\lambda}{(1-\alpha)\lambda+\alpha}\,\dfrac{1}{d_G} & x \sim u_1 \\ \dfrac{\alpha}{(1-\alpha)\lambda+\alpha} & x = u_1 \end{cases}$$

$$= m^{\alpha'}_{u_1}(x)$$

with $\alpha' = \dfrac{\alpha}{(1-\alpha)\lambda+\alpha}$.

Similarly,

$$\frac{1}{(1-\alpha)\lambda+\alpha}\, p_{u_2}(x) = m_{u_2}^{\alpha'}(x)$$

$$\Rightarrow W_1\left(m^{\alpha}_{(u_1,v)},\ m^{\alpha}_{(u_2,v)}\right)$$

$$\geq (1-\alpha)(1-\lambda) + W_1^G\left(m_{u_1}^{\alpha'}(x),\ m_{u_2}^{\alpha'}(x)\right)\cdot\left((1-\alpha)\lambda+\alpha\right)$$

$$\Rightarrow \kappa_\alpha^{G\times H}\left((u_1,v),(u_2,v)\right)$$

$$= 1 - W_1\left(m^{\alpha}_{(u_1,v)},\ m^{\alpha}_{(u_2,v)}\right)$$

$$\leq 1 - (1-\alpha)(1-\lambda) - \cancel{+}\ \left((1-\alpha)\lambda+\alpha\right) W_1^G\left(m_{u_1}^{\alpha'}(x),\ m_{u_2}^{\alpha'}(x)\right)$$

$$= \left((1-\alpha)\lambda+\alpha\right)\left(1 - W_1^G\left(m_{u_1}^{\alpha'}(x),\ m_{u_2}^{\alpha'}(x)\right)\right)$$

$$= \left((1-\alpha)\lambda+\alpha\right)\kappa_{\alpha'}^G(u_1,u_2).$$

$\square$

$$\frac{\kappa_\alpha^{G\times H}\left((u_1,v),(u_2,v)\right)}{1-\alpha} = \frac{(1-\alpha)\lambda+\alpha}{1-\alpha}\,\kappa_{\alpha'}^G(u_1,u_2)$$

$$= \lambda\cdot\frac{(1-\alpha)\lambda+\alpha}{(1-\alpha)\lambda}\,\kappa_{\alpha'}^G(u_1,u_2)$$

$$= \lambda\,\frac{1}{1-\alpha'}\,\kappa_{\alpha'}^G(u_1,u_2)$$