Formula for $Q(x)$.

$$Q(x) = \Big(Q(x)_{y_i, y_j} \Big)_{1 \le i,j \le d_x}$$ with

$i \neq j$

$$Q(x)_{y_i,y_j} = \frac{1}{2} P_{xy_i} P_{xy_j} - \frac{1}{2} P_{xy_i} P_{y_i y_j} - \frac{1}{2} P_{xy_j} P_{y_j y_i} - \sum_{z \in S_2(x)} \frac{P_{xy_i} P_{y_i z} P_{xy_j} P_{y_j z}}{P^{(2)}_{xz}} .$$

$i = j$

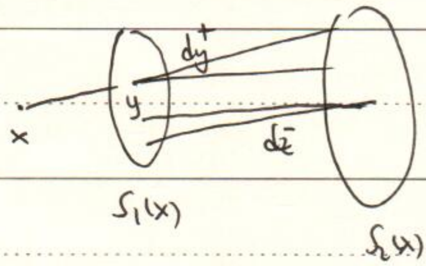
$$Q(x)_{y_i,y_i} = \frac{1}{2} P_{xy_i}^2 + \frac{1}{4} \sum_y P_{xy} P_{yy_i} + \frac{1}{2} P_{xy_i} - \sum_{z \in S_2(x)} \frac{P_{xy_i}^2 P_{y_i z}^2}{P^{(2)}_{xz}} .$$

Formula for $A_\infty(x)$:

Since $A_\infty(x) = 2\, P(x)^{-1} Q(x) P(x)^{-1}$, we have

$$A_\infty(x)_{i,j} = \frac{2\, Q(x)_{y_i,y_j}}{\sqrt{P_{xy_i} \; P_{xy_j}}}$$

Restriction to d-regular graphs.



Define
$$d_y^+ = \sum_{z \in S_2(x)} a_{yz}$$
$$d_y^0 = \sum_{y' \in S_1(x)} a_{yy'} = \sum_{y'} a_{xy'} a_{y'y}$$
$$d_z^- = \sum_{y \in S_1(x)} a_{yz} = \sum_{w} a_{xw} a_{wz}$$

$i \neq j$

$$A_\infty(x)_{i,j} = 2d\, Q(x)_{y_i y_j} = 2d \left(\frac{1}{2d^2} - \frac{a_{y_i y_j}}{2d^2} - \frac{a_{y_j y_i}}{2d^2} - \sum_{z \in S_2(x)} \frac{\frac{1}{d} \cdot \frac{a_{y_i z}}{d} \cdot \frac{1}{d} \frac{a_{y_j z}}{d}}{\frac{1}{d^2} d_z^-} \right)$$

$$= \frac{1}{d} \left(1 - 2 a_{y_i y_j} - 2 \sum_{z \in S_2(x)} \frac{a_{y_i z} a_{y_j z}}{d_z^-} \right)$$

$i = j$

$$A_\infty(x)_{i,i} = 2d\, Q(x)_{y_i y_j} = 2d \left(\frac{1}{2d^2} + \frac{1}{4} \sum_y \frac{a_{xy} a_{yy_i}}{d^2} + \frac{1}{2d} - \sum_{z \in S_2(x)} \frac{\frac{1}{d^2} \cdot \frac{a_{y_i z}^2}{d^2}}{\frac{1}{d^2} d_z^-} \right)$$
$$= \frac{1}{d} \left(1 + \frac{1}{2} d_{y_i}^0 + d - 2 \sum_{z \in S_2(x)} \frac{a_{y_i z}}{d_z^-} \right)$$

Matrix Structure :

$dA_\infty - J$ is a "generalized Laplacian".
The off diagonal entries give "adjacency" relations between y_i and y_j in $S_1(x)$:

$$-2a_{y_iy_j} - 2\sum_{z\in S_2(x)} \frac{a_{y_iz}\,a_{y_jz}}{d_z^-}$$

The on-diagonal entries are

$$d + \frac{1}{2}d_{y_i}^0 - 2\sum_{z\in S_2(x)} \frac{a_{y_iz}}{d_z^-}.$$

If we sum up the "off-diagonal" entries in each row, we have

$$\sum_{j\neq i}\left(-2a_{y_iy_j} - 2\sum_{z\in S_2(x)}\frac{a_{y_iz}\,a_{y_jz}}{d_z^-}\right)$$

$$= -2d_{y_i}^0 - 2\sum_{z\in S_2(x)}\frac{a_{y_iz}\sum_{j\neq i}a_{y_jz}}{d_z^-}$$

$$= -2d_{y_i}^0 - 2\sum_{z\in S_2(x)}\frac{a_{y_iz}(d_z^- - 1)}{d_z^-} = -2d_{y_i}^0 - 2d_{y_i}^+ + 2\sum_{z\in S_2(x)}\frac{a_{y_iz}}{d_z^-}$$

Noticing that $-\sum_{j\neq i}\left(-2a_{y_iy_j} - 2\sum_{z\in S_2(x)}\frac{a_{y_iz}a_{y_jz}}{d_z^-}\right) - \left(d + \frac{1}{2}d_{y_i}^0 - 2\sum_{z\in S_2(x)}\frac{a_{y_iz}}{d_z^-}\right)$

$$= 2d_{y_i}^0 + 2d_{y_i}^+ - d - \frac{1}{2}d_{y_i}^0 = \frac{3}{2}d_{y_i}^0 + \frac{3}{2}d_{y_i}^+ + \frac{1}{2}d_{y_i}^+ - d$$

$$= \frac{3(d-1)}{2} - d + \frac{1}{2}d_{y_i}^+ = \frac{d-3}{2} + \frac{1}{2}d_{y_i}^+,$$

we have

$$(dA_\infty - J)f(y_i) = -\sum_{j\neq i}\left(2a_{y_iy_j} + 2\sum_{z\in S_2(x)}\frac{a_{y_iz}a_{y_jz}}{d_z^-}\right)(f(y_j) - f(y_i))$$
$$- \frac{d-3}{2}f(y_i) - \frac{1}{2}d_{y_i}^+ f(y_i) \qquad \square$$

In matrix form, we have

$$d A_\infty - J = \Delta_{S_1,S_2} - \frac{d-3}{2} I - \frac{1}{2} \operatorname{diag}\left(d_{y_1}^+, \ldots, d_{y_{d_x}}^+\right)$$

where $\Delta_{S_1,S_2} \mathbb{1} = 0$ and

$$(\Delta_{S_1,S_2})_{ij} = -2 a_{y_i y_j} - 2 \sum_{z \in S_2(x)} \frac{a_{y_i z} a_{y_j z}}{d_z^-}.$$

Remark on the "dual" version of curvature matrix.

An interesting dual viewpoint about the curvature matrix method is as follows: We have

$$K_{G,x}(\infty) = \max\left\{ K \;\middle|\; \Gamma_2(f)(x) \geq K \Gamma(f)(x), \ \forall f : B_2(x) \to \mathbb{R} \text{ with } f(x) = 0 \right\}$$

$$= \max\left\{ K \;\middle|\; \Gamma_2(f)(x) \geq K\Gamma(f)(x), \ \forall f : B_2(x) \to \mathbb{R} \text{ with } f(x)=0 \text{ and } \begin{pmatrix} f(z_1) \\ \vdots \\ f(z_m) \end{pmatrix} = -\Gamma_2(x)_{S_2,S_2}^{-1} \Gamma_2(x)_{S_2,S_1} \begin{pmatrix} f(y_1) \\ \vdots \\ f(y_{d_x}) \end{pmatrix} \right\}$$

By previous explicit calculations for $\Gamma_2(x)_{S_2,S_2}$ and $\Gamma_2(x)_{S_2,S_1}$, we derive

$$K_{G,x}(\infty) = \max\left\{ K \;\middle|\; \Gamma_2(f)(x) \geq K\Gamma(f)(x), \ \forall f : B_2(x) \to \mathbb{R} \text{ with } f(x)=0, \text{ and } f(z_k) = \frac{2\sum_y p_{xy} p_{yz_k} f(y)}{p_{xz_k}^{(2)}}, \forall k = 1, \ldots, m \right\}.$$

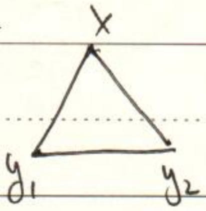
$\square$

Ex 1. K_2

$$dA_\infty(x) - J = \begin{pmatrix}0\end{pmatrix} - \frac{d-3}{2} I = \begin{pmatrix}1\end{pmatrix}$$

$$\Rightarrow A_\infty(x) = (2) \Rightarrow K_{K_2,x}(\infty) = 2 .$$

Ex 2 K_3



$$2A_\infty(x) - J = \begin{pmatrix}2 & -2\\ -2 & 2\end{pmatrix} - \frac{-1}{2} I - \frac{1}{2}\begin{pmatrix}0 & \\ & 0\end{pmatrix}$$

$$= 2\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix} + \frac{1}{2} I$$

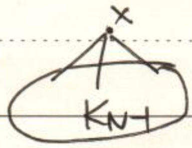
$$\Rightarrow A_\infty(x) = \begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix} + \frac{1}{4} I + \frac{1}{2} J$$

For constant vectors, we get the eigenvalue $0 + \frac{1}{4} + \frac{1}{2}\cdot 2 = \frac{5}{4}$

For vectors vertical to constant ones, we have $2 + \frac{1}{4} + \frac{1}{2}\cdot 0 = \frac{9}{4}$.

Therefore $K_{K_3,x}(\infty) = \frac{5}{4}$.

Ex 3. K_N



$$(N-1)\, A_\infty(x) - J = \begin{pmatrix}2(N-2) & -2 & \cdots & -2\\ -2 & \ddots & & \vdots\\ \vdots & & \ddots & -2\\ -2 & \cdots & -2 & 2(N-2)\end{pmatrix} - \frac{N-1-3}{2} I - \frac{1}{2} 0$$

$$\Rightarrow A_\infty(x) = \frac{1}{N-1}\left[2\begin{pmatrix}N-2 & -1 & \cdots & -1\\ -1 & \ddots & & \vdots\\ \vdots & & \ddots & -1\\ -1 & \cdots & -1 & N-2\end{pmatrix} - \frac{N-4}{2} I + J\right]$$

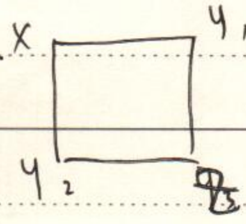
For constant vectors, We derive an eigenvalue $\frac{1}{N-1}\left[0 - \frac{N-4}{2} + N-1\right] = \frac{N+2}{2(N-1)}$

For vectors vertical to constant ones, we have

$$\frac{1}{N-1}\left[2(N-1) - \frac{N-4}{2}\right] = \cancel{\frac{N+2}{2(N-1)}}\ \frac{3N}{2(N-1)}$$

~~(A more easier way: $A_\infty(x) = \frac{1}{N-1}\left[\begin{pmatrix}N-1 & & 0\\ & \ddots & \\ 0 & & N-1\end{pmatrix} - \frac{N-4}{2} I\right]$.~~

Ex 4. C_4

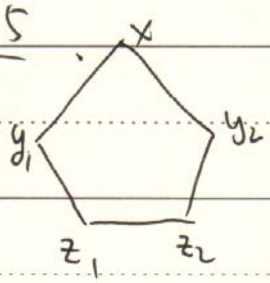


$$2A_\infty(x) - J = \begin{pmatrix}1 & -2\cdot\frac{1}{2}\\ -2\cdot\frac{1}{2} & 1\end{pmatrix} - \frac{2-3}{2} I + \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$$\Rightarrow A_\infty(x) = \frac{1}{2}\left[\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix} + I + J\right] = \frac{1}{2}\left[\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix} + I\right]$$

That is $A_\infty(x) = I$, hence, $K_{G,x}(\infty) = 1$.

Ex 5

C_5 

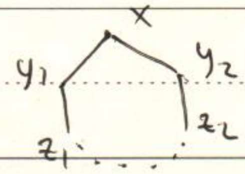
$$2A_\infty(x) + J = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} - \frac{2-3}{2} I \cancel{+} \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow A_\infty(x) = \frac{1}{2}\left[-\frac{1}{2} I - \frac{1}{2} I + J\right] = \frac{1}{2} J$$

Hence the eigenvalues of $A_\infty(x)$ are 1 and 0.

This tells $K_{C_5,x}(\infty) = 0$

Ex 6

C_N, $N \geq 5$ 

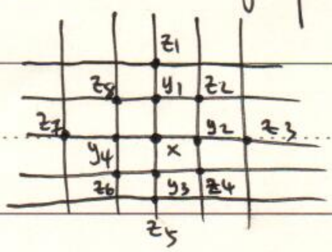
$$2A_\infty(x) + J = 0 + \frac{1}{2} I - \frac{1}{2} I \Rightarrow A_\infty(x) = \frac{1}{2} J$$

$$\Rightarrow K_{C_N,x}(\infty) = 0.$$

Ex 7 For an infinite path graph P_∞ 

Similarly, we have again $A_\infty(x) = \frac{1}{2} J$ and $K_{P_\infty,x}(\infty) = 0$.

Ex 8 lattice graph $\mathbb{Z}^2$.



$$4A_\infty(x) - J = \begin{array}{c|cccc} & y_1 & y_2 & y_3 & y_4 \\ \hline y_1 & & & & \\ y_2 & & & & \\ y_3 & & & & \\ y_4 & & & & \end{array}\!\!\!\!\begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix} - \frac{4-3}{2} I \cancel{+} - \frac{1}{2}\begin{pmatrix} 3 & & & \\ & 3 & & \\ & & 3 & \\ & & & 3 \end{pmatrix}$$

(non-normalized Laplacian for C_4)

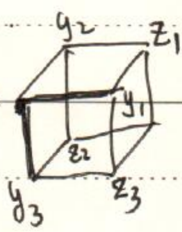
$$\Rightarrow A_\infty(x) = \frac{1}{4}\left[\begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix} - \frac{1}{2} I - \frac{3}{2} I + J\right]$$

$$= \frac{1}{4}\left[\begin{pmatrix} 3 & 0 & 1 & 0 \\ 0 & 3 & 0 & 1 \\ 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 3 \end{pmatrix} - 2I\right] = \frac{1}{4}\begin{array}{cc} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} & \\ \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} & \begin{array}{c} y_1 \\ y_2 \\ y_3 \\ y_4 \end{array} \end{array}$$

which is similar to

$$\frac{1}{4}\begin{array}{cc} \begin{array}{cccc} y_1 & y_3 & y_2 & y_4 \end{array} & \\ \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} & \begin{array}{c} y_1 \\ y_3 \\ y_2 \\ y_4 \end{array} \end{array}$$

Ex 9.



$Q^3_8 = (K_2)^3 = K_2 \times C_4$

$$3A_\infty(x) - J = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} - \frac{3-3}{2} I - \frac{1}{2}\begin{pmatrix} 2 & & \\ & 2 & \\ & & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$$

$$\Rightarrow 3A_\infty(x) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\Rightarrow K_{Q^3,x}(\infty) = \frac{2}{3}.$$

Putting the examples of K_2, C_4, Q^3 together, and the examples of P_∞ ($= \mathbb{Z}$) and $\mathbb{Z}^2$ together, it is quite natural to guess that

for any two ~~d~~ regular graphs G and G', we have

$$(d_G + d_{G'}) A_\infty^{G\times G'}(x,x') = \begin{pmatrix} d_G A_\infty^G(x) & 0 \\ 0 & d_{G'} A_\infty^{G'}(x') \end{pmatrix}.$$

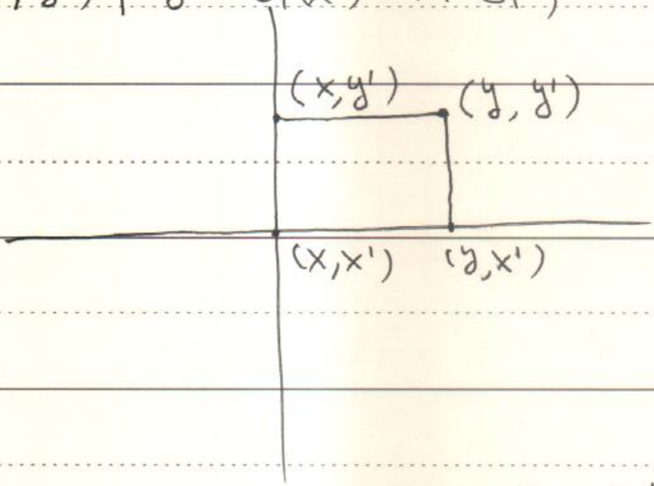
Thm (Cushing-Kamtue-L.-Peyerimhoff 2022). Let G and G' be two regular graphs with vertex degrees d_G and $d_{G'}$ respectively. Then the curvature matrix $A_\infty^{G\times G'}((x,x'))$ of the Cartesian product of G and G' at (x,x') ~~is the direct~~ ~~can be~~ is given by the following direct sums.

$$(d_G + d_{G'}) A_\infty^{G\times G'}((x,x')) = d_G A_\infty^G(x) \oplus d_{G'} A_\infty^{G'}(x').$$

Proof. At the vertex $(x,x') \in V(G) \times V(G')$, we partition all its neighbors into two classes:

horizontal neighbors : $\{(y,x') \mid y \in S_1(x) \text{ in } G\}$.

vertical neighbors $\{(x,y') \mid y' \in S_1(x') \text{ in } G'\}$



Observation: For two neighbors (x,y') and (y,x'), there is precisely one common neighbor in 2-sphere of (x,x') in $G\times G'$, that is, (y,y').

Moreover, (x,y') and (y,x') are NOT adjacent in $G\times G'$.

On the otherhand, the vertex (y,y') has precisely two neighbors in the 1-sphere of (x,x'), namely, (x,y') and (y,x'). We aim at clarify the matrix

$$(d_G+d_{G'})\,A_\infty^{G\times G'}((x,x')) = \begin{array}{cc} \text{horizontal} & \text{vertical} & \\ \left(\begin{array}{c|c} d_G\, A_\infty^{G}(x) & 0 \\ \hline 0 & d_{G'}\, A_\infty^{G'}(x') \end{array}\right) & & \begin{array}{l}\text{horizontal}\\ \text{vertical}\end{array} \end{array}$$

We do it step by step:

① horizontal - vertical block.

$$\left[(d_G+d_{G'})\,A_\infty^{G\times G'}((x,x'))\right]_{(y,x'),(x,y')}$$

$$= \underset{\text{from } J}{1} + \cancel{d_G+d}\left[-2\,\underset{=0}{a_{(y,x')(x,y')}} - 2\underbrace{\sum_{z\in S_2((x,x'))}}_{\text{only one nontrivial } z\text{, namely } (y,y')} \frac{a_{(y,x')z}\,a_{z(x,y')}}{d_z}\right]$$

$$= 1 - 2\cdot\frac{1}{2} = 0.$$

Similarly, the symmetric vertical-horizontal block is also 0.

② h-h block and v-v block

Considering the h-h block, we have first ~~for~~ the off diagonal

$$\left[(d_G+d_{G'})A_\infty^{G\times G'}((x,x'))\right]_{(y_i,x'),(y_j,x')},\quad i\neq j$$

$$= 1+\left[-2a_{(y_i,x')(y_j,x')} - 2\sum_{z\in S_2((x,x'))}\frac{a_{(y_i,x')z}\,a_{z(y_j,x')}}{d_z^-}\right]$$

$$= 1+\left[-2a^G_{y_iy_j} - 2\sum_{z\in S_2^G(x)}\frac{a^G_{y_iz}\,a^G_{zy_j}}{d_z^-}\right] = \left[d_G A_\infty^G(x)\right]_{y_iy_j} \quad (*)$$

For diagonal entries, we calculate

$$\left[(d_G+d_{G'})A_\infty^{G\times G'}((x,x')) - J\right]_{(y_i,x')(y_i,x')}$$

$$= -\left(\sum_{\substack{y_j\in S_1^G(x)\\ j\neq i}}\underbrace{\left[(d_G+d_{G'})A_\infty^{G\times G'}((x,x'))\right]_{(y_i,x')(y_j,x')}}_{=\left[d_GA_\infty^G(x)\right]_{y_iy_j}\text{ due to }(*)} + \sum_{y'_j\in S_1^{G'}(x')}\underbrace{\left[(d_G+d_{G'})A_\infty^{G\times G'}((x,x'))\right]_{(y_i,x')(y_j',x')}}_{=\cdot -1 \text{ due to (1)}}\right)$$

$$-\frac{d_G+d_{G'}-3}{2} - \frac{1}{2}\underbrace{d^+_{(y_i,x')}}_{=d^+_{y_i}+d_{G'}} + 1$$

$$= -\sum_{j\neq i}\left[d_GA_\infty^G(x)\right]_{y_iy_j} + \cancel{d_{G'}} - \frac{d_G-3}{2} - \frac{1}{2}d^+_{y_i} - \cancel{d_{G'}}$$

$$= \left[d_GA_\infty^G(x) - J\right]_{y_iy_i}.$$ This verifies that the h-h block is $d_GA_\infty^G(x)$.

(Indeed, it is easier to use the formula $d+\frac{1}{2}d^0_{y_i} - 2\sum_{z\in S_2(x)}\frac{a_{y_iz}}{d_z^-}$ on (170') + 13[+] to verify the above identity. In fact

$$\left[(d_G+d_{G'})A_\infty^{G\times G'}((x,x')) - J\right]_{(y_i,x')(y_i,x')}$$

$$= d_G+d_{G'} + \frac{1}{2}d^0_{(y_i,x')} - 2\left(\sum_{z\in S_2^G(x)}\frac{a_{(y_i,x')(z,x')}}{d^-_{(z,x')}} + \sum_{y'\in S_1^{G'}(x')}\frac{a_{(y_i,x')(y_i,y')}}{d^-_{(y_i,y')}}\right)$$

(Observation: $d^0_{(y_i,x')} = d^0_{y_i}$, $d^-_{(y_i,y')} = 2$.) ~~Hence~~

$$= d_G + \cancel{d_{G'}} + \frac{1}{2}d^0_{y_i} - 2\sum_{z\in S_2^G(x)}\frac{a_{y_iz}}{d_z^-} - \cancel{d_{G'}} = \left[d_GA_\infty^G(x) - J\right]_{y_iy_i}.)$$

This completes the proof. $\square$

A direct consequence of the above Thm is that

$$K_{G\times G',(x,x')}(\infty) = \frac{1}{d_G+d_{G'}}\min\left\{d_GK_{G,x}(\infty),\ d_{G'}K_{G',x'}(\infty)\right\}.$$

Applying this consequence, we know for hypercubes that

$$K_{Q^n,\cdot}(\infty) = \frac{2}{n}.$$

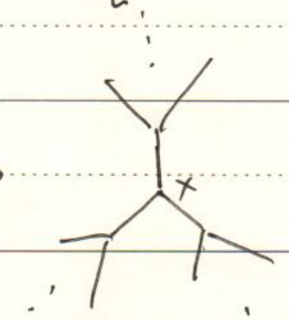
For Hamming graphs $H(n,q) = (K_q)^n$, $q \geq 2$, we have

$$K_{H(n,q),\cdot}(\infty) = K_{(K_q)^n,\cdot}(\infty) = \frac{1}{n} \cdot \frac{q+2}{2(q-1)}$$

Ex 10

Regular Trees

T_d



$$dA_\infty(x) - J$$

$$= \begin{pmatrix} 0 & & 0 \\ & \ddots & \\ 0 & & 0 \end{pmatrix} - \frac{d-3}{2} I - \frac{1}{2} \begin{pmatrix} d-1 & & \\ & \ddots & \\ & & d-1 \end{pmatrix}$$

$$\Rightarrow \quad d\, A_\infty(x) = \left(-\frac{d-3}{2} - \frac{d-1}{2}\right) I + J = -(d-2) I + J$$

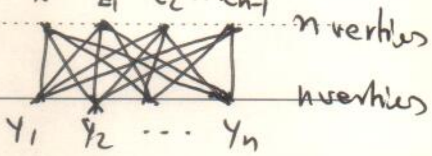
$$\Rightarrow \quad A_\infty(x) = \frac{1}{d}\left(-(d-2) I + J\right)$$

For a constant vector, we derive an eigenvalue $\frac{1}{d}(-(d-2)+d) = \frac{2}{d}$

For a verctor vertical to constant ones, we obtain an eigenvalue $\frac{1}{d}\left(-(d-2)+0\right) = -1 + \frac{2}{d}$.

Therefore, we have $K_{T_d,\cdot}(\infty) = -1 + \frac{2}{d}$.

Ex 11 Complete bipartite graph $K_{n,n}$



$$nA_\infty(x) - J = \begin{pmatrix} 2\frac{(n-1)^2}{n} - 2(n-1)\cdot\frac{1}{n} & \cdots & \cdots & -2\frac{n-1}{n} \\ -\frac{2(n-1)}{n} & \ddots & & \vdots \\ \vdots & & \ddots & -2\frac{n-1}{n} \\ -\frac{2(n-1)}{n} & \cdots & -2\frac{n-1}{n} & 2\frac{(n-1)^2}{n} \end{pmatrix} - \frac{n-3}{2} I - \frac{1}{2}\begin{pmatrix} n-1 & & \\ & \ddots & \\ & & n-1 \end{pmatrix}$$

$$\Rightarrow \quad A_\infty(x)=\frac{1}{n}\left[-(n-2)I+J+\frac{2(n-1)}{n}\begin{pmatrix} n-1 & -1 & \cdots & -1\\ -1 & n-1 & \cdots & \vdots \\ \vdots & & \ddots & -1 \\ -1 & \cdots & \cdots & n-1\end{pmatrix}\right]$$

For a 'constant' vector, we derive an eigenvalue

$$\frac{1}{n}\left[-(n-2)+n+0\right]=\frac{2}{n}$$

When $n\geq 2$, we consider a vector vertical to constant ones, and obtain an eigenvalue

$$\frac{1}{n}\left[-(n-2)+0+\frac{2(n-1)}{n}\cdot n\right]=1$$

Therefore for $n\geq 2$, $K_{K_{n,n},\cdot}(\infty)=\min\left\{\frac{2}{n},1\right\}=\begin{cases}1, & n=1\\ \frac{2}{n}, & n\geq 2\end{cases}$

$$=\frac{2}{n}.$$

When $n=1$, we have $K_{K_{n,n},\cdot}(\infty)=K_{K_2,\cdot}(\infty)=2$

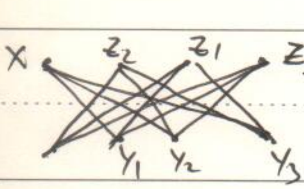
In conclusion, we have

$$K_{K_{n,n},\cdot}(\infty)=\frac{2}{n} \quad \forall n.$$

Rmk. We see $K_{n,n}$ share the same BÉ curvature as Q^n.
Indeed, for any finite dim. parameter N, they share the same N-curvature $K_{K_{n,n},\cdot}(N)=K_{Q^n,\cdot}(N)$.

Ex 12 Crown graph. $\mathrm{Crown}(n,n)=K_{n,n}\setminus\{\text{a perfect matching}\}$.
Let $V=\{x_1,\cdots,x_n,x_{n+1},\cdots,x_{n+n}\}$. Then the edge set is

$$E=\{\{x_i,x_{n+j}\},\ i=1,\cdots,n,\ j\neq i\}.$$



Note Crown(1,1) is empty
Crown(2,2) is a disjoint union of two K_2.

Next, we can suppose $n\geq 3$.

$$(n-1)A_\infty(x)-J=\begin{pmatrix}2(n-3) & -2\frac{n-3}{n-2} & \cdots & -2\frac{n-3}{n-2}\\ -2\frac{n-3}{n-2} & 2(n-3) & \cdots & -2\frac{n-3}{n-2}\\ \vdots & & \ddots & \vdots\\ -2\frac{n-3}{n-2} & \cdots & \cdots & 2(n-3)\end{pmatrix}-\frac{(n-1)-3}{2}I-\frac{1}{2}\begin{pmatrix}n-2 & & \\ & \ddots & \\ & & n-2\end{pmatrix}$$

$$\Rightarrow A_\infty(x) = \frac{1}{n-1}\left[\frac{2(n-3)}{n-2}\begin{pmatrix} n-2 & -1 & \cdots & -1 \\ -1 & n-2 & \cdots & -1 \\ \vdots & & \ddots & \vdots \\ -1 & -1 & \cdots & n-2 \end{pmatrix} - \frac{n-4}{2} I - \frac{n-2}{2} I + J\right]$$

$$= \frac{1}{n-1}\left[\frac{2(n-3)}{n-2}\left((n-1)I - J\right) - (n-3)I + J\right]$$

For a constant vector, we derive an eigenvalue

$$\frac{1}{n-1}\left[\frac{2(n-3)}{n-2}\left[(n-1)-(n-1)\right] - (n-3) + (n-1)\right] = \frac{2}{n-1}$$

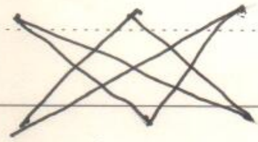
For a vector vertical to constant ones, we obtain an eigenvalue

$$\frac{1}{n-1}\left[\frac{2(n-3)}{n-2}(n-1) - (n-3)\right] = 1 + \frac{2}{n-1} - \frac{2}{n-2}$$

Therefore

$$K_{\mathrm{Crown}(n,n),\cdot}(x) = \begin{cases} \dfrac{2}{n-1} & n \geq 4 \\[2ex] \dfrac{2}{3-1} - 1 = 0 & n = 3 \end{cases}$$

Indeed $\mathrm{Crown}(3,3) = C_6$



Ex 13. Johnson graph $J(n,k)$, is the graph with the set of k-elem. subsets of an n-elem. set as the vertex set V, where two k-elem. subsets are adjacent if they share precisely $k-1$ elements.

In particular, $J(n,1) = K_n$, $J(n,2)$ = the line graph of K_n.

Consider an n-element set $[n] := \{1, 2, \cdots, n\}$.

Let x be the following k-elem. subset $\{i_1, \cdots, i_k\} \subseteq [n]$.

Then $S_1(x)$ is given by

$$y_{\ell,m} := \{i_1, \cdots, \hat{i_\ell}, \cdots, i_k, j_m\} \quad \ell \in [k], \ m \in [n-k]$$

where we use $\{j_1, \cdots, j_{n-k}\}$ to denote the complement $[n] \setminus \{i_1, \cdots, i_k\}$.

Therefore, we see $d_x := k(n-k)$.

Any vertex $y_{\ell,m} \in S_1(x)$ is adjacent to the following vertices in $S_2(x)$:

$$z_{\ell,\ell',m,m'} = \{ i_1, \dots, \hat{i}_\ell, \dots, \hat{i}_{\ell'}, \dots, i_k, j_m, j_{m'} \}, \ \ell' \in [k] \setminus \{\ell\}, \ m' \in [n-k] \setminus \{m\}.$$

Then $d^+_{y_{\ell,m}} = (k-1)(n-k-1)$

On the otherhand, for any $z_{\ell,\ell',m,m'} \in S_2(x)$, there are precisely 4 neighbors in $S_1(x)$, namely, $y_{\ell,m}, y_{\ell,m'}, y_{\ell',m}, y_{\ell',m'}$. In particular $d^-_{z_{\ell,\ell',m,m'}} = 4$.

Then $k(n-k) A_\infty(x) - J = \Delta_{S_1 S_2} - \frac{k(n-k)-3}{2} I - \frac{1}{2}(k-1)(n-k-1) I$

$$= \Delta_{S_1,S_2} - \frac{2k(n-k)-n-2}{2} I$$

It Remains to figure out the "Laplacian" Δ_{S_1,S_2} for the graph with $V = \emptyset S_1(x)$, and the weight between y_{ℓ_1,m_1} and y_{ℓ_2,m_2} are

$$2 a_{y_{\ell_1,m_1} y_{\ell_2,m_2}} + 2 \sum_{z \in S_2(x)} \frac{a_{y_{\ell_1,m_1} z}\, a_{y_{\ell_2,m_2} z}}{4}$$

<u>Observation 1</u>: $a_{y_{\ell_1,m_1} y_{\ell_2,m_2}} = 1$ iff $\ell_1 = \ell_2, m_1 \neq m_2$ or $\ell_1 \neq \ell_2, m_1 = m_2$

So the induced subgraph of $S_1(x)$ is $K_k \times K_{n-k}$.

<u>Observation 2</u>.

$$2 \sum_{z \in S_2(x)} \frac{a_{y_{\ell_1,m_1} z}\, a_{y_{\ell_2,m_2} z}}{4} = \frac{1}{2} \cdot \#\{\text{common neighbors of } y_{\ell_1,m_1} \text{ and } y_{\ell_2,m_2} \text{ in } S_2(x)\}.$$

$$= \begin{cases} \frac{1}{2} \# \{ Z_{l_1, l_1', m_1, m_2} : l_1' \in [k] \setminus \{l_1\} \} = \frac{1}{2}(k-1), & \text{if } l_1 = l_2,\ m_1 \neq m_2 \\ \frac{1}{2}(n-k-1), & \text{if } l_1 \neq l_2,\ m_1 = m_2 \\ \frac{1}{2} \# \{ Z_{l_1, l_2, m_1, m_2} \} = \frac{1}{2}, & \text{if } l_1 \neq l_2\ m_1 \neq m_2 . \end{cases}$$

Therefore,

$\Delta_{S_1,S_2} =$ the (positive) non-normalized Laplacian of the product

$$\Big(K_{n-k},\ \underbrace{2 + \frac{k-1}{2} - \frac{1}{2}}_{= \frac{k+2}{2}}\Big) \times \Big(K_k,\ \underbrace{2 + \frac{n-k-1}{2} - \frac{1}{2}}_{= \frac{n-k+2}{2}}\Big)$$

+ the (positive) non-normalized Laplacian of

$$\Big(K_{k(n-k)},\ \frac{1}{2}\Big).$$

Recall the eigenvalues of $(K_{n-k}, \frac{k+2}{2})$ are $\{0, \frac{(n-k)(k+2)}{2}\}$

of $(K_k, \frac{n-k+2}{2})$ are $\{0, \frac{k(n-k+2)}{2}\}$

of $(K_{k(n-k)}, \frac{1}{2})$ are $\{0, \frac{k(n-k)}{2}\}$.

Taking a constant vector, we have an eigenvalue of $A_\infty(x)$

$$\frac{1}{k(n-k)} \Big[k(n-k) - \frac{2k(n-k)-n-2}{2} \Big] = \frac{1}{k(n-k)} \cdot \frac{n+2}{2}$$

Considering vectors vertical to constant ones, we get the corresponding minimal eigenvalue of $A_\infty(x)|_{1^\perp}$

$$\frac{1}{k(n-k)} \Big[\min \big(\Delta_{S_1 S_2}|_{1^\perp} \big) - \frac{2k(n-k)-n-2}{2} \Big]$$

$$= \frac{1}{k(n-k)} \Big[\min \Big\{ \frac{(n-k)(k+2)}{2} + \frac{k(n-k)}{2},\ \frac{k(n-k+2)}{2} + \frac{k(n-k)}{2} \Big\} - k(n-k) + \frac{n+2}{2} \Big]$$

$$= \frac{1}{k(n-k)} \Big[\underbrace{\min \{ (n-k)(k+1),\ k(n-k+1) \} - k(n-k)}_{>0} + \frac{n+2}{2} \Big]$$

Therefore $K_{J(n,k),\cdot}(\infty) = \lambda_{\min}(A_\infty(x)) = \frac{n+2}{2} \cdot \frac{1}{k(n-k)}$.