Perpectual cutoff semigroup

Let G be a locally finite graph. If we only know the $CD(K,\infty)$ holds at every vertex out side of a finite subset $\emptyset \neq W \subset V$, ~~can~~ can we determine whether the graph is finite or not? For a general subset $\emptyset \neq W \subset V$, can we estimate the maximal distance to W?
For that purpose, we develop the theory of perpectual cutoff semigroup due to Münch (Bull. Lond. Math. Soc.)

Let us consider the closure of W defined as

$$cl(W) := \{ v \in V : d(v,W) \leq 1 \}$$

and the cutoff operator: for any $f \in \ell^\infty(V)$,

$$S^W f(x) = f(x) \vee \sup_{cl(W)} f, \quad \forall x \in V.$$

The resulting function $S^W f : V \to \mathbb{R}$ is a function in $\ell^\infty(V)$ such that $S^W f|_{cl(W)} \equiv$ a constant s_0 and $S^W f(x) \geq s_0$ holds for any $x \in V$.
We define

$$\ell^\infty_W(V) := \{ f \in \ell^\infty(V) : f|_{cl(W)} \equiv \text{const}, f \geq f|_{cl(W)} \}$$
$$= \{ f \in \ell^\infty(V) : f \geq f(w) \text{ for all } w \in cl(W) \}$$

Subsequently, we define $Q_t^W : \ell^\infty(V) \to \ell^\infty_W(V)$ as

$$Q_t^W f := S^W P_t f = P_t f \vee \sup_{cl(W)} P_t f .$$

The <u>perpectual cutoff semigroup</u> $P_t^W : \ell^\infty(V) \to \ell^\infty_W(V)$ is defined as

$$P_t^W f := \sup_{t_1 + \cdots + t_n = t} Q_{t_1}^W \cdots Q_{t_n}^W f ,$$

where the supremum is taken over the set $\left\{ (t_1, \dots, t_n) \,\middle|\, \begin{array}{l} n \in \mathbb{Z}, t_i > 0, \\ t_1 + \cdots + t_n = t \end{array} \right\}$

Notice that ~~$P_t^W f$~~ $\|P_t^W f\|_\infty \le \|f\|_\infty$ (see property (ii) below).
$\|Q_t^W f\|_\infty \le \|f\|_\infty$

and $P_t^W f$ is the supremum of a set of functions in $\ell_W^\infty(V)$ with a uniform infinity-norm upper bound $\|f\|_\infty$, and hence $P_t^W f \in \ell_W^\infty(V)$.

Recall the characterization that $CD(K,\infty) \iff$

$$\Gamma(P_t f) \le e^{-2Kt} P_t(\Gamma f), \quad \forall f \in \ell^\infty(V).$$

We aim at establishing the following gradient estimate.

<u>Thm</u> (Münch). Let G be a locally finite graph. Suppose that $CD(K,\infty)$ holds on $V \setminus W$ for some $\emptyset \neq W \subset V$. Then

$$\Gamma(P_t^W f) \le e^{-2Kt} P_t(\Gamma f), \quad \forall f \in \ell_W^\infty(V)$$

The 1st difficulty of such kind of result lie in the unknown curvature property on W. However for cutoff semigroup P_t^W, we have $\Gamma(P_t^W f)\big|_W = 0$ since $P_t^W f \equiv$ const on $cl(W)$

Recall the general result that

$$\Gamma_2(f)(x) = Hf(x) + \tfrac{1}{2}(\Delta f(x))^2 - \Gamma(f)(x).$$

Applying the above identity yields

$$\Gamma_2(P_t^W f)\big|_W \ge 0.$$

Therefore $\Gamma_2(P_t^W f) \ge 0 = K\Gamma(P_t^W f)$ holds on W.
for any K

That is, $\Gamma_2(f)(x) \ge K\Gamma(f)(x)$ holds for any $x \in W$ and $f \in \ell_W^\infty(V)$.

This "solves" the 1st difficulty and the next thing to do is to show necessary analogues of P_t^W to P_t.

In last lecture, we have show the following three properties of P_t^W:

<u>Thm</u>. Let $G=(V,E)$ be a locally finite graph.

Let $\phi \neq W \subset V$. Let $f \in \ell_\infty^W(V)$.

Then for $x \in V$ and $s,t \geq 0$,

(i) $P_s^W \circ P_t^W = P_{s+t}^W$. (semigroup property)

(ii) $\|P_t^W f\|_\infty \leq \|f\|_\infty$.

(iii) $\|P_t^W f - f\|_\infty \leq 2t \|f\|_\infty$.

<u>Proof</u>. (i) Let $S = \{0 < \sigma_1 < \cdots < \sigma_m = s\}$ be a partition of $[0,s]$ and $T = \{0 < \tau_1 < \cdots < \tau_n = t\}$ be a partition of $[0,t]$.

We denote $Q_S^W := Q^W_{\sigma_m - \sigma_{m-1}} \circ \cdots \circ Q^W_{\sigma_1}$, $Q_T^W = Q^W_{\tau_n - \tau_{n-1}} \circ \cdots \circ Q^W_{\tau_1}$.

[<u>Monotonicity</u>: If T' is a refinement of T, then

$$Q^W_{T'} f \geq Q^W_T f.$$

For the proof, we only need to show

$$Q^W_{t_2} Q^W_{t_1} f \geq Q^W_{t_1+t_2} f$$

This is because

$$P_{t_2}(P_{t_1} f \vee \sup_{cl(W)} P_{t_1} f) \geq P_{t_2} P_{t_1} f = P_{t_2+t_1} f$$

$$\Rightarrow \quad S^W(P_{t_2}(P_{t_1} f \vee \sup_{cl(W)} P_{t_1} f)) \geq S^W(P_{t_1+t_2} f)$$

That is $Q^W_{t_2} Q^W_{t_1} f \geq Q^W_{t_1+t_2} f$. □]

Next, we show $P_s^W \circ P_t^W = P_{s+t}^W$. On the one hand,

$$Q_S^W \circ Q_T^W f = Q^W_{(t+S)\cup T} f \leq P^W_{t+s} f$$

Taking supremum over all partitions S of $[0,s]$, and all partitions

T of $[0,t]$ in the left hand side, yields

$$P_s^W \circ P_t^W f \leq P_{t+s}^W f . \quad (1)$$

On the other hand, for any partition U of $[0, s+t]$, we consider its refinement $U' := U \cup \{t\}$, and the partition $T = U' \cap (0,t]$, and partition $S := (U-t) \cap (0,s]$. Then

$$P_s^W \circ P_t^W f \geq Q_S^W \circ Q_T^W f = Q_{U'}^W f \geq Q_U^W f .$$

Taking supremum over all partitions U of $[0, s+t]$ yields

$$P_s^W \circ P_t^W f \geq P_{s+t}^W f . \quad (2) .$$

Combining (1) and (2) leads to $P_{s+t}^W f = P_s^W \circ P_t^W f$. $\square$

(ii) On the one hand, we have

$$\sup_V Q_t^W f = \sup_V (P_t f \vee \sup_{c\ell(W)} P_t f) \leq \sup_V f$$

Morever, using the above estimate iteratively, we have

$$\sup_V Q_{t_1}^W \circ \cdots Q_{t_n}^W f \leq \sup_V f .$$

This leads to $\sup_V P_t^W f \leq \sup_V f$. (A)

Similarly, we have on the other hand

$$\inf_V Q_t^W f = \inf_V (P_t f \vee \sup_{c\ell(W)} P_t f) \geq \inf_V f .$$

This leads to $\inf_V P_t^W f \geq \inf_V f$ (B)

Combining (A) and (B) yields $\| P_t^W f \|_\infty \leq \| f \|_\infty$.

Remk: In (i) and (ii), we only need $f \in \ell^\infty(V)$. (not necessarily $f \in \ell^\infty_W(V)$)

(iii) First recall the analogous result for P_t:

$$| P_t f - f | \leq t \cdot | \Delta_\Theta P_s f | \leq 2t \, \| f \|_\infty$$

for some $s \in (0,t)$

~~Hence~~ Next, we consider $Q_t^w f - f = P_t f \vee (\sup_{Cl(w)} P_t f) - f$.

At $x \in V$ s.t. $P_t f(x) > \sup_{Cl(w)} P_t f$, we have

$$|Q_t^w f(x) - f(x)| = |P_t f(x) - f(x)| \leq 2t \|f\|_\infty.$$

At $x \in V$ s.t. $P_t f(x) \leq \sup_{Cl(w)} P_t f$, we have

$$Q_t^w f(x) - f(x) \geq P_t f(x) - f(x) \geq -2t \|f\|_\infty \quad (\mathrm{I})$$

and $Q_t^w f(x) - f(x) = \inf_V Q_t^w f - f(x) \leq \inf_V Q_t^w f - \inf_V f$

For $f \in l_w^\infty(V)$, we have

$$\inf_V Q_t^w f - \inf_V f = \sup_{Cl(w)} P_t f - \sup_{Cl(w)} f \leq \sup_{Cl(w)} (P_t f - f)$$

$$\leq 2t \|f\|_\infty.$$

Therefore, $Q_t^w f(x) - f(x) \leq 2t \|f\|_\infty \quad (\mathrm{II})$

Combining (I) and (II) leads to $\|Q_t^w f - f\|_\infty \leq 2t \|f\|_\infty$

Iteratively applying this inequality, for any $f \in l_w^\infty(V)$,

$$\|Q_{t_1}^w Q_{t_2}^w f - f\|_\infty = \|Q_{t_1}^w Q_{t_2}^w f - Q_{t_2}^w f + Q_{t_2}^w f - f\|_\infty$$

$$\leq \|Q_{t_1}^w Q_{t_2}^w f - Q_{t_2}^w f\|_\infty + \|Q_{t_2}^w f - f\|_\infty$$

$$\leq 2t_1 \underbrace{\|Q_{t_2}^w f\|_\infty}_{\leq \|f\|_\infty \text{ by (ii)}} + 2t_2 \|f\|_\infty \leq 2(t_1 + t_2) \|f\|_\infty.$$

$$\Rightarrow \|Q_{t_1}^w Q_{t_2}^w \circ \cdots \circ Q_{t_n}^w f - f\|_\infty \leq 2(t_1 + t_2 + \cdots + t_n) \|f\|_\infty.$$

Taking supremum, we derive

$$\|P_t^w f - f\|_\infty \leq 2t \|f\|_\infty. \qquad \square$$

We next show three properties about the time-derivative of $P_t^w f$.

<u>Thm</u>: Let $f \in \ell_w^\infty(V)$. Then for $x \in V$, and $t \geq 0$,

(iv) $$\partial_t^+ P_t^w f(x)\Big|_{t=0} = \partial_t^+ Q_t^w f(x)\Big|_{t=0} = \begin{cases} S^w \Delta f(x) : & f(x) = \inf_V f \\ \Delta f(x) & : \text{otherwise}. \end{cases}$$

(v) $$\|\partial_t^+ P_t^w f\|_\infty \leq \|\Delta P_t^w f\|_\infty$$

(vi) $$\partial_t^+ \Gamma P_t^w f \leq 2\Gamma(P_t^w f, \Delta P_t^w f).$$

<u>Proof</u>. Due to the semigroup property $P_s^W \circ P_t^W = P_{s+t}^W$ and $P_t^W f \in \ell_W^\infty(V)$, Property (iv) implies immediately that

$$\partial_t^+ P_t^W f(x) = \begin{cases} S^W \Delta P_t^W f(x) : & P_t^W f(x) = \inf_V P_t^W f \\ \Delta P_t^W f(x) & : \text{otherwise}. \end{cases}$$

Hence, property (v) follows immediately from (iv).

Next, we prove (iv). ~~Observe~~ Recall that

$$Q_t^W f(x) = P_t f(x) \vee \sup_{cl(W)} P_t f$$

By mean-value theorem, $\exists\ s \in [0,t]$ s.t.

$$P_t f(x) = f(x) + t\Delta P_s f. \qquad (*)$$

Moreover, $\|\Delta P_s f - \Delta f\|_\infty \leq 2\|P_s f - f\|_\infty \leq 4s\|f\|_\infty$.

This leads to

$$\begin{aligned} &|P_t f(x) - (f + t\Delta f)(x)| \\ &= |f(x) + t\Delta P_s f(x) - f(x) - t\Delta f(x)| = t|\Delta P_s f(x) - \Delta f(x)| \\ &\leq 4t^2\|f\|_\infty. \end{aligned} \qquad (**)$$

<u>Claim</u>: $$\left|\sup_{cl(W)} P_t f - \left(t\sup_{cl(W)} \Delta f + \sup_{cl(W)} f\right)\right| \leq 4t^2\|f\|_\infty.$$

<u>Proof</u>: $$\sup_{cl(W)} P_t f - \underbrace{\sup_{cl(W)} f}_{\text{const.}} = \sup_{cl(W)} (P_t f - f) \overset{(*)}{\leq} t\sup_{cl(W)} \sup_{[0,t]} \Delta P_s f.$$

On the other hand

$$\sup_{Cl(W)} P_t f - \sup_{Cl(W)} f = \sup_{Cl(W)} (P_t f - f) \overset{(*)}{\geq} t \sup_{Cl(W)} \inf_{[0,t]} \Delta P_s f .$$

Applying $\|\Delta P_s f - \Delta f\|_\infty \leq 4s \|f\|_\infty$, we arrive at

$$\sup_{Cl(W)} P_t f - \sup_{Cl(W)} f \leq \sup_{Cl(W)} \sup_{[0,t]} \Delta P_s f \leq t \sup_{Cl(W)} \sup_{[0,t]} (\Delta f + 4s\|f\|_\infty)$$

$$\overset{(*)}{=} t \sup_{Cl(W)} \Delta f + 4t^2 \|f\|_\infty \qquad (A)$$

and

$$\sup_{Cl(W)} P_t f - \sup_{Cl(W)} f \geq t \sup_{Cl(W)} \inf_{[0,t]} \Delta P_s f$$

$$\geq t \sup_{Cl(W)} \inf_{[0,t]} (\Delta f - 4s\|f\|_\infty)$$

$$= t \sup_{Cl(W)} (\Delta f - 4t\|f\|_\infty)$$

$$= t \sup_{Cl(W)} \Delta f - 4t^2 \|f\|_\infty . \qquad (B)$$

Combining (A) and (B) yields

$$\left| \sup_{Cl(W)} P_t f - \left(t \sup_{Cl(W)} \Delta f + \sup_{Cl(W)} f\right) \right| \leq 4t^2 \|f\|_\infty ,$$

as claimed. □

Putting (**) and Claim together, we obtain

$$(f + t\Delta f)(x) \vee \left(t \sup_{Cl(W)} \Delta f + \sup_{Cl(W)} f\right) - 4t^2\|f\|_\infty \leq \underbrace{P_t f(x) \vee \sup_{Cl(W)} P_t f}_{= Q_t^W f(x)} \leq (f + t\Delta f)(x) \vee \left(t \sup_{Cl(W)} \Delta f + \sup_{Cl(W)} f\right) + 4t^2 \|f\|_\infty$$

That is

$$\left| Q_t^W f(x) - (f + t\Delta f)(x) \vee \left(t \sup_{Cl(W)} \Delta f + \sup_{Cl(W)} f\right) \right| \leq 4t^2 \|f\|_\infty$$

Note that $f(x) = Q_0^W f(x)$ for $f \in \ell_W^\infty(V)$. Let us compare

$(f + t\Delta f)(x)$ and $t \sup_{Cl(W)} \Delta f + \sup_{Cl(W)} f$.

<u>Case 1</u>. $f(x) = \sup_{Cl(W)} f$. Then we have $|Q_t^W f(x) - f(x) - t S^W \Delta f(x)| \leq 4t^2 \|f\|_\infty$

Case 2. If $f(x) > \sup_{Cl(W)} f = \inf_V f$. Then for small enough $t>0$, we have $f(x) + t\Delta f(x) \geq t \sup_{Cl(W)} \Delta f + \inf_V f$, and hence

$$\left| Q_t^W f - f(x) - t\Delta f(x) \right| \leq 4t^2 \|f\|_\infty.$$

Now, we arrive at

$$\lim_{t\to 0^+} \frac{Q_t^W f - f(x)}{t} = \begin{cases} S^W \Delta f & \text{if } f(x) = \inf_V f \\ \Delta f(x) & \text{if } f(x) > \inf_V f \end{cases}.$$

Hence $\partial_t^+ Q_t^W f$ exists and

$$(\ast) \quad \partial_t^+ Q_t^W f = \begin{cases} S^W \Delta f & \text{if } f(x) = \inf_V f \\ \Delta f(x) & \text{otherwise.} \end{cases} := L^W f(x).$$

In order to derive the result on $P_t^W f$, we need apply $(\ast)$ iteratively. the following estimate

$$\left| Q_t^W f - f(x) - t L^W f(x) \right| \leq 4t^2 \|f\|_\infty$$

For any $t_1 + t_2 + \cdots + t_n = t$, we derive

$$\Big| \underbrace{Q_{t_n}^W \circ \cdots \circ Q_{t_1}^W f(x)}_{\shortparallel \; f_n} - \underbrace{Q_{t_{n-1}}^W \circ \cdots \circ Q_{t_1}^W f(x)}_{\shortparallel \; f_{n-1}(x)} - t_n L^W Q_{t_{n-1}}^W \circ \cdots \circ Q_{t_1}^W f(x) \Big| \leq 4t_n^2 \underbrace{\| Q_{t_{n-1}}^W \circ \cdots \circ Q_{t_1}^W f \|_\infty}_{\leq \|f\|_\infty}$$

$$| f_n(x) - f_{n-1}(x) - t_n L^W f_{n-1}(x) | \leq 4t_n^2 \|f\|_\infty.$$

$$| f_{n-1}(x) - f_{n-2}(x) - t_{n-1} L^W f_{n-2}(x) | \leq 4t_{n-1}^2 \|f\|_\infty$$

$$\vdots$$

$$| f_1(x) - f(x) - t_1 L^W f(x) | \leq 4t_1^2 \|f\|_\infty$$

Summing up, we obtain

$$\left| f_n(x) - f(x) - \sum_{k=0}^{n-1} t_{k+1} L^W f_k(x) \right| \leq 4 \underbrace{(t_1^2 + \cdots + t_n^2)}_{\leq (t_1 + t_2 + \cdots + t_n)^2 = t^2} \|f\|_\infty$$

<u>Observation</u>: $L^W f_k(x) - L^W f(x) \leq \cancel{2\|f_k - f\|_\infty}$, $4t\|f\|_\infty$ for small t.

Recall $L^w f(x) = \begin{cases} \Delta f(x) \vee \sup_{Cl(W)} \Delta f, & \text{if } f(x) = \inf_V f. \\ \Delta f(x), & \text{if } f(x) > \inf_V f. \end{cases}$

Case 1. $f(x) = \inf_V f$.

Then $L^w f_k(x)$ = either $\Delta f_k(x) \vee \sup_{Cl(W)} \Delta f_k$ or $\Delta f_k(x)$

$$\leq \Delta f_k(x) \vee \sup_{Cl(W)} \Delta f_k$$

and $Lf(x) = \Delta f(x) \vee \sup_{Cl(W)} \Delta f$.

Since $|\Delta f_k(x) - \Delta f(x)| \leq 2\|f_k - f\|_\infty \overset{\text{Proof of (iii) on page (3)}}{\leq} 4t\|f\|_\infty$, we have

$$\Delta f_k(x) \leq \Delta f(x) + 4t\|f\|_\infty \text{ and}$$

$$\sup_{Cl(W)} \Delta f_k - \sup_{Cl(W)} \Delta f \leq \sup_{Cl(W)} (\Delta f_k - \Delta f) \leq 4t\|f\|_\infty.$$

Combing leads to

$$L^w f_k(x) \leq \Delta f_k(x) \vee \sup_{Cl(W)} \Delta f_k \leq \Delta f(x) \vee \sup_{Cl(W)} \Delta f + 4t\|f\|_\infty$$
$$= L f(x) + 4t\|f\|_\infty.$$

as claimed.

Case 2. $f(x) > \inf_V f$.

Since $\|f_k - f\| \leq 2t\|f\|_\infty$, we know for small enough t,

$$f_k(x) > \inf_V f_k \text{ holds.}$$

Hence for such t, we have $L^w f_k(x) = \Delta f_k(x)$ and $L^w f(x) = \Delta f(x)$ and

$$L^w f_k(x) - L^w f(x) = \Delta f_k(x) - \Delta f(x) \leq 4t\|f\|_\infty$$

as claimed. □

Employing this observation, we continue to estimate

$$f_n(x) - f(x) \leq \sum_{k=0}^{n-1} t_{k+1} L^w f_k(x) + 4t^2\|f\|_\infty$$
$$\leq \sum_{k=0}^{n-1} t_{k+1} (L^w f(x) + 4t\|f\|_\infty) + 4t^2\|f\|_\infty. \text{ for } t \text{ small.}$$

$$= t L^w f(x) + 8t^2 \|f\|_\infty \quad \text{for } t \text{ small}.$$

Therefore, we obtain by taking supremum that

$$\frac{P_t^W f(x) - f(x)}{t} \leq L^W f(x) + 8t\|f\|_\infty,$$

and hence

$$\overline{\partial_t^+} P_t^W f\Big|_{t=0} = \limsup_{t \to 0^+} \frac{P_t^W f(x) - f(x)}{t} \leq L^W f(x). \quad (*\star 1)$$

On the other hand, we have $P_t^W f(x) \geq Q_t^W f(x)$ by definition, and hence

$$\underline{\partial_t^+} P_t^W f\Big|_{t=0} = \liminf_{t \to 0^+} \frac{P_t^W f(x) - f(x)}{t} \geq \liminf_{t \to 0^+} \frac{Q_t^W f(x) - f(x)}{t}$$

$$= \partial_t^+ Q_t^W f(x) = L^W f(x). \quad (*\star 2).$$

Combining $(*\star 1)$ and $(*\star 2)$ yields

$$L^w f(x) \leq \underline{\partial_t^+} P_t^w f\Big|_{t=0} \leq \overline{\partial_t^+} P_t^w f\Big|_{t=0} \leq L^w f(x),$$

which implies that $\partial_t^+ P_t^W f\big|_{t=0}$ exists and

$$\partial_t^+ P_t^W f\Big|_{t=0} = L^w f(x), \quad \text{proving (iv), and hence (v).}$$

~~Remark. Due to~~ Next we show (vi).

It is direct to check

$$\partial_t^+ \Gamma(P_t^w f) = \partial_t^+ \left(\tfrac{1}{2} \Delta (P_t^w f)^2 - P_t^W f \, \Delta P_t^W f \right)$$

$$\overset{\partial_t^+ \text{ and } \Delta \text{ commute}}{=} \tfrac{1}{2} \Delta \left(2 P_t^W f \cdot \partial_t^+ P_t^w f\right) - \partial_t^+ P_t^W f \, \Delta P_t^W f - P_t^W f \, \Delta \partial_t^+ P_t^W f)$$

$$= 2\Gamma(P_t^W f, \partial_t^+ P_t^W f)$$

Due to the semigroup property (i), it is enough to check

$$\partial_t^+ \Gamma P_t^W f \leq 2\Gamma(P_t^W f, \Delta P_t^W f) \text{ at } t=0.$$ ⑨

By Property (iv), we observe

$$\begin{aligned}\partial_t^+ \Gamma P_t^W f \big|_{t=0} &= 2\Gamma(P_t^W f, \partial_t^+ P_t^W f)\big|_{t=0} \\ &= 2\Gamma(f, \partial_t^+ P_t^W f|_{t=0}) \\ &= 2\Gamma(f, \partial_t^+ Q_t^W f|_{t=0}) = \partial_t^+ \Gamma Q_t^W f\big|_{t=0}.\end{aligned}$$

(We used $P_t^W f|_{t=0} = Q_t^W|_{t=0} = f$).

~~By mean value theorem, we~~ We have

$$\underset{\substack{\uparrow \\ \text{existence guaranteed} \\ \text{by (iv)}}}{\partial_t^+ \Gamma Q_t^W f\big|_{t=0}} = \lim_{t\to 0^+} \frac{\Gamma(Q_t^W f) - \Gamma(f)}{t} = \lim_{t\to 0^+} \frac{\Gamma(S^W P_t f) - \Gamma(f)}{t}$$

<u>Observation</u>: $\Gamma(S^W f) \leq \Gamma(f)$ for all $f \in \ell^\infty(V)$.

This can be checked directly by $\Gamma(f)(x) = \frac{1}{2}\frac{1}{d_x}\sum_{y:\{x,y\}\in E}(f(y)-f(x))^2$.

Therefore, we have

$$\partial_t^+ \Gamma Q_t^W f\big|_{t=0} \leq \limsup_{t\to 0^+} \frac{\Gamma(P_t f) - \Gamma(f)}{t} = 2t\Gamma(f, \Delta f).$$

Putting together, we obtain

$$\partial_t^+ \Gamma P_t^W f\big|_{t=0} = \partial_t^+ \Gamma Q_t^W f\big|_{t=0} \leq 2t\,\Gamma(f, \Delta f),$$

as claimed. □

Finally, we check the following key ~~ob~~ property.

<u>Thm</u>. ~~Let~~ For any $f \in \ell_W^\infty(V)$, we have

$P_\infty^W f := \lim_{t\to\infty} P_t^W f$ is a constant function on V.

<u>Proof</u>. Observe that $\inf_V P_t f \overset{\text{for } \delta>0}{\leq} P_{t+\delta} f = P_\delta(P_t f) \leq \sup_V P_t f$

Recall in the proof of (ii), we check that this is still true for P_t^W:

$$\inf_V P_t^W f \le P_{t+\delta}^W f = P_\delta^W(P_t^W f) \le \sup_V P_t^W f$$

That is, $\inf_V P_t^W f$ is nondecreasing, and $\sup_V P_t^W f$ is non-increasing in t. Let us denote by

$$c := \lim_{t\to\infty} \inf_V P_t^W f \ , \quad \text{and} \quad C := \lim_{t\to\infty} \sup_V P_t^W f \ .$$

<u>Claim</u>. $P_t f \le Q_t^W f \le P_t f + \inf_V Q_t^W f - \inf_V f$

$$P_t f \le P_t^W f \le P_t f + \inf_V P_t^W f - \inf_V f \ , \quad \forall f \in \ell_W^\infty(V).$$

<u>Pf</u>. $P_t f \le Q_t^W f \le P_t^W f$ is by definition. On the other hand, we have

$$\begin{aligned} Q_t^W f &= P_t f \vee \sup_{\mathrm{cl}(W)} P_t f \\ &= P_t f \vee \inf_V Q_t^W f \le P_t f + \inf_V Q_t^W f - \inf_V f . \end{aligned}$$

The last inequality is due to the fact that both $P_t f - \inf_V f$ and $\inf_V Q_t^W f - \inf_V f$ are non-negative. $\square$

Applying the ineq. about Q_t^W iteratively, we have

$$\begin{aligned} Q_{t_1}^W Q_{t_2}^W f &\le P_{t_1} Q_{t_2}^W f + \inf_V Q_{t_1}^W Q_{t_2}^W f - \inf_V Q_{t_2}^W f \\ &\le P_{t_1}\big(P_{t_2} f + \underbrace{\inf_V Q_{t_2}^W f - \inf_V f}_{\text{constant}}\big) + \inf_V Q_{t_1}^W Q_{t_2}^W f \\ &\qquad - \inf_V Q_{t_2}^W f \\ &\le P_{t_1} \circ P_{t_2} f + \inf_V \cancel{Q_{t_2}^W f} - \inf_V f + \inf_V Q_{t_1}^W Q_{t_2}^W f - \cancel{\inf_V Q_{t_2}^W f} \\ &= P_{t_1+t_2} f + \inf_V Q_{t_1}^W Q_{t_2}^W f - \inf_V f . \end{aligned}$$

Hence, for any $t_1 + \cdots + t_n = t$, we derive

$$Q^W_{t_1}\cdots Q^W_{t_n} f \le P_t f + \inf_v Q^W_{t_1}\cdots Q^W_{t_n} f - \inf_v f.$$

Taking supremum, we obtain

$$P^W_t f \le P_t f + \inf_v P^W_t f - \inf_v f,$$ as claimed. □

Look at the inequality again,

$$P_t f \le P^W_t f \le P_t f + \underbrace{\inf_v P^W_t f} - \inf_v f.$$

Recall that

$\lim_{t\to\infty} \inf_v P^W_t f = c$, $\lim_{t\to\infty} P_t f$ is a const fct. (?)

We hope to controll $\inf_v P^W_t f - \inf_v f$.

~~Pick T>0~~ For any $\varepsilon>0$, $\exists\, T>0$ s.t.

$$|c - \inf_v P^W_T f| < \varepsilon.$$

Hence

$$P_t(P^W_T f) \le P^W_t P^W_T f \le P_t(P^W_T f) + \inf_v P^W_t(P^W_T f) - \inf_v P^W_T f$$

$$= P_t(P^W_T f) + \underbrace{\inf_v P^W_{t+T} f}_{\le c\ \cancel{\text{as } t\to\infty}} - \inf_v P^W_T f$$

$$\Rightarrow \quad P_t(P^W_T f) \le P^W_{t+T} f \le P_t(P^W_T f) + c - \inf_v P^W_T f$$

$$\le P_t(P^W_T f) + \varepsilon.$$

~~$\exists\, T'>0$ s.t.~~

$$\cancel{|P_{T'}(P^W_T f) - \lim_{t\to\infty} P_t(P^W_T f)| < \varepsilon.}$$

$$\cancel{\Rightarrow \lim_{t\to\infty} P_t(P^W_T f) - \varepsilon \le P^W_{T'+T} f \le \lim_{t\to\infty} P_t(P^W_T f) + 2\varepsilon.}$$

That is, for any $\varepsilon>0$, ~~$\exists T'$~~ we have

$$\lim_{t\to\infty} P_t(P^W_T f) \le C = \limsup_{t\to\infty} \sup_v P^W_{t+T} f \le \lim_{t\to\infty} P_t(P^W_T f) + \varepsilon$$

$$\lim_{t\to\infty} P_t(P^W_T f) \le c = \liminf_{t\to\infty} \inf_v P^W_{t+T} f \le \lim_{t\to\infty} P_t(P^W_T f) + \varepsilon$$

$$\Longrightarrow \quad -\varepsilon \le C - c \le \varepsilon.$$ This means $C = c$. □

Now, we are prepared to show the gradient estimate. (12) on page (0)$_b$.

__Proof__ for $x \in V$, Let $F(s) = e^{2Ks} P_{t-s}(\Gamma P_s^W f)(x)$ Then

$$\partial_s^+ F(s) = 2KF(s) + e^{2Ks} \partial_s^+ \left(P_{t-s} \Gamma P_s^W f \right)(x). \quad \triangleleft 1 \triangleright$$

Observe

$$\frac{P_{t-s-\varepsilon} \Gamma(P_{s+\varepsilon}^W f)(x) - P_{t-s} \Gamma(P_s^W f)(x)}{\varepsilon}$$

$$= \frac{P_{t-s-\varepsilon} \Gamma(P_{s+\varepsilon}^W f)(x) - P_{t-s} \Gamma(P_{s+\varepsilon}^W f)(x) + P_{t-s}\left(\Gamma(P_{s+\varepsilon}^W f) - \Gamma(P_s^W f)\right)(x)}{\varepsilon}$$

$$= \frac{P_{t-s-\varepsilon} \Gamma(P_{s+\varepsilon}^W f)(x) - P_{t-s} \Gamma(P_{s+\varepsilon}^W f)(x)}{\varepsilon} + P_{t-s}\left(\frac{\Gamma(P_{s+\varepsilon}^W f) - \Gamma(P_s^W f)}{\varepsilon}\right)(x)$$

By property (vi')
$$\left|\partial_s^+ \Gamma(P_s^W f)\right| \leq 2\left|\Gamma(P_s^W f, \Delta P_s^W f)\right|$$
$$\leq 2 \cdot 2 \, \|P_s^W f\|_\infty \|\Delta P_s^W f\|_\infty$$
$$\leq 4 \, \|f\|_\infty \cdot 2\|f\|_\infty = 8 \|f\|_\infty .$$

Then we derive from dominant convergence theorem that

$$\lim_{\varepsilon \to 0} P_{t-s}\left(\frac{\Gamma(P_{s+\varepsilon}^W f) - \Gamma(P_s^W f)}{\varepsilon}\right)(x) = P_{t-s}\left(\partial_s^+ \Gamma(P_s^W f)\right)(x)$$

For the 1^{st} term, by mean-value theorem, we have $\exists \, \delta \doteq \delta(\varepsilon) \in [0, \varepsilon]$

s.t.
$$\lim_{\varepsilon \to 0} \frac{P_{t-s-\varepsilon} \Gamma(P_{s+\varepsilon}^W f)(x) - P_{t-s} \Gamma(P_{s+\varepsilon}^W f)(x)}{\varepsilon}$$
$$= \lim_{\varepsilon \to 0} \left[- \Delta P_{t-s-\delta(\varepsilon)} \Gamma(P_{s+\varepsilon}^W f)(x)\right] = - \Delta P_{t-s} \Gamma(P_s^W f)(x)$$

In conclusion, we arrive at

$$\partial_s^+ (P_{t-s} \Gamma P_s^W f)(x) = - \Delta P_{t-s} \Gamma(P_s^W f)(x) + P_{t-s}(\partial_s^+ \Gamma(P_s^W f))(x)$$

⊛

For $f \in \ell^\infty(V)$, we have the community ~~that~~ of Δ and P_t.

i.e. $P_t \Delta f = \Delta P_t f$. Therefore

$$\partial_s^+ (P_{t-s} \Gamma P_s^W f)(x) = P_{t-s}\left((\partial_s^+ - \Delta) \Gamma (P_s^W f)\right)(x).$$

Inserting into <1>, we have

$$\partial_s^+ F(s) = 2K F(s) + e^{2Ks} P_{t-s} \left((\partial_s^+ - \Delta) \Gamma P_s^W f \right)(x)$$

$$\overset{(vi)}{\leq} 2K F(s) + e^{2Ks} P_{t-s} \Big(\underbrace{2\Gamma(P_s^W f, \Delta P_s^W f) - \Delta \Gamma P_s^W f}_{= -2\Gamma_2(P_s^W f)} \Big)(x)$$

$$= 2K F(s) - 2e^{2Ks} P_{t-s}(\Gamma_2(P_s^W f))(x).$$

$CD(K,\infty)$ on $V \setminus W \Rightarrow \Gamma_2(P_s^W f) \geq K \Gamma(P_s^W f)$ on $V\setminus W$.

$P_s^W f \in \ell_W^\infty(V) \Rightarrow \Gamma_2(P_s^W f) \geq 0 = K\Gamma(P_s^W f)$ on W

Hence, we have $\Gamma_2(P_s^W f) \geq K \Gamma(P_s^W f)$ on the whole V.

$$\Rightarrow \partial_s^+ F(s) \leq 2KF(s) - 2e^{2Ks} P_{t-s}(K \Gamma(P_s^W f))(x)$$

$$= 2KF(s) - 2KF(s) = 0.$$

Hence, $F(s)$ is nonincreasing which implies that

$P_t(\Gamma f)(x) = F(0) \geq F(t) = e^{2Kt} \Gamma(P_t^W f)(x)$. □

<u>Thm</u> (Münch). Let G be a locally finite graph. Suppose that G satisfies $CD(K,\infty)$ on $V\setminus W$ for some $K>0$ and $\phi \neq W \subset V$.

Then $$d(\cdot, W) \leq \frac{2D}{K} + 1.$$

<u>Proof</u>. It remains to show

$$d(\cdot, cl(W)) \le \frac{2}{K}.$$

Let $f := d(\cdot, cl(W)) \wedge \frac{3}{K} \in \ell^\infty_W(V)$. We have for $x \in V$, $d(x, cl(W)) \wedge \frac{3}{K} = f(x) = f(x) - f(w)$ for some $w \in cl(W)$ that

$$f(x) = f(x) - f(w) = f(x) - P_t^W f(x) + P_t^W f(x) - P_t^W f(w) + P_t^W f(w) - f(w)$$

$$\le \overline{\int_0^\infty} |\partial_t^+ P_t^W f(x)| \, dt + |P_t^W f(x) - P_t^W f(w)| + \overline{\int_0^\infty} |\partial_t^+ P_t^W f(w)| \, dt$$

[Here, we use the following result:

For $F : I \to \mathbb{R}$ a continuous function, if $|\partial_t^+ F| < \infty$ everywhere, then

$$F(b) - F(a) \le \overline{\int_a^b} \partial_t^+ F(t) \, dt$$

on every interval $[a, b]$ where $\overline{\int}$ stands for the upper Riemann integral. See Hagood and Thomson, Recovering a function from a Dini derivative, Amer. Math. Monthly 113 (2006), no. 1, 34-46.]

By gradient ~~$\le 2 \int_0^\infty$~~ estimate, we have

$$|\partial_t^+ P_t^W f(x)|^2 \underset{(V)}{\le} \|\Delta P_t^W f\|_\infty^2 \underset{|\Delta f|^2 \le 2\Gamma(f)}{\le} 2 \|\Gamma P_t^W f\|_\infty \le 2 \cdot e^{-2Kt} \|P_t \Gamma f\|_\infty$$

$$\le 2 \cdot e^{-2Kt} \|\Gamma f\|_\infty \underset{f \text{ is 1-Lipschitz}}{\le} 2 e^{-2Kt} \cdot \frac{1}{2} = e^{-2Kt}.$$

Hence, $f(x) \le \underbrace{2 \int_0^\infty e^{-Kt} \, dt}_{= \frac{2}{K}} + |P_t^W f(x) - P_t^W f(w)|$.

Since $|\Gamma P_t^W f(x)| \le \frac{1}{2} e^{-2Kt} \longrightarrow 0$ as $t \to \infty$ (since $K > 0$).

We derive from connectivity of G that

$$|P_t^w f(x) - P_t^w f(w)| \to 0 \text{ as } t \to \infty.$$

Hence, $f(x) = d(\cdot, cl(w)) \wedge \frac{3}{K} \leq \frac{2}{K}$.

This tells $d(\cdot, cl(w)) \leq \frac{2}{K}$.