Lectures on Riemannian Geometry

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Introduction

On June 10, 1854, Georg Friedrich Bernhard Riemann delivered a lecture entitled "*ü*ber die Hypothesen, welche der Greometrie zu liegen"(On the Hypotheses which lie at the Foundations of Geometry) to the faculty of G*ö*ttingen University. This lecture was published later in 1866, and gives birth to Riemannian geometry.

This lecture was given by Riemann as a probationrary inaugural lecture for seeking the position of "Privatdocent". Privatdocent is a position in the German university system. It is a lecturer who receive no salary, but is merely forwarded fees paid by those students who elected to attend his lectures. To attain such a position, one has to submit an inaugural paper(Habilitationsschrift) and give a probationary inaugural lecture on a topic chosen by the faculty, from a list of 3 proposed by the coordidates. The first 2 topics which Riemann submitted were ones on which he has already worked; The 3rd topic ha chose was the foundations of geometry. Usually, the faculty chooses the first topic proposed by the candidate. However, contrary to all traditions, Gauss passed over the first two and chose instead the 3rd of Riemann's topics. So Riemann has to prepare a lecture on a topic that he had not worked on before. In the end, Riemann finished his lecture in about seven more weeks.

Why did Gauss choose the 3rd topic? In fact, that is topic in which Gauss had been interested in for many years.

The single most important work in the history of differential geometry is Gauss' paper, in Latin, of 1827: "Disquisitiones generales circa super ficies curvas"(General Investigations of Curved Surfaces). The most influential result in Gauss' paper is the so-called "Theorem Egregium". Roughly speaking, this theorem asserts that the Gauss cvrvature of a surface is determined by its first fundamental form. This opens the door to "Intrinsic geometry" and provides possibility of studying more abstract spaces other than surfaces in E^3. For example, one can study the geometry of a "flat torus", which is a topological torus associated with a "flat metric".

What Riemann did in his lecture is developing higher-dim intrinsic geometry. Geometry presupposes the concept of space. In this course of Riemannian geometry, the space we study is a C^∞-manifold M(Hausdorff and second countable) associated a Riemannian metric g. A vital step is to understand what is the extension of Gauss curvature in higher dimensional manifolds. The original definition of Gauss curvature using Gauss map is not available in higher dimensional manifolds. The expression of Gauss curvature in the Gauss equation is possibly extended to higher dimensional spaces.

Actually, before discussing "curvature", we can already see a lot of information of the geometry of the underlying spaces from the Riemannian metric.

We will roughly follow the scheme below:

(I). Riemannian metric

From the Riemannian metric, we can calculate the length of curves, and moreover, we obtain a natural volume measure.

(II). Geodesics

It's natural to look for shortest curves connecting two points. Geodesics are curves which are "locally" shortest. They can be obtained from the First variation of the length functional. In order to explore the problem whether a geodesic is shortest, we will discuss exponential maps, normal coordinates, Hopf-Rinow theorem(1931), etc.

(III).Connections, parallelism, and Covariant Derivatives

We reinterpret the geodesic equations in terms of connections and covariant derivatives, or parallelism. In this process, we develop (Abstract) calculus on Riemmanian manifolds.

(IV).Curvature

When we consider the Second Variation of the length functional. In this variational formula, a "curvature" term will appear, which is a generalization of Gaussian curvature of surfaces. We will discuss properties of Riemannian curvature tensor and various curvature notions.

(V).Spaces forms and Jacobi fields

We will discuss the complete Riemannian manifolds with constant curvature, which are referred to as space forms. Those will be model spaces when we study the geometry of a general Riemannian manifold. In this process, we discuss the theory of Jacobi fieldsvariational vector fields of family of geodesics.

(VI).Comparison theorems

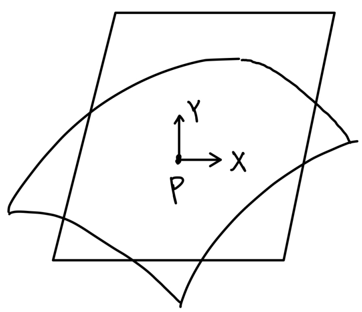
We explore geometry of Riemannian manifolds with curvature bounds via comparing them with spaces forms".

Chapter 1

Riemannian Metric

1.1 Definition

Recall in the theory of surfaces in E^3: For a surface $S \subset E^3$, $\forall\ p \in S$, and any two tangent vectors X,Y$\in T_pS$, we have the inner product $\langle X, Y\rangle_p$. ($\langle X, Y\rangle_p$ is the inner product of E^3)



This inner product $\langle X, Y\rangle_p$ corresponds to the first fundenmental forms of S at p. Based on this inner product, one can compute the lengths of a curve in S, the area of a domain in S, etc.

Now, let us consider a C^∞-manifold $M^n(dimM = n)$.

Definition 1.1 (Riemannian Metric)**.** *A Riemannian metric g on M is a "C^∞ assignment": For each tangent vector space $T_pM(p \in M)$ of M, we assign an inner product $g_p(\cdot,\cdot)$=$\langle\cdot,\cdot\rangle_p$, which is smoothly dependent on p in the following sense: f(p):=$\langle X_p, Y_p\rangle_p = g_p(X_p, Y_p)$ is a smooth function on $U \subset M$ for any smooth tangent vector fields X,Y on $U \subset M$.*

Remark 1.1. *Recall that by inner product, we mean $g_p(\cdot,\cdot)$ is symmetric, positive definite and bilinear.*

What it looks like in local coordinates: Given $p \in M$. For any coordinate neighborhood $U \ni p$, let its coordinate functions be $x^1, x^2, \cdots, x^n$. Then the tangent vector space T_pM is spanned by $\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n}$. Its dual spacethe cotangent vector spaces T_p^*M is spanned by $dx^1, \cdots, dx^n$. Then we denote

$$\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \rangle_p = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})(p) := g_{ij}(p),$$

and for any smooth tangent vector fields

$$X = X^i \frac{\partial}{\partial x^i}, Y = Y^j \frac{\partial}{\partial x^j}$$

we have $\langle X_p, Y_p \rangle_p = X^i(p)Y^j(p) \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle_p = g_{ij}(p)X^i(p)Y^j(p)$. Here we use the Einstein summation: an index occuring twice in a product is to be summed from 1 up to the space dimension. Therefore, in local coordinates, we can consider the Riemannian metric g as

$$g = g_{ij} dx^i \otimes dx^j,$$

where (1) "$g_p(\cdot, \cdot)$ depends smoothly on p" is equivalent to say "$g_{ij}(p)$ is smooth on $U \ni p, \forall i, j$".

(2) "$g_p(\cdot, \cdot)$ is an inner product" is equivalent to say

$\cdot$ $g_{ij} = g_{ji}$, i.e the matrix $(g_{ij}(p))$ is symmetric at any $p \in U$.

$\cdot$ The matrix $(g_{ij}(p))$ is positive definite at any $p \in U$.

Hence, we can reformulate the definition of a Riemannian metric g as belows.

Definition 1.2 (Riemannian Metric)**.** *A Riemannian metric g on a C^∞-manifold M is a smooth (0,2)-tensor satisfying*

$$g(X, Y) = g(Y, X), g(X, X) = 0 \text{ and } g_p(X, X) = 0 \Leftrightarrow X(p) = 0$$

for any smooth tangent vector fields X,Y.

Definition 1.3. *A Riemannian manifold is a differentiable(we will always assume C^∞) manifold equipped with a Riemannian metric.*

Remark 1.2. *a couple (M,g).*

1.2 Examples

(1)$M = R^n$, $\forall p \in M$, $T_pR^n = R^n$. The standard inner product on R^n gives a standard Riemannian metric g_0:

$$g_0(X, Y) = \sum_i X^i Y^i = X^T Y.$$

Another way to see it: R^n is covered by a single coordinate $(x^1, \cdots, x^n)$.

$$Matrix : (g_{ij}) = (\delta_{ij}) = I_{n\times n}.$$
$$Tensor : g = dx^1 \otimes dx^1 + \cdots + dx^n \otimes dx^n.$$

More generally, (g_{ij}) can be any positive definite and symmetric $n \times n$ matrix $A = (a_{ij})$. Then $g = a_{ij}dx^i \otimes dx^j$ and $g(X, Y) = X^T AY$.

(2) Induced Metric: Let $f : M^n \to N^{n+k}$ be a smooth immersion(i.e. $df_p : T_pM \to T_{f(p)}N$ is injective for any $p \in M$). Let (N, g_N) be a Riemannian manifold(e.g. $(N, g_N) = (R^{n+k}, g_0)$). We can define the pull-back metric f^*g_N on M as belows

$$(f^*g_N)_p(X_p, Y_p) = (g_N)_{f(p)}(df_p(X_p), df_p(Y_p)).$$

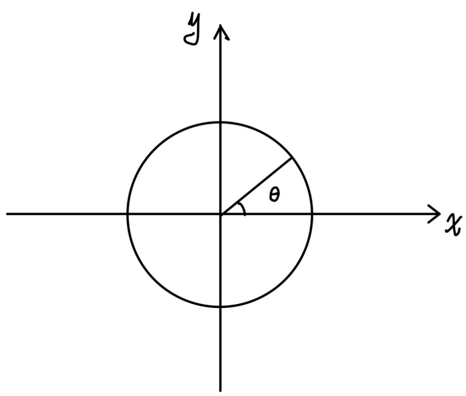
One can verify that f^*g_N is a Riemannian metric on M. $((f^*g_N)_p(X_p, X_p) \Leftrightarrow df_p(X_p) = 0 \overset{df_p \text{ is injective}}{\Longleftrightarrow} X_p = 0)$

Definition 1.4. *We call f^*g_N an induced metric on M with respect to the smooth immersion $f : M^n \to N^{n+k}$.*

A special case: $M \subset N$ is an immersed submanifold. Then the inclusion map $i : M \to N$ is an immersion. In this case, the induced metric $(i^*g)_p$ is just the restriction of $(g_N)_p$ on $T_pM \subset T_pN$.

Example 1.1. *Let $M = S^1$ be the unit circle in R^2. Choose a coordinate neighborhood $\{\theta : 0 < \theta < 2\pi\}$. Then the inclusion map is given by*

$$\begin{cases} x = \cos\theta \\ y = \sin\theta \end{cases}$$

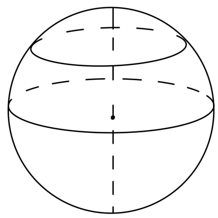


$g_{S^1} = (dx \otimes dx + dy \otimes dy)|_{S^1} = d\theta \otimes d\theta$. Then we have $dx = -\sin\theta d\theta$, $dy = \cos\theta d\theta$ and $g_{S^1} = (dx \otimes dx + dy \otimes dy)|_{S^1} = d\theta \otimes d\theta$.

Example 1.2. *Let $M = S^2$ be the unit sphere in R^3. Choose a coordinate neighborhood $\{(\theta, z) : 0 < \theta < 2\pi, -1 < z < 1\}$*

$$\begin{cases} x = \sqrt{1 - z^2}\cos\theta \\ y = \sqrt{1 - z^2}\sin\theta \\ z = z \end{cases}$$

Then the induced metric on S^2 is $g_{S^2} = \frac{1}{1-z^2} dz \otimes dz + (1-z^2) d\theta \otimes d\theta$. (Exercise!)



(3)Product metric: Let (M, g_M), (N, g_N) be two Riemannian manifolds. Let $M \times N$ be the Cartesian product. Let $\pi_1 : M \times N \to M$, $\pi_2 : M \times N \to N$ be the natural projections, Then $M \times N$ has the following product metric g:

$$g_{p,q}(X, Y) = (g_M)_p(d\pi_1(X), d\pi_1(Y)) + (g_N)_p(d\pi_2(X), d\pi_2(Y))$$

$\forall (p, q) \in M \times N, \forall X, Y \in T_{(p,q)}(M \times N)$. For example, the torus $T^n = S^1 \times \cdots \times S^1$ has a product metric based on the induced metric on S^1. Such tori are flat tori.

(4)If g_1, g_2 are two Riemannian metrics of M, so does $ag_1 + bg_2$, $\forall a, b > 0$.

1.3 When are two Riemannian manifolds "equivalent"?

Definition 1.5 (Isometry)**.** *Let (M, g_M), (N, g_N) be two Riemannian manifolds. Let $\phi : M \to N$ be a diffeomorphism(i.e. ϕ is bijective, C^∞ and π^{-1} is also C^∞). If $\phi^* g_N = g_M$. (That is, $(g_M)_p(X, Y) = (g_N)_{\phi(p)}(d\phi_p(X), d\phi_p(Y))$, $\forall p \in M$, $\forall X, Y \in T_pM$), then we call ϕ an isometry.*

1.4 Existencce of Riemannian Metrics

Theorem 1.1. *A C^∞-manifold M (Hausdorff,second countable) has a Riemannian metric.*

Proof. Let $\{U_\alpha\}$ be a locally finite covering of M by coordinate neighborhoods.That is ,any point p of has a neighborhood U such that $U \cap U_\alpha \neq \emptyset$ at most for a finite number of indices.

Let $\{\phi_\alpha\}$ be a C^∞ partition of unity on M subordinate to the covering $\{U_\alpha\}$ That is

(1) $\phi_\alpha \geq 0, \phi_\alpha = 0$ on M $\setminus \overline{U_\alpha}$.

(2) $\sum_\alpha \phi_\alpha(p) = 1, \forall p \in$M.

On each U_α we can define a Riemannian metric $g_\alpha(\cdot, \cdot)$ induced by the local coordinates.(e.g. $(U_\alpha, x^i_\alpha) \to g_\alpha = \sum_i dx^i_\alpha \otimes dx^i_\alpha$)

Let us set

$$g_p(X, Y) := \sum_\alpha \phi_\alpha(p)(g_\alpha)_p(X, Y), \forall p \in M, \forall X, Y \in T_pM. \tag{1.4.1}$$

It is direct to verify that this construction defines a Riemannian metirc on M .(In fact ,the main point is to check that g is positive definite.This is because the summation in (1.4.1) is actually a finite sum,and $\sum_{\alpha} \phi_\alpha \alpha = 1 \implies \exists \beta$ s.t.$\phi_\beta(p) > 0$. Hence $g_p \geq \phi_\beta(p) g_\beta > 0$.

■ □

Remark 1.3. *Whitney (1936) showed that any n-dimensional C^∞ manifold M^n can be embedded into $\mathbb{R}^{2n+1}$ .Thus M^n always has a Riemannian metric induced from the standard Riemannian metric g_0 of $\mathbb{R}^{2n+1}$.*

On the other hand ,given a Riemannian manifold (M^n, g_M) ,the Riemannian metric g_M is usually different from the metric induced from g_0 of $\mathbb{R}^{2n+1}$ .In fact ,Nash's embedding theorem tells that for any Riemannian manifold (M^n, g_M) ,there exists a N,s.t. (M^n, g_M) can be underlineisometrically embedded into $(\mathbb{R}^N, g_0)$. In other words,there exists an embedding $\varphi : M^n \to \mathbb{R}^N$ s.t. $g_M = \varphi^* g_0$. Nevertheless, the intrisic point of view in the above proof offers great conceptual and technical advantages over the approach of submanifold geometry of Euclidean space.

1.5 The Metric Structure

The Riemannian metirc g on M induces a natural distance function d. That is a function $d : M \times M \to \mathbb{R}$ satisfying for any $p, q, r \in M$

(1)$d(p, q) \geq 0$, and $d(p, q) = 0 \iff p = q$.

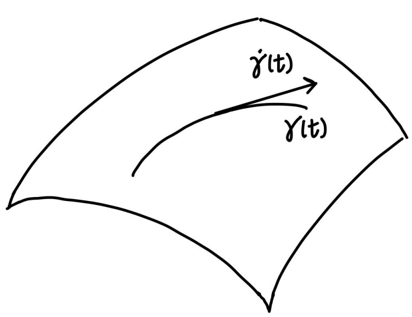
(2)$d(p, q) = d(q, p)$.

(3)$d(p, q) \leq d(p, r) + d(r, q)$.

To show this fact,let's consider the length of curves in M.

Let $\gamma : [a, b] \to M$ be a smooth (parametrized) curve in M. For any $t \in [a, b]$, we have the tangent vector

$$\dot{\gamma}(t) := d\gamma(\frac{d}{dt}) \in T_{\gamma(t)} M.$$



We always assume the parametrization is regular ,i.e. $\dot{\gamma}(t) \neq 0, \forall t$. Then the length of γ is

$$\begin{aligned}\text{Length}(\gamma) :&= \int_a^b |\dot{\gamma}(t)| dt \\ &= \int_a^b \sqrt{\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle_{\gamma(t)}} dt.\end{aligned}$$

Lemma 1.1. *The quantity* Length(γ) *does not depend on the choice of parametirzation.*

Proof. Let $\gamma_1 : [c,d] \to M$ $(c < d)$ is another regular parametrization of the same curve. Then there exist a smooth function $t_1 = t_1(t) : [a,b] \to [c,d]$ s.t.

$$\gamma_1(t_1(t)) = \gamma(t).$$

Since both parametrization are regular, we have$\dot{\gamma_1}(t_1) \neq 0,\ \dot{\gamma} 0(t) \neq 0$

Obseve that

$$\dot{\gamma}(t) = \dot{\gamma_1}(t_1) \cdot \frac{dt_1}{dt}. \tag{1.5.1}$$

By definition, (1.5.1) can be checked as belows:for any smooh function f,

$$\begin{aligned}\dot{\gamma}(t) \cdot f = d\gamma(\frac{d}{dt}) \cdot f &= \frac{d}{dt} f(\gamma(t)) = \frac{d}{dt} f(\gamma_1(t_1(t))) \\ &= \frac{d}{dt_1} f(\gamma_1(t_1)) \frac{dt_1}{dt} = d\gamma_1(\frac{d}{dt_1}) f \cdot \frac{dt_1}{dt}\end{aligned}$$

Hence we have $\frac{dt_1}{dt} \neq 0$, which means either $\frac{dt_1}{dt} > 0$ or $\frac{dt_1}{dt} < 0$. As we assume $a \leq b,\ c \leq d$, we have here $\frac{dt_1}{dt} > 0$.

Now we caculate

$$\begin{aligned}\text{Length}(\gamma_1) &= \int_c^d \sqrt{\langle \dot{\gamma}_1(t), \dot{\gamma}_1(t) \rangle_{\gamma_1(t_1)}} dt_1 \\ &= \int_a^b \sqrt{\langle \frac{dt}{dt_1} \dot{\gamma}(t), \frac{dt}{dt_1} \dot{\gamma}(t) \rangle_{\gamma(t)}} \frac{dt_1}{dt} dt \\ &= \int_a^b \sqrt{\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle_{\gamma(t)}} \left| \frac{dt}{dt_1} \right| \left| \frac{dt_1}{dt} \right| dt \\ &= \int_a^b \sqrt{\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle_{\gamma(t)}} dt = \text{Length}(\gamma).\end{aligned}$$

■ □

Exercise 1.1. *Let* $\varphi : (M, g_M) \to (N, g_N)$ *be an isometry,*γ *be a smooth curve in M. Show that* $\text{Length}_M(\gamma) = \text{Length}_N(\varphi(\gamma))$.

Arclength parametrization. Now we look for a ”standard” parametrization for a given curve. Consider smooth curve $\gamma : [a,b] \to M$. We can define the arclength function of γ:

$$s(t) := \int_a^t \sqrt{\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle_{\gamma(t)}} dt$$

Then $s = s(t) : [a, b] \to [0, \text{Length}(\gamma)]$ is a strictly increasing function. Denote by$t = t(s)$ its inverse function. Then we can reparametrize $\gamma(t)$ as

$$\gamma_1(t) = \gamma(t(s)), \ s \in [0, \text{Length}(\gamma)]$$

Proposition 1.1. $\langle \dot{\gamma_1}(s), \dot{\gamma_1}(s) \equiv 1$

Proof. We caculate

$$\begin{aligned} \langle \dot{\gamma_1}(s), \dot{\gamma_1}(s) \rangle_{\gamma_1(s)} &= \langle \dot{\gamma}(t) \frac{dt}{ds}, \dot{\gamma}(t) \frac{dt}{ds} \rangle_{\gamma(t(s))} \\ &= (\frac{dt}{ds})^2 \cdot \langle \dot{\gamma}(t), \dot{\gamma}(t)_{\gamma(t)} \\ &= 1 \end{aligned}$$

■ □

Remark 1.4. *The length of a (continuous and) piecewise smooth curve is defined as the sum of the smooth pieces.*

On a Riemannian manifold (M, g),the distance between two points p, q can be defined:

$$d(p, q) := \inf\{\text{Length}(\gamma) : \gamma \in C_{p,q}\}$$

where $C_{p,q} := \{\gamma : [a, b] \to M : \gamma \textit{ piecewise smooth curve with } \gamma(a) = p, \ \gamma(b) = q\}$

It can be checked immediately that $d : M \times M \to \mathbb{R}$ satisfies

(1)$d(p, p) = 0, d(p, q) \geq 0$

(2)$d(p, q) = d(q, p)$

(3)$d(p, q) \leq d(p, r) + d(r, q)$ (By definition)

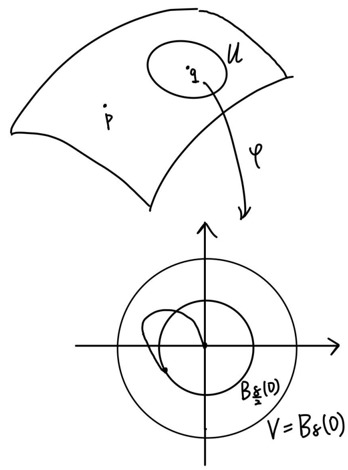
To show that d is indeed a distance function , it remains to prove $d(p, q) = 0 \Longrightarrow p = q$ or equivalently, $p \neq q \Longrightarrow d(p, q) > 0$

Theorem 1.2. (M, d) *is a metric space.*

Proof. It remains to show $p \neq q \Longrightarrow d(p, q) > 0$

There exists a coordinate neighborhood U of q, with coordinate map φ such that

$$\begin{cases} \varphi(q) = 0 \in B_\delta(0) := V \subset \mathbb{R}^n \\ p \notin U \end{cases}$$



Denote $\varphi^{-1} : B_\delta(0) \to U$. Then on $B_\delta(0)$,we have the pull-back metric $(\varphi^{-1})^* g := h$

Let $\gamma : [a,b] \to B_\delta(0)$ be a curve connecting $0 = \varphi(q)$ and a point on $\partial B_{\frac{\delta}{2}}(0)$. Let c be the smallest number with $\gamma(c) \in \partial B_{\frac{\delta}{2}}(0)$. Then we have

$$\text{Length}_h(\gamma) \geq \text{Length}_h(\gamma|_{[a,c]}) = \int_a^c \sqrt{\langle\dot\gamma, \dot\gamma\rangle_h} dt.$$

Obeserve that there exists a positive constant ϵ s.t.

$$\langle\dot\gamma, \dot\gamma\rangle_h \geq \epsilon\langle\dot\gamma, \dot\gamma\rangle_{g_0} \ \ on\ B_\delta(0)$$

(**Exercise**: Let K be an open set of $\mathbb{R}^n$, $[g_{ij}(x)]$ be n^2 continuous functions on K such that the matrix $[g_{ij}(x)]$ is symmetric for any $x \in K$.

(1)Denote by $\lambda(x), \Lambda(x)$ the smallest, largest eigenvalues of $[g_{ij}(x)]$. Show that both λ and Λ are continuous functions on K.

(2)Suppose K is compact and $[g_{ij}(x)]$ is positive definite for any $x \in K$. Show that there exist positive constants ϵ_1, ϵ_2 such that

$$\epsilon_1|v|^2 \leq \sum_{i,j} g_{ij}(x)v^i v^j \leq \epsilon_2|v|^2$$

for any $x \in K$ and any $(v^1, \cdots, v^n) = v \in R^n$.)

Therefore, we have

$$\text{Length}_h(\gamma) \geq \sqrt{\epsilon}\int_a^c \langle\dot\gamma, \dot\gamma\rangle_{g_0} dt = \sqrt{\epsilon}\text{Length}_{g_0}(\gamma|_{[a,c]}) \geq \frac{\sqrt{\epsilon}\delta}{2}.$$

Any curve connecting $p, q \in M$ must intersect with $\phi^{-1}(\partial B_{\frac{\delta}{2}}(0))$ at some point. Hence, we have

$$d(p,q) \geq \frac{\sqrt{\epsilon}\delta}{2} > 0.$$

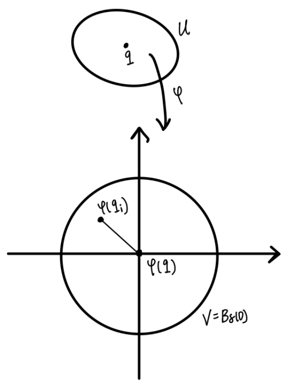
■ □

Moreover, we have the following property.

Proposition 1.2. *Let (M,g) be a Riemannian manifold. For given $p \in M$, the function $f(\cdot) := d(\cdot, p)\ : M \to R$ is continuous.*

Proof. We need to show $f(q_i) \to f(q)$ while q_i tends to q (in the sense of the manifold topology). By triangle inequality, $|f(q_i) - f(q)| \leq d(q_i, q)$. So it suffices to prove $d(p, q) \to 0$ as $i \to \infty$. Pick a coordinate neighborhood U of q such that

$$\exists\, \phi : U \to B_\delta(0) =: V \subset R^n,\ \phi(q) = 0.$$



Without loss of generality, we assume $\phi(q_i) \in B_{\frac{1}{i}}(0)$. Denote by $h := (\phi^{-1})^* g$. Choose $\widetilde{\gamma_i} : [0, 1] \to V$ be the curve $\widetilde{\gamma_i}(t) = t\phi(q_i)$. (Note $\widetilde{\gamma_i}(0) = \phi(q), \widetilde{\gamma_i}(1) = \phi(q_i)$). Then $\exists \epsilon_2 > 0$ such that

$$\text{Length}_h(\widetilde{\gamma_i}) = \int_0^1 \sqrt{\langle \dot{\gamma}, \dot{\gamma} \rangle_h} dt \leq \sqrt{\epsilon_2} \int_0^1 \langle \dot{\gamma}, \dot{\gamma} \rangle_{g_0} dt \leq \frac{\sqrt{\epsilon}}{i}.$$

Therefore, we have

$$d(p_i, q_i) \leq \text{Length}_h(\widetilde{\gamma_i}) \leq \frac{\sqrt{\epsilon_2}}{i} \to 0,\ as\ i \to \infty.$$

■ □

Corollary 1.1. *The topology on M induced by the distance function d coincides with original manifold topology of M.*

Proof. The continuity of $f(\cdot) = d(\cdot, p)$ tells every open set of the topology induced by d is again open of the manifold topology. By the proof of **1.5.2**, one can see the other way around every open set of the manifold topology is open in the topology induced by d. ■ □

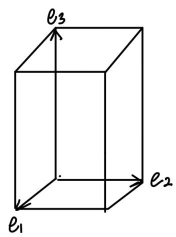
Remark 1.5. *It actually suffice to show that the topology induced by d coincides with the one in R^n in each coordinate neighberhood, which is induced by the Euclidean distance. We know for any point in this coordinate, exist positive constant ϵ_1, ϵ_2 with*

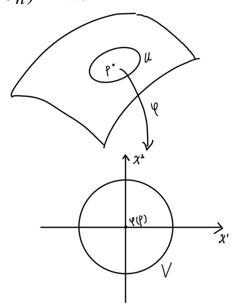
$$\epsilon_1 |v|^2 \leq g_{ij} v^i v^j \leq \epsilon_2 |v|^2, \forall v \in R^n.$$

By our precious argument, this implies the Riemannian distance d and the Euclidean distance control each other. Hence the two topology coincides.

1.6 Riemannian Measure, Volume

For any $p \in M$, (T_pM, g) is an vector space with inner products. Consider an orthonormal basis $\{e_1, \cdots, e_n\}$ of (T_pM, g). The volume of the parallelepiped spanned by $e_1, \cdots, e_n$, $vol(e_1, \cdots, e_n) = 1$.





Now we hope to develop a natural notion of intergration on a Riemannian manifold. Locally, such an intergration should be the intergration over on Euclidean subset V:

$$\int_V ()dx^1 \cdots dx^n.$$

So we need to know the length of the tangent vecters $\{\frac{\partial}{\partial x^i}(p), i = 1, 2, \cdots, n\}$ and the volume of the parallelepiped spanned by them.

We can write $\frac{\partial}{\partial x^i}(p) = a_i^j e_j$.

Then $g_{ik}(p) = \left\langle \frac{\partial}{\partial x^i}(p), \frac{\partial}{\partial x^k}(p) \right\rangle = \left\langle a_i^j e_j, a_k^l e_l \right\rangle (p) = a_i^j a_k^l \delta_j^l = \sum_l a_i^l a_k^l$.

Matrix form:$[g_{ik}] = AA^T$, with $A = \begin{pmatrix} a_1^1 & a_1^2 & \cdots & a_1^n \\ a_2^1 & a_2^2 & \cdots & a_2^n \\ \vdots & \vdots & \ddots & \vdots \\ a_n^1 & a_n^2 & \cdots & a_n^n \end{pmatrix}$.

Therefore, we have

$$\begin{aligned} vol(\frac{\partial}{\partial x^1}(p), \cdots, \frac{\partial}{\partial x^n}(p)) &= det(a_i^j)(p) vol(e_1, \cdots, e_n) \\ &= \sqrt{det(g_{ij})(p)}. \end{aligned}$$

Volume of "small" compact domain.

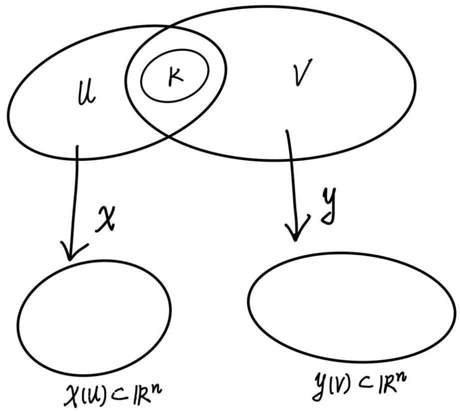
$\forall p \in M$, let $(U, x^1, \cdots, x^n)$ be a coordinate neighberhood: $x : U \to R^n$. Consider compact set $K \subset U$ such that $x(K)$ is measurable(in R^n). Then we define its volume as

$$vol(K) := \int_{x(K)} \sqrt{det(g_{ij}) \circ x^{-1}} dx^1 \cdots dx^n \quad (dx^1 \cdots dx^n : Lebesgue\ measure\ on\ R^n). \tag{1.6.1}$$

Well-definedness?

Proposition 1.3. *The difinition (1.6.1) does not depends on the choice of the coordinate.*

Proof. Suppose we have another coordinate neighborhood $(V, y^1, \cdots, y^n)$ containing K: $y : V \to R^n$.



$y \circ x^{-1} : x(U) \to y(V)$ is a diffeomorphism. Observe that $\frac{\partial}{\partial x^i}(p) = \frac{\partial(y^k \circ x^{-1})}{\partial x^i}(x(p))\frac{\partial}{\partial y^k}(p)$. We have

$$\begin{aligned} g^x_{ij}(p) :&= \left\langle \frac{\partial}{\partial x^i}(p), \frac{\partial}{\partial x^j}(p) \right\rangle \\ &= \left\langle \frac{\partial}{\partial y^k}(p), \frac{\partial}{\partial y^l}(p) \right\rangle \frac{\partial(y^k \circ x^{-1})}{\partial x^i}(x(p)) \frac{\partial(y^l \circ x^{-1})}{\partial x^j}(x(p)) \\ &= \frac{\partial(y^k \circ x^{-1})}{\partial x^i}(x(p)) \frac{\partial(y^l \circ x^{-1})}{\partial x^j}(x(p)) g^y_{kl}(p). \end{aligned}$$

Denote the jacobian matrix of the map $y \circ x^{-1}$ by

$$J(x(p)) = \begin{pmatrix} \frac{\partial(y^1 \circ x^{-1})}{\partial x^1} & \frac{\partial(y^1 \circ x^{-1})}{\partial x^2} & \cdots & \frac{\partial(y^1 \circ x^{-1})}{\partial x^n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial(y^n \circ x^{-1})}{\partial x^1} & \frac{\partial(y^n \circ x^{-1})}{\partial x^2} & \cdots & \frac{\partial(y^n \circ x^{-1})}{\partial x^n} \end{pmatrix}.$$

We obtain $[g^x_{ij}(p)] = J^T(x(p))[g^y_{kl}(p)]J(x(p))$.
Hence $\sqrt{det(g^x_{ij})(p)} = |det(J(x(p)))| \sqrt{det(g^y_{kl})(x(p))}$.
Therefore we have

$$\begin{aligned} \int_{y(K)} \sqrt{det(g^y_{ij}) \circ y^{-1}} dy^1 \cdots dy^n &\overset{y = y \circ x^{-1}}{=} \int_{x(K)} \sqrt{det(g^y_{ij}) \circ y^{-1}(y \circ x^{-1})} |det(J(x(p)))| dx^1 \cdots dx^n \\ &= \int_{x(K)} \sqrt{det(g^x_{ij}) \circ x^{-1}} dx^1 \cdots dx^n \end{aligned}$$

■ □

Volume of "larger" conmpact domain.

Now let us consider the case when K can not be contained in a single coordinate nerghborhood. Let $U_\alpha, x_\alpha^1, \cdots, x_\alpha^n$ be a locally finite covering of M by coordinate neighborhoods. Let $\{\phi_\alpha\}_\alpha$ be a C^∞ partition of unity on M subordinate to the covering $\{U_\alpha\}$. Then we define

$$vol(K) := \sum_\alpha \int_{x_\alpha(K\bigcap U_\alpha)} (\phi_\alpha \circ x_\alpha^{-1}) \sqrt{det(g_{ij}^{x_\alpha}) \circ x_\alpha^{-1}} dx_\alpha^1 \cdots dx_\alpha^n.$$

Proposition 1.4. *This definition does not depend on the choice of the covering of coordinate neighborhoods and partiton of unity.*

Proof. Let $\{V_\beta, y_\beta^1, \cdots, y_\beta^n\}_\beta$ be another locally finite covering of M by coordinate neighborhood and $\{\psi_\beta\}$ be a partition of unity of M subordinate to $\{V_\beta\}$. Then we compute

$$\begin{aligned} &\sum_\beta \int_{y_\beta(K\bigcap V_\beta)} (\psi_\beta \circ y_\beta^{-1}) \sqrt{det(y_{ij}^{y_\beta}) \circ y_\beta^{-1}} dy_\beta^1 \cdots dy_\beta^n \\ =&\sum_\beta \int_{y_\beta(K\bigcap V_\beta)} \sum_\alpha (\phi_\alpha \circ y_\beta^{-1})(\psi_\beta \sqrt{det(y_{ij}^{y_\beta})}) \circ y_\beta^{-1} dy_\beta^1 \cdots dy_\beta^n. \end{aligned}$$

We can exchange the order of the two summations, since each is a finite sum.

$$\begin{aligned} &\sum_\beta \int_{y_\beta(K\bigcap V_\beta)} \sum_\alpha (\phi_\alpha \circ y_\beta^{-1})(\psi_\beta \sqrt{det(y_{ij}^{y_\beta})}) \circ y_\beta^{-1} dy_\beta^1 \cdots dy_\beta^n \\ =&\sum_\alpha \int_{y_\beta(K\bigcap V_\beta)} \sum_\beta (\psi_\beta \circ y_\beta^{-1})(\phi_\alpha \sqrt{det(y_{ij}^{y_\beta})}) \circ y_\beta^{-1} dy_\beta^1 \cdots dy_\beta^n \\ \overset{change\ of\ variables}{=}&\sum_\alpha \int_{x_\alpha(K\bigcap U_\alpha)} \sum_\beta (\psi_\beta \circ x_\alpha^{-1})(\phi_\alpha \sqrt{det(y_{ij}^{x_\alpha})}) \circ x_\alpha^{-1} dx_\alpha^1 \cdots dx_\alpha^n \\ =&\sum_\alpha \int_{x_\alpha(K\bigcap U_\alpha)} (\phi_\alpha \circ x_\alpha^{-1}) \sqrt{det(y_{ij}^{x_\alpha}) \circ x_\alpha^{-1}} dx_\alpha^1 \cdots dx_\alpha^n \end{aligned}$$

■ □

Let us denote by $C_0^0(M)$ the vector space of continuous functions on M with compact support. For any $f \in C_0^0(M)$, we define

$$\int_M f := \sum_\alpha \int_{x^\alpha(U_\alpha)} (\phi_\alpha f) \circ x_\alpha^{-1} \sqrt{det(g_{ij}^{x_\alpha}) \circ x_\alpha^{-1}} dx_\alpha^1 \cdots dx_\alpha^n.$$

From the above discussion, we know this is well-defined. Moreover, since $\phi_\alpha \geq 0$, we know

$$f \geq 0 \Rightarrow \int_M f \geq 0.$$

Therefore, we obtain a positive linear functional $\urcorner : C_0^0(M) \to R$, $\urcorner(f) := \int_M f$.

By riesz representation theorem, these exists a unique Borel measure *dvol* such that

$$\urcorner(f) = \int_M f = \int_M f dvol$$

for any $f \in C_0^0(M)$.

Remark 1.6. *In each coordinate neighborhood, the integration with respect to dvol can be considered an the integration with respect to the n-form*

$$\Omega_0 = \sqrt{det(tg_{ij})} dx^1 \wedge \cdots \wedge dx^n.$$

Notice that when we change the coordinate, we have

$$dy^1 \wedge \cdots \wedge dy^n = det(J(x(p)))dx^1 \wedge \cdots \wedge dx^n.$$

That is, Ω_0 may change sign when we change from one coordinate to the other one.

Particularly, we can have a globally defined n-form Ω_0 when M is orientable. In this case,

$$\int_M f dvol = \int_M f \Omega_0.$$

(Recall: On an orientable manifold M, let $\{e_1, \cdots, e_n\}$ be an orthonormal frame fields, and $\{\omega^1, \cdots, \omega^n\}$ be its dual. Then

$$\Omega_0 = \omega^1 \wedge \cdots \wedge \omega^n.$$

)

Once the measure *dvol* is obtained, the machine of measure theory is intiated. We define the $L^p (1 \le p \le \infty)$ norm of $f \in C_0^0(M)$ as

$$||f||_{L^p} := (\int_M |f|^p dvol)^{\frac{1}{p}}.$$

We can take the completion of $C_0^\infty(M)$ with respect to L^p-norm, the resulting space is called $L^p(M)$.

·In particular for $p = 2$, we can define inner product:

$$\langle f_1, f_2 \rangle_{L^2} := \int_M f_1 f_2 dvol, \forall f_1, f_2 \in L^2(M).$$

This verifies $L^2(M)$ to be a Hilbert Space.

1.7 Divergence Theorem

Let X be a smooth tangent vector fields of M. The divergence of X is defined as a C^∞ function $div(X)$ on M as below: Let $(U, x^1, \cdots, x^n)$ be a coordinate neighborhood, we have the volume element

$$\Omega_0 = \sqrt{det(g_{ij})}dx^1 \wedge \cdots \wedge dx^n.$$

The divergence $divX : M \to R$ is a C^∞ function on M such that

$$(divX)_{\Omega_0} = d(i(X)\Omega_0)$$

where i(X) is the interior product with respect to X(i.e. the contraction of a differential form with the vector field X).

That is, for any vector fields $Y_1, \cdots, Y_{n-1}$, we have

$$i(X)\Omega_0(Y1, \cdots, Y_{n-1}) := \Omega_0(X, Y_1, \cdots, Y_{n-1}).$$

Remark 1.7. *(1)When we change coordinates, Ω_0 may change sign but this does not matter for the definition of divX. The global definition of divX does not require the orientability of M.*

(2)Let us consider the expression of divX in local coordinate. Let $X = X^i \frac{\partial}{\partial x^i}$,

$$i(X)\Omega_0 = i(X^i \frac{\partial}{\partial x^i}) \sqrt{det(g_{kl})}dx^1 \wedge \cdots \wedge dx^n.$$

Lemma 1.2. $i(\frac{\partial}{\partial x^i}(dx^1 \wedge \cdots \wedge dx^n) = (-1)^{i-1}dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n.$

Proof. Let $Y_1, \cdots, Y_{n-1}$ be any (n-1) smooth vector fields. We compute

$$\begin{aligned}
&i(\frac{\partial}{\partial x^i}(dx^1 \wedge \cdots \wedge dx^n)(Y_1, \cdots, Y_{n-1}) \\
=&dx^1 \wedge \cdots \wedge dx^n(\frac{\partial}{\partial x^i}, Y_1, \cdots, Y_{n-1}) \\
=&\sum_{\sigma \in S(n)} (sgn\sigma)dx^{\sigma(1)} \otimes \cdots \otimes dx^{\sigma(n)}(\frac{\partial}{\partial x^i}, Y_1, \cdots, Y_{n-1}) \\
=&\sum_{\substack{\sigma \in S(n) \\ \sigma(1)=i}} (sgn\sigma)dx^{\sigma(2)} \otimes \cdots \otimes dx^{\sigma(n)}(Y_1, \cdots, Y_{n-1})
\end{aligned}$$

■ □

Hence $i(X)\Omega_0 = \sum_i X^i \sqrt{det(g_{kl})}(-1)^{i-1}dx^1 \wedge \cdots \wedge dx^n$. Let $\sqrt{G} = \sqrt{det(g_{kl})}$. Furthermore, we obtain

$$\begin{aligned}
d(i(X)\Omega_0) &= \sum_i (-1)^{i-1} \sum_k \frac{\partial}{\partial x^k}(\sqrt{G}X^i)dx^k \wedge dx^1 \wedge \cdots \wedge \widehat{dx^i} \cdots \wedge dx^n \\
&= \sum_i \frac{\partial}{\partial x^i}(\sqrt{G}X^i)dx^1 \wedge \cdots \wedge dx^n \\
&= (divX)\Omega_0.
\end{aligned}$$

Hence $divX = \frac{1}{\sqrt{G}}\frac{\partial}{\partial x^i}(\sqrt{G}X^i)$.

Notice that in particular, when $M = R^n,\ (g_{ij}) = (\delta_{ij})$, we have for $X = X^i\frac{\partial}{\partial x^i}$,

$$divX = \sum_{i=1}^{n}\frac{\partial}{\partial x^i}X^i = \sum_{i=1}^{n}\frac{\partial X^i}{\partial x^i}X^i.$$

This reduces to the classical divergence.

(3)By Cartan's magical formula

$$L_X\omega = i(X)d\omega + div(X)\omega.\ (L_X\omega : Lie\ derivative\ of\ differential\ forms)$$

we have $L_X\Omega_0 = i(X)d\Omega_0 + div(X)\Omega = div(X)\Omega_0$.

This tells us that the divergence of a vector field is "infinitesional" changing rate of the volume element along the vecter field.

Theorem 1.3. *[Divergence Theorem] Let X be a smooth vector fields on (M,g).Then*

$$\int_M div(X)\ dvol = 0$$

Proof. Let $\{U_\alpha\}$ be a locally finite covering of M by coordinate neighborhood, $\{\phi_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}$. Then we have

$$X = \sum_\alpha \phi_\alpha X$$

Since X has compact support and the summation above is finite, we have

$$\int_M div(\sum_\alpha \phi_\alpha X)\ dvol = \sum_\alpha \int_{U_\alpha} div(\phi_\alpha X)\ dvol.$$

So it is enough to show $\int_{U\alpha} div(\phi_\alpha X)\ dvol$ holds for each α. Without loss of generality, we assume the support of X is contained in a coordinate neighborhood $(U, x^1, \cdots, x^n)$, and $X = X^i\frac{\partial}{\partial x^i}$. By definition, we have

$$\begin{aligned}\int_M div(X)\ dvol &= \int_U \frac{1}{\sqrt{G}}\frac{\partial}{\partial x^i}(X^i\sqrt{G})\ dvol\\ &= \int_{x(U)} (\frac{1}{\sqrt{G}}\frac{\partial}{\partial x^i}(X^i\sqrt{G})\sqrt{G})\circ x^{-1}dx^1\cdots dx^n\\ &= \int_{x(U)} \frac{\partial}{\partial x^i}(X^i\sqrt{G}\circ x^{-1})dx^1\cdots dx^n\\ &= 0\end{aligned}$$

■ □

Gradient vector fields of a function

Let $f \in C^\infty(M)$. The gradient vector field of f, grad f, is defined as a smooth vector field such that

$$\langle grad\ f, Y\rangle (= g(grad\ f, Y)) = Y(f).$$

expressions in local coordinates: $(U, x^1, \cdots, x^n)$, $Y = Y^j \frac{\partial}{\partial x^j}$, suppose $grad\ f = X^i \frac{\partial}{\partial x^i}$.

Then by definition

$$\langle grad\ f, Y\rangle = g_{ij}X^iY^j = Y(f) = Y^k \frac{\partial f}{\partial x^k}.$$

That is

$$(g_{ij}X^i)Y^j = \frac{\partial f}{\partial x^k}Y^k, \forall Y \Rightarrow g_{ij}X^i = \frac{\partial f}{\partial x^j}.$$

Recall $[g_{ij}]$ is a positive definite matrix. Denote by $[g^{ij}]$ its inverse matrix, i.e.

$$g^{ik}g_{kj} = \delta^i_j = \begin{cases} 0, if\ i \neq j \\ 1, if\ i = j \end{cases}$$

Next, we compute

$$\begin{aligned} g_{ij}X^i = \frac{\partial f}{\partial x^j} &\Rightarrow g^{kj}g_{ij}X^i = g^{kj}\frac{\partial f}{\partial x^j} \\ &\Rightarrow X^k = \delta^k_i X^i = g^{kj}g_{ij}X^i = g^{kj}\frac{\partial f}{\partial x^j}. \end{aligned}$$

Hence $grad\ f = g^{kj}\frac{\partial f}{\partial x^j}\frac{\partial}{\partial x^k}$.

Remark 1.8. *(1)For the case $M = R^n$, $(g_{ij}) = (\delta_{ij})$, we have*

$$grad\ f = \sum_i \frac{\partial f}{\partial x^i}\frac{\partial}{\partial x^i} = (\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n}).$$

.

(2)"The gradient vector field is vertical to the level set of a function."

Proposition 1.5. *Let $f \in C^\infty(M)$, c be a regular value of f. Then the vector field grad f is vertical to the level set $f^{-1}(c)$.*

Proof. Since c is a regular value of f, $f^{-1}(c)$ is a submanifold of M. Let $X \in Tf^{-1}(c) \subset TM$. We know

$$X(f) = 0\ on\ f^{-1}(c).$$

Therefore, $\langle grad\ f, X\rangle = X(f) = 0\ on\ f^{-1}(c)$. ■ □

(3)We say a one form η is dual to a vector field X if

$$\langle X, Y\rangle = \eta(Y),\ for\ any\ vector\ field\ Y.$$

In particular, we see

$$\langle grad\ f, Y\rangle = Y(f) = df(Y), \forall Y.$$

That is, df is dual to $grad\ f$.

Corollary 1.2. *Let $f \in C_0^\infty(M)$. Then* $\int\limits_M div(grad\ f)\ dvol = 0.$

Definition 1.6. *The Laplacian of a smooth function f is* $\Delta f := div(grad\ f)$.

Remark 1.9. *(1)In local coordinate* $(U, x^1, \cdots, x^n)$*, we have*

$$\begin{aligned}\Delta f &= \frac{1}{\sqrt{G}}\frac{\partial}{\partial x^i}((grad\ f)^i\sqrt{G})\\ &= \frac{1}{\sqrt{G}}\frac{\partial}{\partial x^i}(g^{ij}\frac{\partial f}{\partial x^j}\sqrt{G})\\ &= \frac{1}{\sqrt{G}}\frac{\partial}{\partial x^i}(\sqrt{G}g^{ij}\frac{\partial f}{\partial x^j}).\end{aligned}$$

In particular, for the case $M = R^n$, $g_{ij} = \delta_{ij}$*, we have* $\Delta f = \sum\limits_i \frac{\partial^2 f}{(\partial x^i)^2}$.

(2). $\int\limits_M \Delta f\ dvol = 0,\ \forall \in C_0^\infty(M)$.

(3).For any smooth function f and vector field X, we have

$$div(fX) = fdiv(X) + \langle grad\ f, XX\rangle .$$

Proof.

$$\begin{aligned}div(fX) &= \frac{1}{\sqrt{G}}\frac{\partial}{\partial x^i}(fX^i\sqrt{G})\\ &= \frac{1}{\sqrt{G}}\frac{\partial}{\partial x^i}X^i\sqrt{G} + f\frac{1}{\sqrt{G}}\frac{\partial}{\partial x^i}(X^i\sqrt{G})\\ &= X(f) + fdiv(X)\\ &= \langle grad\ f, X\rangle + fdivX.\end{aligned}$$

■ □

Theorem 1.4. *[Green's formula] Let f,h be two smooth functions,, at least one of which has ccompact support. Then*

$$\begin{aligned}\int\limits_M f\Delta h\ dvol &= -\int\limits_M \langle grad\ f, grad\ h\rangle\ dvol\\ &= \int\limits_M (\Delta f)h\ dvol.\end{aligned}$$

Proof. Applying $div(fX) = fdivX + \langle grad\ f, X\rangle$ to $X = grad\ h$, we have

$$div(f(grad\ h)) = fdiv(grad\ h) + \langle grad\ f, grad\ h\rangle = f\Delta h + \langle grad\ f, grad\ h\rangle .$$

Since $f \cdot grad\ h$ has compact support, we can apply divergence theorem to derive the Green's formula. ■ □

Remark 1.10. *(1)Δ is called the Laplace-Beltrami operator. On a compact manifold (M,g), we have*

$$\langle \Delta f, h\rangle_{L^2} = \langle f, \Delta h\rangle_{L^2}, \forall f, h \in C_0^\infty(M).(i.e. \Delta\ is\ self-adjoint)$$

$$\langle \Delta f, f\rangle_{L^2} = -\langle grad\ f, grad\ f\rangle_g = -\int_M |grad\ f|^2\ dvol \leq 0.(i.e. -\Delta\ is\ positive)$$

(2)Divergence theorem and Green's formulas can be extended to compact Riemannian manifolds with boundary.

Theorem 1.5. *Let M be a compact Riemannian manifolds with C^∞ boundary ∂M. Let ν be the outward normal vector field on ∂M, X be a smooth vector field on M. Then*

$$\int_M (divX)\ dvol_M = \int_{\partial M} g(X, \nu)\ dvol_{\partial M}.$$

Remark 1.11. *Let (M, g) be a compact submanifold of (N, g_N). Then ∂M has a Riemannian metric induced from g_N. Therefore we have a natural $dvol_{\partial M}$. As a corallary, we have*

$$\int_M f\Delta h\ dvol_M = -\int_M g\langle grad\ f, grad\ h\rangle\ dvol_M + \int_{\partial M} g(grad\ h, \nu) f\ dvol_{\partial M}.$$

Last lecture: More explanation on the calculation of

$$\sum_{\substack{\sigma\in S(n)\\ \sigma(1)=i}} sgn(\sigma) dx^{\sigma(2)}\otimes\cdots\otimes dx^{\sigma(n)}(Y_1,\cdots,Y_{n-1}).$$

Notice that $\{\sigma(2),\cdots,\sigma(n)\} = \{1,2,\cdots,\widehat{i},\cdots,n\}$. So $\sigma(2),\cdots,\sigma(n)$ is produced by a permutation τ of $\{1,2,\cdots,\widehat{i},\cdots,n\}$. Moreover, $sgn(\sigma) = (-1)^{i-1} sgn(\tau)$.

Hence

$$\begin{aligned}
&\sum_{\substack{\sigma\in S(n)\\ \sigma(1)=i}} sgn(\sigma) dx^{\sigma(2)}\otimes\cdots\otimes dx^{\sigma(n)}(Y_1,\cdots,Y_{n-1})\\
=&\sum_{\tau\in S(n-1)} (-1)^{i-1} sgn(\tau) dx^{\tau(1)}\otimes\cdots\otimes dx^{\tau(i-1)}\otimes dx^{\tau(i+1)}\otimes\cdots\otimes dx^{\tau(n)}(Y_1,\cdots,Y_{n-1})\\
=&(-1)^{i-1} dx^1\wedge\cdots\wedge\widehat{dx^i}\wedge\cdots\wedge dx^n(Y_1,\cdots,Y_{n-1}).
\end{aligned}$$

Chapter 2

Geodesics

2.1 Geodesic equations and Christoffel symbols

Let $\gamma : (a, b) \to (M, g)$ be a rugular smooth curve(i.e.$\dot{r}(t) \neq 0, \forall t \in (a, b)$). Recall its length is defined as

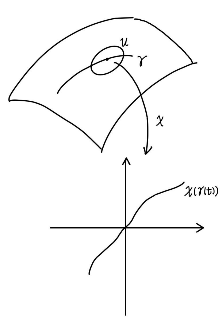
$$L(\gamma) := \text{Length}(\gamma) := \int_a^b \sqrt{\langle \dot{(\gamma)}(t), \dot{(\gamma)}(t) \rangle_{\gamma(t)}} dt.$$

In a local coordinate neignborhood $(U, x^1, \cdots, x^n)$. The curve can be written as

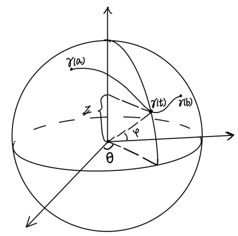
$$(x^1(\gamma(t)), \cdots, x^n(\gamma(t))).$$

When $\gamma|_{(a,b)}$ is contained in U, we have $\dot{\gamma}(t) = x^i(\gamma(t)) \frac{\partial}{\partial x^i}$ and

$$L(\gamma) = \int_a^b \sqrt{g_{ij}(x(\gamma(t)))\dot{x}^i(t)\dot{x}^j(t)} dt.$$



Example 2.1. *Consider the unit sphere $S^2 \subset \mathbb{R}^3$.*



Consider the coordinate neighborhood $(\phi, \theta) : \phi \in (-\frac{pi}{2}, \frac{pi}{2}), \theta \in (0, 2\pi)$. *We have the induced Riemannian metric*

$$g = d\phi \otimes d\phi + \cos^2 \phi d\theta \otimes d\theta.$$

Consider a smooth curve $\gamma(t),\ t \in (a, b)$ *on* S^n *with the spherical coordinate* $(\phi(t), \theta(t))$. *Then*

$$L(\gamma) = \int_a^b \sqrt{\dot{\phi}^2(t) + \cos^2 \phi(t)\dot{\theta}^2(t)}dt.$$

Observe that

$$L(\gamma) \geq \int_a^b |\dot{\phi}(t)|dt \geq |\int_a^b \dot{\phi}(t)dt| = |\phi(b) - \phi(a)|.$$

where "=" holds iff $\dot{\theta}(t) = 0(\Leftrightarrow \theta(t) \equiv const)$ *and* ϕ *is monotonic.Therefore, where* $\gamma(a)$ *and* $\gamma(b)$ *has the same coordinate* θ*, the shortest curve connecting them is the great circle passing through them.* ■

A natural question is then: given two points $p, q \in M$,

(1)does there exist a shortest curve connecting p,q?

(2)if it exists, is it unique.

In order to find the shortest curve, we consider the critical point of the Length functional, Length(γ). Note that Length(γ) is a bit massy to bundle with since it has a $\sqrt{\cdot}$ term as the integrand. In fact, we can consider the Energy functional instead:

$$E(\gamma) := \frac{1}{2}\int_a^b \langle\dot{\gamma}(t), \dot{\gamma}(t)\rangle = \frac{1}{2}\int_a^b g_{ij}(x(\gamma(t)))\dot{x}^i(t)\dot{x}^j(t)dt.$$

(In phisics, $E(\gamma)$ is usually called "action of γ" where γ is contained as the orbit of a mass point. In physics, we have the so-called "least action principle").

In the following, we will explain why we can consider the critical value of $E(\gamma)$ instead of that of $L(\gamma)$.

Lemma 2.1. *For each smooth curve* $\gamma : (a, b) \to M$, *we have*

$$L^2(\gamma) \leq 2(b - a)E(\gamma)$$

and "=" holds if and only if $\sqrt{\langle\dot{\gamma}(t), \dot{\gamma}(t)\rangle} =: \|\dot{\gamma}(t)\| \equiv const.$

Proof. By Hölder's inequality,

$$L(\gamma) = \int_a^b \|\dot{\gamma}(t)\|dt \leq (\int_a^b 1^2 dt)^{\frac{1}{2}}(\int_a^b \|\dot{\gamma}(t)\|^2 dt)^{\frac{1}{2}} = \sqrt{b-a}\sqrt{2E(\gamma)}.$$

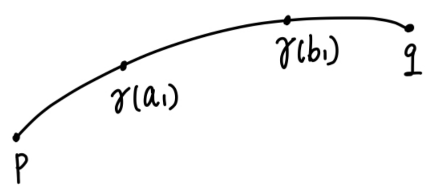
with equality precisely if $\|\dot{\gamma}(t)\| \equiv$ const. ■ □

Recall the length of a curve does not depend on the choice of the parametrized by arc length, i.e. $\|\dot{\gamma}\| = 1$, in order to find shortest curves. In this case, we have

$$b - a = L(\gamma), \ and \ L(\gamma)^2 = 2(b-a)E(\gamma) \Rightarrow L(\gamma) = 2E(\gamma).$$

Hence, after parametrizing curves by arc length, it is enough to minimize $E(\gamma)$.

Moreover, observe that if $\gamma \in C_{p,q}$ is a shortest curve from p to q. $\gamma : (a,b) \to M$. Then for any $a \le a_1 \le b_1 \le b$, γ is also a shortest curve from $\gamma(a_1)$ to $\gamma(b_1)$,



otherwise, we can shorten $\gamma\|_{(a,b)}$ further.

So we can localize our problem, and consider the case when $p, q \in U$.

Lemma 2.2. *The Euler-Lagrange equations for the energy E are*

$$\ddot{x}^i(t) + \Gamma^i_{jk}(x(t))\dot{x}^j(t)\dot{x}^k(t) = 0, \ i = 1, 2, \cdots, n \qquad (2.1.1)$$

with

$$\Gamma^i_{jk} = \frac{1}{2} g^{il}(g_{lk,j} + g_{jl,k} - g_{jk,l}),$$

and

$$g_{jl,k} = \frac{\partial}{\partial x^k} g_{jl}.$$

Definition 2.1. *(geodesics)A smooth curve $\gamma : [a,b] \to M$ which satisfies(with $\dot{x}^i(t) = \frac{d}{dt}x^i(\gamma(t))$)*

$$\ddot{x}^i(t)) + \Gamma^i_{jk}(x(t))\dot{x}^j(t)\dot{x}^k(t) = 0, \ i = 1, 2, \cdots, n$$

is called a geodesics.

Remark 2.1. *Christoffel is a German mathematician. He studied in Berlin, and worked in ETH Zūrich, Strasburg. Riemann's 1854 lecture was only published in 1868. Christoffel published in Crelles Journal(Journal für die reine and and) in 1869 an article discussing the necessary condition when two quadratic differential forms*

$$F = \sum_{i,k} \omega_{i,k} dx^i dx^k, \ F' = \sum_{i,k} w'_{i,k} dx'^i dx'^k.$$

can be transfirmed into eaach other via independent variable changes. It was there be introduced the "Christoffel symbols".

Influced by Christoffel's work, italian mathematician Gregorio Ricci-Curbasto published 6 articles during 1883-1888 on Christoffel's method, and introduced a new calculus: He interpreted Christoffel's algorithm into "covariant differentiations". Ricci(1893) called it "absolute differential calculus".

Later in 1901, Ricci and his student Levi-Civita published Ricci's calculus in French in Klein's journal(Mathematische Annalen). It is nowed called "tensor analysis".

Einstein(1914) derives the geodesic equation using Christoffel symbols in his Berlin lecture.

Levi-Civita(1916/17) realized the geometric meaning of Christoffel symbols: it determines the "parallel transport" of vectors. This pull Christoffel and Ricci's discussion back to the track of geometry.

Proof. Let us first look at a general functional

$$I(x) = \int_b^a f(t, x(t), \dot{x}(t))dt$$

where $x(t) := (x^1(t), \cdots, x^n(t))$.
Claim: the Euler-Lagrange equation of I(x) is

$$\frac{d}{dt}\frac{\partial f}{\partial \dot{x}^i} - \frac{\partial f}{\partial x^i} = 0,\ i = 1, \cdots, n.$$

proof of claim: Consider $y(t) = (y^1(t), \cdots, y^n(t))$ with $y(a) = y(b) = 0$. Solving $\frac{d}{d\epsilon}|_{\epsilon=0} I(x + \epsilon y) = 0$, we have

$$\begin{aligned} 0 &= \int_a^b (\frac{\partial f}{\partial x^i} y^i(t) + \frac{\partial f}{\partial \dot{x}^i} y^i(t))dt \\ &= \int_a^b (\frac{\partial f}{\partial x^i} - \frac{d}{dt}\frac{\partial f}{\partial \dot{x}^i}) y^i(t)dt \end{aligned}$$

By the fundamental lemma of calculus of variations, we have

$$\frac{d}{dt}\frac{\partial f}{\partial \dot{x}^i} - \frac{\partial f}{\partial x^i},\ i = 1, \cdots, n.$$

This is the E-L equation of I. ■

For our energy functional

$$E(\gamma) = \int_a^b g_{ij}(x(t))\dot{x}^j(t)\dot{x}^k(t)dt,$$

where $f(t, x(t), \dot{x}(t)) = g_{ij}(x(t))\dot{x}^j(t)\dot{x}^k(t)$.

We have

$$\frac{d}{dt}[g_{ik}(x(t))\dot{x}^k(t) + g_{ji}(x(t))\dot{x}^j(t)] - g_{jk,i}(x(t))\dot{x}^j(t)\dot{x}^k(t) = 0,\ i = 1, 2, \cdots, n.$$

Hence, $g_{ik,l}\dot{x}^l\dot{x}^k + g_{ik}\ddot{x}^k + g_{ji,l}\dot{x}^l\dot{x}^j + g_{ji}\ddot{x}^j - g_{jk,i}\dot{x}^j\dot{x}^k = 0,\ i = 1, 2, \cdots, n.$
$\Rightarrow$

$$2g_{im}\ddot{x}^m + (g_{ik,j} + g_{ji,k} - g_{jk,l})\dot{x}^j\dot{x}^k = 0,\ l = 1, \cdots, n. \tag{2.1.2}$$

Multiply both sides by g^{il} and sum up over i, we have

$$\ddot{x}^l + \frac{1}{2}g^{il}(g_{ik,j} + g_{ji,k} - g_{jk,l})\dot{x}^j\dot{x}^k = 0,\ l = 1, \cdots, n.$$

■ □

Remark 2.2. *:*

1.When we calculated the term $\Gamma^i_{jk}\dot{x}^j\dot{x}^\cdot$*. pay attention to the fact that*

$$\begin{aligned}\Gamma^i_{jk}\dot{x}^j\dot{x}^k &= \frac{1}{2}g^{il}(g_{jl,k} + g_{kl,j} - g_{lk,j})\dot{x}^j\dot{x}^k \\ &= \frac{1}{2}g^{il}(g_{jl,k} + g_{kl,j} - g_{lk,j})\dot{x}^j\dot{x}^k.\end{aligned}$$

2.As mentioned before, we only need to consider curves parametrized by arc length when looking for shortest curves. Now, we explain the other aspect: The solution of the E-L equation(2.1.1) on page 25 i.e. every geodesic, is parametrized proportionally to arc length.

Explanation:

$$\begin{aligned}&\frac{d}{dt}(g_{ij}(x(t))\dot{x}^i(t)\dot{x}^j(t)) \\ =&g_{ij}\ddot{x}^i\dot{x}^j + g_{ij}\dot{x}^i\ddot{x}^j + g_{ij,k}\dot{x}^i\dot{x}^j\dot{x}^k \\ =&2g_{ij}\ddot{x}^i\dot{x}^j + g_{ij,k}\dot{x}^i\dot{x}^j\dot{x}^k \\ =&2g_{lm}\ddot{x}^m\dot{x}^l + g_{lj,k}\dot{x}^l\dot{x}^j\dot{x}^k(change\ the\ indexes) \\ =&-(g_{lk,j} + g_{jl,k} - g_{jk,l}\dot{x}^l\dot{x}^i\dot{x}^k + g_{lj,k}\dot{x}^l\dot{x}^j\dot{x}^k(use\ (2.1.2)) \\ =&(g_{jk,l} - g_{lk,j})\dot{x}^l\dot{x}^j\dot{x}^k \\ =&0\end{aligned}$$

Hence $\langle\dot{x},\dot{x}\rangle \equiv const.$

That is, every geodesic is parametrized proportionally arc length.

3(Curves in TM).We explain another viewpoint about the geodesic. First, any smooth curve in M gives a cvrve in its tangent bundle TM.

(1)Systems of coordinates. The toral space of TM is the set of pairs (q, v), $q \in M$, $v \in T_qM$*. Let* $(U, x^1, \cdots, x^n)$ *be a coordinate neighborhood of M.* $\forall q \in U$*, any vector in* T_qM *can be written as*

$$y^i\frac{\partial}{\partial x^i}$$

Recall locally we have $TU{=}U\times R^n$*. Then* $(U\times R^n, x^1, \cdots, x^n, y^1, \cdots, y^n)$ *is a coordinate neighborhood of* $(q, v) \in TM$*. Then one can show that we obtain a differentiable structure for TM.*

(2)Let $t \to \gamma(t)$ *be a* C^{infty} *curve in M, then it determines curve* $t \to (\gamma(t), \dot{\gamma}(t)) \in TM$*. If, moreover,* γ *is a geodesic in M, then the curve* $t \to (\gamma(t), \dot{\gamma}(t))$ *in terms of coordinates* $(x^1, \cdots, x^n, y^1, \cdots, y^n)$

$$t \to (x^1(t), \cdots, x^n(t), y^1(t), \cdots, y^n(t))$$

satisfies

$$\begin{cases}\dot{x}^k(t) = y^k(t) \\ \dot{y}^k(t) + \Gamma^k_{ij}(x(t))y^iy^j = 0\end{cases} k = 1, 2, \cdots, n. \tag{2.1.3}$$

Local Existence and uniqueness of geodesics

From ODE theory: (See proposition 2.5 in do carmo, chapter 3 and discussions before that proposition.)

Theorem 2.1. *For any $p \in M$, there exists*

. *an open set $V \subset M,\ p \in V$*

. *numbers $\delta > 0$ and $\epsilon > 0$*

. *a C^∞ mapping: $\gamma:\ (-\epsilon, \epsilon) \times U \to M$. $U = (q, v) : q \in V,\ v \in T_qM,\ \|v\| < \delta$.*

Remark 2.3. *Let's have a closer look at the relations between the domain $(-\epsilon, \epsilon)$, and the length of the velocity $\|v\| < \delta$. Fix $q \in M$, let's denote $\gamma_v(t)$ as the geodesics with*

$$\gamma_v(0) = q,\ \dot\gamma_v(0) = v.$$

Then we can claim $\gamma_{\lambda v}(t) = \gamma_v(\lambda t)$.

This is because: in local coordinates, $\gamma_v(t)$ is written as $(x^1(t), \cdots, x^n(t))$.

Then satisfy

$$\begin{cases} (x^1(t), \cdots, x^n(t)) = v \\ \ddot x^k(t) + \Gamma^k_{ij}(x(t))\dot x^i \dot x^j = 0 \end{cases} \tag{2.1.4}$$

$$\Rightarrow \begin{cases} (x^1(\lambda t), \cdots, x^n(\lambda t)) = \lambda v \\ \ddot x^k(\lambda t) + \Gamma^k_{ij}(x(\lambda t))x(\dot{\lambda t})^i x(\dot{\lambda t})^j = \lambda^2(\ddot x^i + \Gamma^k_{ij}(x)\dot x^i \dot x^j)|_{\lambda t} = 0 \end{cases} \tag{2.1.5}$$

Hence $\gamma_{\lambda v}(t) = (x^1(\lambda t), \cdots, x^n(\lambda t)) = \gamma_v(\lambda t)$.

$\Rightarrow$Lemma: If $\gamma(t, q, v)$ is defined for $t \in (-\epsilon, \epsilon)$ and $\|v\| < \delta$, then $\gamma(t, q, \lambda v)$ is defined for $t \in (-\frac{\epsilon}{\lambda}, \frac{\epsilon}{\lambda})$ and $\|v\| < \delta$.

Corollary 2.1. *Let $p \in M,\ v \in T_pM$. Then $\exists\ \epsilon > 0$ and a unique geodesic $\gamma : [0, \epsilon] \to M$ with $\gamma(0) = p,\ \dot\gamma(0) = v$.*

Proof. Assign $s = \frac{\delta}{2\|v\|}$, then $\|sv\| < \delta$. By theorem 2.1, $\exists\ \epsilon_0 > 0$, and a unique geodesic $\gamma_{sv} : [0, \epsilon_0] \to M$ with $\gamma_{sv}(0) = p,\ \dot\gamma_{sv}(0) = sv$. Hence $\gamma_v(t) = \gamma_{sv}(\frac{t}{s})$ is a geodesic defined on $[0, s\epsilon_0]$. Hence $\epsilon = s\epsilon_0$, by the uniqueness of Thm 2.1, we show this corollary. □

Exercise 2.1. *Compute the geodesic equation of S^2 in sphererical coordinates.*

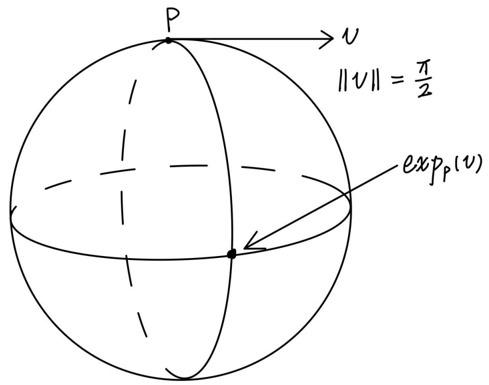
Exercise 2.2. *What is the transformation behavior of the Christofell symbols? Do they define a tensor*

2.2 Minimizing Properties of Geodesics

Next, we explain that a geodesic is "locally" shortest curve. For that purpose, we first discuss the important concept Exponential map.

Let (M, g) be a Riemannian manifold, $p \in M$. Roughly speaking, the exponential map of M at p maps $v \in T_pM$, with $g_p(v, v) = \|v\|$, to a point q on the geodesic

$\gamma_v : [0, b] \to M$ with $\gamma(0) = p, \dot{\gamma}(0) = v$, such that the arc length $\widehat{pq} = \|v\|$. This means, we should pick $q = \gamma_v(1)$.(Since as a geodesic,$\|\dot{\gamma}t\| = \|\dot{\gamma}(0)\| = \|v\|$.)



Definition 2.2. *(Exponential Map) Let* (M, g) *be a Riemannian manifold,* $p \in M$*. Denote* $V_p := \{v \in T_pM :$ *the geodesic* γ_v *with* $\gamma_v(0) = p, \dot{\gamma}_v = v$ *is defined on* $[0, 1]\}$*.* $exp_p : V_p \to M, v \mapsto \gamma_v(1)$ *is called the exponential map of M at p.*

In the following we use $\gamma_{p,v}$ to denote the geodesic with $\gamma_{p,v} = p, \dot{\gamma}_{p,v} = v$.(Oftenly, p is *omitted*.)

What does V_p look like?

(1)Star-sharped around $0 \in T_pM$.

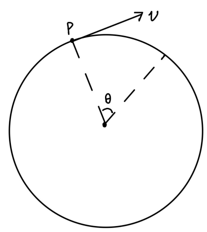
If $v \in V_p$, i.e. γ_v is defined on $[0, 1]$, then $\gamma_{\lambda v}(0 < \lambda < 1)$ is defined on $[0, \frac{1}{\lambda}]$, and, in particular, on$[0, 1]$. Hence $v \in V_p \Rightarrow \lambda v(0 \leq \lambda \leq 1) \in V_p$.

(2)$\forall p \in M$, $\exists\ \epsilon_0\ s.t.\ B(0, \epsilon_0) \subset V_p$ i.e. $\forall\ \omega \in T_pM, \|\omega\| \leq \epsilon_0$, we have $\gamma_{p,\omega}$ is defined on $[0, 1]$.

By Theorem **??**, $\exists\ \epsilon, \delta > 0,\ s.t.\ \forall v \in T_pM, \|v\| < \delta$, $\gamma_{p,v}$ is defined on $[0, \epsilon]$, hence $\gamma_{p,\epsilon v}$ is defined on $[0, 1]$.

$\Rightarrow \forall \omega \in T_pM$ with $\|\omega\| \leq \epsilon\|v\| < \epsilon\delta$, we have $\gamma_{p,\omega}$ is defined on $[0, 1]$. That is, $\omega \in V_p$. ■

Example 2.2.
(1) $M = R^n,\ g_{ij} = \delta_{ij}$*.*
The geodesic equation is $\ddot{x}^i(t) = 0$*. The geodesics are straight lines parametrized proportionally to arc length.* $\forall p \in R^n,\ v \in T_pR^n.\ exp_p(v) = p + v.\ V_p = T_pR^n = R^n$*.*
(2) Circle $(S^1, d\theta \otimes d\theta)$*.*
$\forall p \in M$, T_pS^1 *can be identified with R. Then* $exp_p(v) = e^{iv}$, $V_p = T_pS^1 = R$ *for* $p = e^{i0} = 1$*.(In local coordinates,* $exp_p : v \to v$*.). This is the simplest example explaining why the terminology "exponential map" is used. It actually comes from Lie theory.*



(3) Open disc in $R^2 : D_0 = \{(x^2+y^2) \in R^2 | x^2+y^2 = 1\}$ *with a Riemannian metric induced from the canonical Euclidean metric on* R^2.

$exp_0(v) = 0 + v = v$. *But* $V_0 \neq R^2,\ V_0 = D_0$*(we identify* T_0D_0 *with* R^2*).*

Theorem 2.2. *The exponential map* exp_p *maps a neighborhood of* $0 \in T_pM$ *diffeomorphically onto a neighborhood of* $p \in M$.

Remark 2.4. *Reason for restricting to a neighborhood:*

(1)exp_p *may not be defined on the whole* T_pM.

(2)even if exp_p *is defined on the whole* T_pM*, it is not necessarilly a diffeomorphism.(Example of* $(S^1, d\theta \otimes d\theta),\ exp_p$ *is not injective.)*

Proof. $0 \in T_pM,\ decp_p(0) : T_0(T_pM) \to T_pM$. Since T_pM is a vector space, we may identify $T_0(T_pM)$ with T_pM.

$\Rightarrow\ dexp_p(0) : T_pM \to T_pM$. Now we can calculate $dexp_p(0)(v)$ for a $v \in T_0(T_pM) = T_pM$.

recall: for a c^∞ map $f : M \to N, x \mapsto y$. One way to calculate $df : T_xM \to T_yN$ is the following :

For any $v \in T_xM$, consider a curve γ with $\gamma(0) = x, \dot\gamma(0) = v$. Then $\xi = f(\gamma)$ is a curve in N, and $df(v) = \dot\xi(0)$.

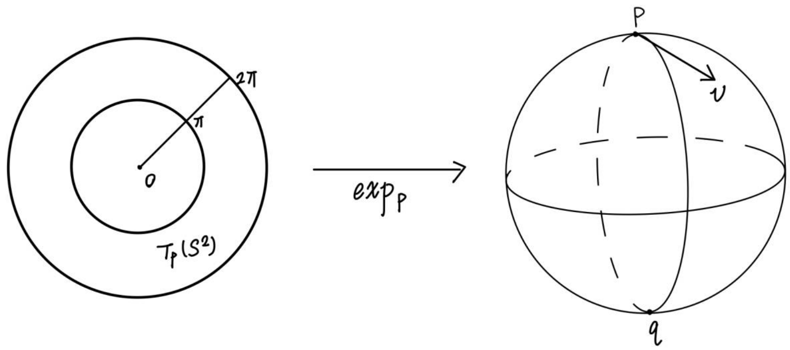
Here, $exp_p : V_p \subset T_pM \to M, 0 \mapsto p$. For $v \in T_0(T_pM) = T_pM$, consider $\gamma(t) = tv$.

We have

$$\begin{aligned} dexp_p(0)(v) &= \frac{d}{dt} exp_p(tv)|_{t=0} \\ &= \frac{d}{dt}\gamma_{tv}(1)|_{t=0} = \frac{d}{dt}\gamma_v(t)|_{t=0} = \dot\gamma_v(0) = v. \end{aligned}$$

That is $dexp_p(0) = id|_{T_pM}$.

In particular, $dexp_p(0)$ has maximal range, and by "inverse function theorem", there exists a neighborhood of $0 \in T_pM$, which is mapped diffeomorphically onto a neighborhood of $p \in M$. □



Example 2.3.

$S^2 \subset R^3$. q is a antipodalpoint of p. exp_p is defined on the whole T_pS^2. Let $B(0,\pi) \subset T_p(S^2)$ be the open ball around 0 in $T_p(S^2)$.(with the scalar product given by the Riemannian metric of S^2). $exp_p : B(0,\pi) \to S^2\backslash\{q\}, \overline{B(0,\pi)}\backslash B(0,\pi) = \partial B(0,\pi) \to \{q\}$ diffeomorphically. However, $exp_p(B(0,2\pi)\backslash\overline{B(0,\pi)}) = S^2\backslash\{p,q\}$ and $exp_p(\overline{B(0,2\pi)}\backslash B(0,2\pi)) = \{p\}$.

And we can identify T_pM with R^n via $\Phi : T_pM \to R^n, v = v^ie_i \Rightarrow (v^1,\cdots,v^n)$. Thm 2,2,1(page 28) tells that $\exists$ a neighborhood $U \ni p$, such that exp_p^{-1} map U diffeomorphically onto a neighborhood of $0 \in T_pM \overset{=}{via\Phi} R^n$. In particular, $p \mapsto 0$.

Definition 2.3. *(Normal coordinates) The local coordinates defined by (U, exp_p^{-1}) are called (Riemannian) normal coordinates with center p.*

The advantage of such a choice of coordinates is presented in the following result:

Theorem 2.3. *In normal coordinates, we have*

$$g_{ij}(0) = \delta_{ij} \tag{2.2.1}$$

$$\Gamma^i_{jk}(0) = 0. \tag{2.2.2}$$

for the Riemannian metric, and all i,j,k.

Proof. (2.2.1) follows from the identification Φ of T_pM and R^n(Recall $g_p(e_i,e_j) = \delta_{ij}$).

Next, we show (2.2.2), recall in the local coordinate $(U, exp_p^{-1}) = (U, v^1,\cdots,v^n)$, the geodesic equation is

$$\ddot{v}^i + \Gamma^i_{jk}(v(t))\dot{v}^j(t)\dot{v}^k(t) = 0,\ i = 1,2,\cdots,n. \tag{2.2.3}$$

On the other hand, in $exp_p^{-1}(U) \subset R^d$, the line $tv, t \in R, v \in R^d$ is $exp_p^{-1}(\gamma_{tv}(1)) = exp_p^{-1}(\gamma_v(t))$, i.e. is the image of a geodesic in M via the coordinate map. □

Remark 2.5. *Even if exp_p is defined on the whole T_pM, it may be not a global diffeomorphism. Suppose $exp_p : B(0,\rho) \to exp_p(B(0,\rho))$ is diffeomorphic, how large can ρ be?*

Here we mention the following concept of injectivity radius.

Definition 2.4. *Let M be a Riemannian manifold,$p \in M$. The injectivity radius of p is*

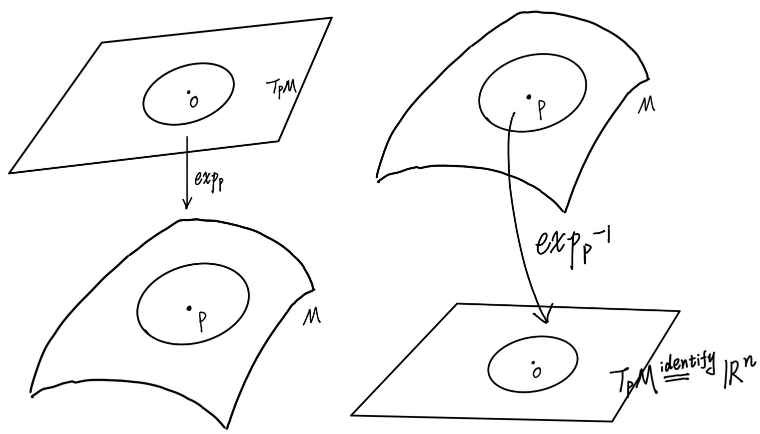
$$i(p) := sup\rho > 0 : exp_p \ is \ a \ diffeomorphism \ on \ B(0,\rho) \subset T_pM.$$

The injectivity radius of M is

$$i(M) := \inf_{p \in M} i(p).$$

The above example shows that $i(S^2) = \pi$.

Normal coordinates



T_pM has an inner product defined by g. Let $e_1, \cdots, e_n (n = dim\ M)$ be an orthonormal basis of T_pM(w.r.t. the inner of product given by g). Then for each $v \in T_pM$, we can write $v = v^i e_i$. Therefore, $v(t) = tv$ satisfies (2.2.3). This implies $\Gamma^i_{jk}(tv)v^jv^k = 0,\ i =, \cdots, n,\ \forall v \in R^d$.

In particular, for t=0

$$\Gamma^i_{jk}(0)v^jv^k = 0,\ i = 1, 2, \cdots, n,\ \forall v \in R^d. \tag{2.2.4}$$

For any indices l and m, pick $v = e_l + e_m$, we have

$$\Gamma^i_{lm}(0) = 0,\ i = 1, 2, \cdots, n.$$

That is $\Gamma^i_{jk}(0) = 0,\ \forall i, j, k$.

Recall the definition of $\Gamma^i_{jk}(0)$, we obtain at $0 \in R^d$: $g^{il}(g_{jl,k} + g_{lk,j} - g_{jk,l}) = 0,\ \forall i, j, k$.

$\Rightarrow g_{jl,k} + g_{lk,j} - g_{jk,l} = 0,\ \forall j, k, l$.

By a cyclic permutation of the indices: $j \to k,\ k \to l,\ l \to j$, we have $g_{kj,l,k} + g_{jl,k} - g_{kl,j} = 0$. Notice that $g_{lk,j} = g_{kl,j},\ g_{jk,l} = g_{kj,l}$, we get $2g_{jl,k} = 0$ at $0 \in R^d$. ■

Remark 2.6. *In general, the second derivatives of the metric cannot be made to vanish at a given point by a suitable choice of local coordinates. The obstruction is given by the "curvature tensor".*

On R^n we can introduce the standard polar coordinates $(r, \varphi^1, \cdots, \varphi^{n-1})$ where $\varphi = (\varphi^1, \cdots, \varphi^{n-1})$ parametrizes the unit sphere S^{n-1}.

Then via Φ, we obtain polar coordinate on T_pM. We can write the metric g in polar coordinate:

$$g_{rr} := g_{11} = g(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}), g_{r\varphi} := (g_{1,l})_{l=2,\cdots,n}, g_{\varphi\varphi} = (g_{kl})_{k,l=2,\cdots,n}$$

In particular at $0 \in T_pM$, we have

$$g_{rr}(0) = 1, \ g_{r\varphi}(0) = 0. \tag{2.2.5}$$

(The reason as (2.2.1) in Theorem 2.2.2 on page 29.)

The point is that in this case we can show (2.2.5) holds true not only at $0 \in T_pM(= R^n)$, but in the whole coordinate neighborhood.

Theorem 2.4. *For the polar coordinates, obtained by transforming the Euclidean coordinates of R^d, on which the normal coordinates with centre p are based, into polar coordinates, we have*

$$g_{ij} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & g_{\varphi\varphi}(r,\varphi) & \\ 0 & & & \end{pmatrix}$$

where $g_{\varphi\varphi}(r, \varphi)$ is the $(n-1) \times (n-1)$ matrix of the components of the matrix w.r.t angular variables $(\varphi^1, \cdots, \varphi^{n-1}) \in S^{n-1}$.

Proof. In this case, $tv, v \in R^n$ is transformed to be $\varphi \equiv const$. That is, $\varphi \equiv const$ are geodesic when parametrized by arc length in the local coordinates. They are given by

$$x(t) = (t, \varphi_0). \ \varphi_0 \ fixed.$$

Geodesic equation gives

$$\Gamma^i_{rr}(x(t))\dot{t}\dot{t} = \Gamma^i_{rr}(x(t)) = 0, \ \forall i \ (\dot{\varphi}_0 = 0).$$

Compare with the situation in (2.2.4) on page 30. Hence at $x(t) \in T_pM = R^n$,

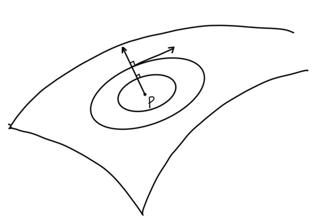
$$g^{il}(g_{rl,r} + g_{lr,r} - g_{rr,l}) = 0, \ \forall i$$

$\Rightarrow$

$$2g_{rm,r} - g_{rr,m} = 0, \ \forall m. \tag{2.2.6}$$

For $m = r$, we have $g_{rr,r} = 0$, combining with $g_{rr}(0) = 1 \Rightarrow g_{rr} \equiv 1$.

Hence $g_{rr,m} \equiv 0, \forall m \Rightarrow g_{r\varphi,r} = 0, \forall \varphi \Rightarrow g_{r\varphi} \equiv 0$. ■ □



Remark 2.7. *$dr \otimes dr + g_{\varphi\varphi}(r,\varphi)d\varphi \otimes d\varphi$ is not a proguct metric, since $g_{\varphi\varphi}$ may depend on r.*

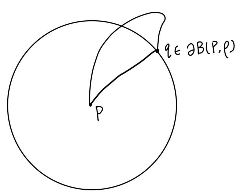
Remark 2.8. *Since g is positive definite, we have*

$$(g_{\varphi\varphi}(r,\varphi))$$

is also positive definite.

Corollary 2.2.
(1)For any $p \in M$, $\exists \rho > 0$ s.t. the (Riemannian) polar coordinates may be introduced on $\overline{B(p,\rho)} := \{q \in M, d(\varphi,q) = \rho\}$.
(2)For any such that ρ, and $q \in \partial B(p,\rho)$, there is a unique normal geodesic whose length(=ρ) is the shortest one among all curves that belongs to $C_{p,q}$.

Proof.
(1).By theorem 2.3, polar coordinates can be introduced on a neighborhood U of p. Since manifold topology and metric topology coincides, such a ρ can be found.



(2).Consider any $c \in C_{p,q}$. Without loss of generality, let's assume it's smooth. c may leave our polar coordinate neighborhood. Let $t_0 := \inf\{t \in \tau : d(c(t),p) \geq \rho\}$, then $c|_{[0,t_0]} \subset \overline{B(p,\rho)}$. In polar coordinates, write $c(t) = (r(t),\varphi(t))$, $c(t_0) = (\rho,\varphi(t_0))$. We calculate

$$\begin{aligned} L(c|_{[0,t_0]}) &= \int_0^{t_0} \sqrt{g_{ij}(c(t))\dot{c}^i(t)\dot{c}^j(t)}dt \\ &= \int_0^{t_0} \sqrt{g_{rr}(c(t))\dot{r}\dot{r} + g_{\varphi\varphi(c(t))\dot{\varphi}\dot{\varphi}}}dt \\ &\geq \int_0^{t_0} |\dot{r}|dt \geq |\int_0^{t_0} \dot{r}dt| = |r(t_0) - r(0)| = \rho \end{aligned}$$

Moreover, "=" holds iff. $g_{\varphi\varphi}\dot{\varphi}\dot{\varphi} \equiv 0 (\Leftrightarrow \dot{\varphi} = 0 \Rightarrow \varphi \equiv const)$ and $\dot{r} \geq 0$ or $\dot{r} \leq 0$. Hence the "=" holds iff $c(t) = (t,\varphi_0)$, *where* $q = (\rho,\varphi_0)$. Recall (t,φ_0) is the geodesic in the polar coordinates. □

Remark 2.9.
(1)From the proof, we see $\forall c \in C_{p,q}$, $L(c) \geq L(\gamma)$, where γ is the radical geodesic. And "=" holds iff c is a monotone reparametrization of γ.
(2)There may exists other geodesics from p to q, whose length is longer. That is the "shortestness" property of a geodesic is not global! From Corollary 2.2.1, we see $\forall p,q \in M$, where they are close "enough" to each other, then there exists precisely one geodesic of shortest length. Can we have a uniform description of the "closedness"

which ensure the existence of shortest geodesics, at least when M is compact? For this surpose, we first discuss a refinement of Theorem 2.2.1(page 28)($dexp_p(0) = I$) and the "totally normal neighborhood".

Theorem 2.5. *(totally normal neighborhood) For any point $p \in M$, there exists a neighborhood M of p and a number $\delta > 0$, such that, for every $q \in W$,(in other words, injectivity radius $i(q) \geq \delta$), and $exp_q(B(0, \delta)) \supset W$.*

Remark 2.10. *(Terminologies) If exp_p is a diffeomorphism of a neighborhood V of the origin in T_pM. Then we call $exp_p(V) := U$ a normal neighborhood of p. Theorem 2.2.4 tells, $\exists$ a neighborhood of p such that W is a normal neighborhood of each q of W. W is then called a totally normal neighborhood of $p \in M$.*

If $B(0, \epsilon)$ is such that $\overline{B(0, \epsilon)} \subset V$, we call $exp_p(B(0, \epsilon))$ the normal ball with center p and radius ϵ. The geodesic in $exp_p(B(0, \epsilon))$ that begin at p are referred to as radical geodesics.

Remark 2.11. *By Corollary 2.2.1, any 2 point in W can be connected by a cunique minimizing geodesic. For the proof, we first discuss a revision.*

Theorem 2.6. *There exists a neighborhood U of p, $U := \{(q, v) \in TM : q \in V, v \in T_qM, \|v\| < \epsilon\}$ such tvhat $exp : U \to M, (q, v) \mapsto exp_q v$ is well defined. Consider the following map.*

$$F : U \to M \times M, (q, v) \mapsto (q, exp_q)$$

In particular, we see $F(p, 0) = (p, p)$. Then $dF(p.0) : T_{(p,0)}(TM) \to (M \times M)$.

Lemma 2.3. *For each $p \in M$ and with it the zero vector $0 \in T_pM$, $dF(p, 0)$ is nonsingular.*

Proof.
Proof of lemma 2.2.1:

First note that, we can identify the tangent space $T_{(p,p)}(M \times M)$ to $T_pM \times T_pM$, $T_{(p,0)}(TM)$ to $T_pM \times T_0(T_pM) \cong T_pM \times T_pM$. $F : U \to M \times M$. In local coordinates, this map centre considered as $(x^1, \cdots, x^n, v^1, \cdots, v^n) \to (x_1^1, \cdots, x_1^n, x_2^1, \cdots, x_2^n)$.

We consider $dF_{(p,0)}$ as a linear map $T_pM \times T_pM \to T_pM \times T_pM$,

. varying p, F is identity in the first coordinate. Hence on the first factor to the factor $dF_{(p,0)}$ is identity.

. fix p and vary v in T_pM, the first coordinate of F is fixed and the second coordinate is $exp_p v$.

Hence, $dF_{(0,p)}$ is identically 0 from the second factor to the first factor and identity from the second factor to the second factor.(Theorem 2.2.1, page 28)

$$\begin{pmatrix} dF^1_{v=0} & dF^1_{q=p} \\ dF^2_{v=0} & dF^2_{q=p} \end{pmatrix} = \begin{pmatrix} I & 0 \\ I & 0 \end{pmatrix} \tag{2.2.7}$$

∎

proof of theorem 2.6:

By lemma and the inverse function theorem, we know that F is a local diffeomorphism. This means that $\exists$ a neighborhood $U' \subset U$ of $(p, 0) \in TM$ s.t. F maps U' diffeomorphically onto a neighborhood W' of $(p, exp_p 0_p = p) \in M \times M$.

By shrinking U' if necessary, we can take U' to be the form

$$U' = \{(q, v) : q \in V', v \in T_qM, \|v\| < \delta\}$$

where $V' \subset V$ is a neighborhood of $p \in M$.

Now choose a neighborhood W of p in M so that $W \times W \subset W'$. Then form the definition of F, we see $exp_p(B(0, \delta)) \supset W$. ■ □

Now, we have some immediate consequence.

Corollary 2.3. *Let Ω be a compact subset of a Riemannian manifold M. There exists $\rho_0 > 0$ with the propert for any $p \in \Omega$, Riemmanian polar coordinates may be introduced on $B(p, \rho_0)$*

Proof. $\forall p \in \Omega$, we can find a totally normal neighborhood W_p of p. By compactness, we have a finite subcover $\{W_{p_i}\}_{i=1}^N$ of $\{W_p\}_{p\in\Omega}$ of Ω. Since for each W_p, $\exists$ a ρ_p s.t. Riemmanian polar coordinates may be defined on $B(q.\rho_p)$, $\forall q \in W_p$. We pick $\rho_0 = \min\limits_{i=1,\cdots,N}\{\delta_{p_i}\}$ ■ □

Corollary 2.4. *Let Ω be a compact subset of a Riemannian manifold M. Then there exists $\rho_0 > 0$ with the property that for any two points $p, q \in M$ with $d(p, q) \le \rho_0$ can be connected by precisely one geodesic of shortest path.*

The geodesic depends continuously on (p, q):

Proof. ρ_0 from Corollary 2.2.2 satisfies the first claim by Corollary 2.2.1. Morever, given $(p, q), d(p, q) \le \rho_0$, there exists a unique $v \in T_pM$(given by $F^{-1}(p, q) = (p, v)$) that depends continuously on (p, q) and is s.t. $\gamma(v) = v$. □

Corollary 2.5. *Let M be a compact Riemannian manifold, $i(M) > 0$.*

Local isometries map geodesics to be geodesics

Recall that a C^∞ differentiable map $h : M \to N$ is a local isometry, if $\forall p \in M$, $\exists$ a neighborhood U for which $h|_U : U \to h(U)$ is an isometry and $h(U)$ is open in N and $g|_U = h^*(\gamma|_{h(U)})$ where $(g_{ij}(p), (\gamma_{\alpha\beta}(h(p)))$ are the metrics on $U, h(U)$ respectively. In fact, $g_{ij}(p) = \gamma_{\alpha\beta}(h(p))\frac{\partial h^\alpha(p)}{\partial x^i}\frac{\partial^\alpha(p)}{\partial x^j}$.

A local isometry has the same effect as a coordinate change. We have already see in the Homework Exercise 2, that the geodesic equations

$$\ddot{x} + \Gamma^i_{jk}\dot{x}^j\dot{x}^k = (\dot{y}^\alpha + \widetilde{\Gamma}^\alpha_{\eta\gamma}\dot{y}^\eta\dot{y}^\gamma)\frac{\partial x^i}{\partial y^\alpha}.$$

Hence geodesic is mapped to be geodesic($det(\frac{x^i}{\partial y^\alpha}) \neq 0.$). Intuitively, Isometries leave the lengths of tangent vectors and therefore also the lengths and energies of curves invariant. Thus, critical points, i.e. geodesics, are mapped geodesics.

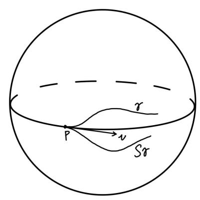
This observation has interesting consequences.

Example 2.4. *(geodesics of S^n)*

The orthogonal group $O(n+1)$ operates isomotrically on $\mathbb{R}^{n+1}$, and since it maps S^n into S^n, it also operates isometricaly on S^n.

Let now $p \in S^n, v \in T_p(S^n)$. Let E be the two dimensions plane through the origin of $\mathbb{R}^{n+1}$ containing v.

Claim:the geodesic γ_v pass through p with tangent vector v is the great circle through p with tangent vector v(parametrized proportionally to arc length), i.e. the intersection of S^n and E.



Proof. Let $S \in O(n+1)$ be the reflection acress E, then $Sv = v, Sp = p$.

γ_v is a geodesic$\Rightarrow$ $S\gamma_v$ is also a geodesic through p with tangent vector v. By uniqueness result, $\gamma_v = S\gamma_v$.

Hence image of γ_v is the great circle. ■ □

Example 2.5. *(geodesics on $\mathbb{T}^2$)*

$\omega_1 = (1,0) \in \mathbb{R}^2, \omega_2 = (0,1) \in \mathbb{R}^2$. *Consider $z_1, z_2 \in \mathbb{R}^2$ as equivalent if $\exists\, m_1, m_2 \in \mathbb{Z}$ s.t. $z_1 - z_2 = m_1\omega_1 + m_2\omega_2$.*

The covering map $\pi : \mathbb{R}^2 \to \mathbb{T}^2, z \mapsto [z]$, differentiable structure:

$\Delta_\alpha \subset$ is open and does not contain equivalent points, then $U_\alpha := \pi(\Delta_\alpha), z_\alpha = (\pi|_{\Delta_\alpha}^{-1})$.

For each chart $(U, (\pi|_U)^{-1})$ we use the Euclidean metric on $\pi^{-1}(U)$. Since the translations

$$z \mapsto z + m_1\omega_1 + m_2\omega_2,\ m_1, m_2 \in \mathbb{Z}$$

are Euclidean isometries, the Euclidean metrics on the different components of $\pi^{-1}(U)$(which are obtained from each other by translations.) yield tyhe same metric on U. Hence the Riemmanian metric on $\mathbb{T}^2$ is well defiend, and $\pi : \mathbb{R}^2 \to \mathbb{T}^2$ is a local isometry. Therefore, Euclidean geodesics of R^2 are mapped onto geodesics of T^2.

2.3 Global Properties:Hopf-Rinow Theorem

In the last section, we know when two points $p, q \in M$ are close enough to each other, there exists precisely one geodesic with the shortest length. Naturally, one would ask the following questions.

Question 1: If a curve γ is of shortest lengths, is γ a geodesic?

Question 2: Let $\gamma : [0,1] \to M$ be a geodesic, is it the shortest curve from $\gamma(0)$ to $\gamma(1)$?

Question3: Given $p, q \in M$, does there exists a curve from p to q with the shortest length?

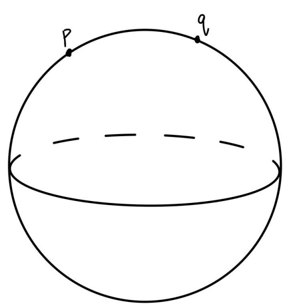
Recall that a geodesic γ has to be parametrized proportionally to its arc length. Hence, a proper way to formulate Question 1 is:

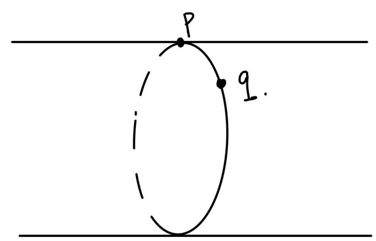
Question 1': Let $\gamma : [0,1] \to M, \|\dot{\gamma}\| = 1$ and $\forall \xi \in C_{p,q}$(piecewise C^∞ curve from p to q), $Length(\gamma) \leq Length(\xi)$. Is γ necessarily a geodesic ?

Proposition 2.1. *If a piecewise C^∞ $\gamma : [a,b] \to M$ with parameter proportional to arc length, has length less than or equal to the length of any other piecewise C^∞ curve from $\gamma(a)$ to $\gamma(b)$, then γ is a geodesic.*

Proof. Let $t \in [a,b]$, and let W be a totally normal neighborhood of $\gamma(t)$. There exists a closed interval $I \subset [a,b]$ with nonempty interior, $t \in I$ s.t. $\gamma(I) \subset W$. By the global "shortestness" property of γ, we know $\gamma(I)$ is a piecewise smooth curve connecting two points in W which the shortest length. By corollary 2.2.1, noticing further γ is parametrized proportionally to arc length, we konw $\gamma|_I$ is a geodesic. ■ □

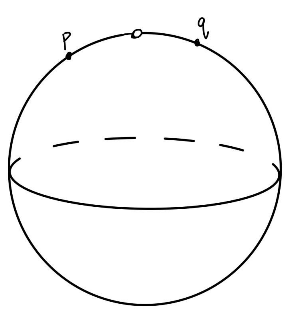
Concerning Question 2, we have found several counter-examples, like





So the answer to Question 2 is "No!" Then one may ask when is a geodesic γ also a minimizing curve? We will discuss this issue in later lecture.

The answer to Question 3 is also "No". If a curve γ from p to q is of the shortest curve, after choosing the parameter proportional to arc length, Proposition tells that γ must be a geodesic.



Then there is no curve from p to q with the shortest length.(but you have a minimizing sequence of curves)

When is the answer to Question 3 "Yes"? It turns out, one have to require M to be compact!!

Given a Riemannian manifold (M, g), recall that (M, g) with the distance function d derived from g is a metric space (M, d). And the topology of (M, d) coincides with original topology of M. Therefore (M, d) is a complete metric space iff M is complete as a topological space with regard to its original topologies. So we do not need to distinguish this two Completeness.

Hopf-Rinow theorem tells completeness implies the existence of minimizing geodesic between any two points. Morover, H-R thm also gives two several equivalent descriptions of completeness.

Theorem 2.7. *(Hopf-Rinow 1931).(Über den Begriff der vollständigen differential geometrischen Flaäche, Commentarii Mathematici Helvetici, 1931)*

Let M be a Riemannian manifold. The following statements are equivalent:

(i) M is a complete metric space.

(ii) The closed and bounded subsets of M is compact.

(iii) $\exists p \in M$ *for which* exp_p *is defined on all of* T_pM.

(iv) $\forall p \in M,\ exp_p$ *is defined on all of* T_pM.

Each of the statement (1)-(4) implies

(v) Any two points $p, q \in M$ *can be joined by a geodesic of length* $d(p, q)$*, i.e. by a geodesic of shortest length.*

Digest of the theorem:

(1) (i)⇒(v) but bot vise versa. Conuterexample: open disc is not complete, but satisfies property (v).

(2) **Definition 2.3.1**(geodesically complete): A Riemannian manifold M is geodesically complete if for all $p \in M$, exp_p is defined on all of T_pM, or, in other words, if any geodesic $gamma(t)$ with $\gamma(0) = p$ is defined for all $t \in \mathbb{R}$.

H-R theorem tells completeness(manifold topology, apriori independent of the metric)⇔geodesical completeness.(depends on the Riemannian metric)

(3) (M, g) as a complete metric space is a very special one as shown in (ii).

Consider a countable set $A = \{a_i; i = 1, 2, \cdots\}$ with a discrete metric, i.e. $d : A \times A \to [0, \infty)$ s.t. $d(a_i, a_j) = \delta_{ij}$. Then (A, d) is complete and bounded. But A is not compact.

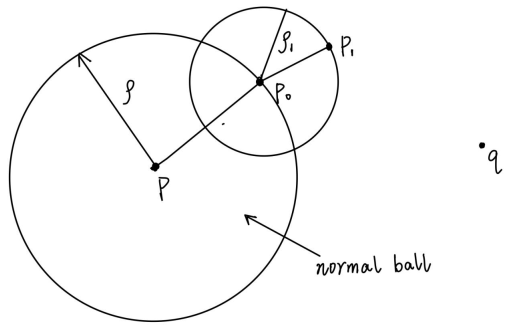
(4) **Corollary 2.3.2**: Let (M, g) be a complete Riemannian manifold. Then $exp_p : T_pM \to M$ is surjective for any $p \in M$.

Proof. **The "core" of result is (iv)⇒(v):**

Given $p, q \in M$, we hope to find a shortest geodesic γ from p to q. We know $\gamma(0) = p$, but how to decide $\dot\gamma(0)$?

Consider a normal ball $B(p, q)$. Since $\partial B(p, \rho)$ is compact, and $d(p, .)$ is a continuous function, there exists $p_0 \in \partial B(p, \rho)$ s.t. $d(p, .)$ attain its minimum on $\partial B(p, \rho)$ at p_0.

Now the idea is the following:



At p_0, consider the normal ball $B(p_0, \rho_1)$, find p_1 to be the point at which $d(p, .)$ attain its minimum on $\partial B(p_0, \rho_1)$. And, we continue this procedure, and hopefully we arrive at q.

Two issues in this arguement:

(1) Can the piecewise geodesics(or broken geodesic) be a single geodesic?

(2) Can we arrive at q eventually?

To solve this two issues, we argue as below:

In $B(p, \rho)$, we know the radical geodesic from p to p_0 is

$$c(t) = exp_p tV, \ for\ some\ V \in T_p M$$

with $p_0 = exp_p \rho V$.(That is c is parametrized by arc length.)

We consider the curve

$$c(t) = exp_p tV, \ t \in [0, \infty)(by\ (iv)\ we\ can\ do\ this)$$

we hope to show $c(r) = q$, where $r = d(p, q)$. If this was shown to be true, we know $c(r)$ is the shortest one and we are done.

In other words, we hope to prove

$$d(c(r), q) = 0 \tag{2.3.1}$$

Next we know

$$d(c(0), q) = d(p, q) = r \tag{2.3.2}$$

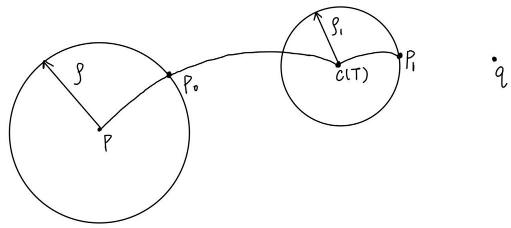
Consider the set

$$I := \{t \in [0, r], d(c(t), q) = r - t\}.$$

(ii)$\Leftrightarrow$ $0 \in I$. Hence $I \neq \Phi$. Moreover, since $f(t) := d(c(t), q) - r + t$ is continuous, and $I = f^{-1}(0) \bigcap [0, r]$, I is closed. Let $T = \sup_{t \in I} t$. Since I is closed, we see $T \in I$. If $T = r$, then we are done.

Suppose $T < r$, consider the normal ball $B(c(T), \rho_1)$(Without loss of generality, we can assume $\rho_1 < r - T$.).

Let $p \in \partial B(c(T), \rho_1)$ be the point at which $d(q, .)$ attain its minimum on $\partial B(c(T), \rho_1)$.



Consider the three points: $c(T),\ p_1,\ q$. By definition, $d(q, c(T)) = r - T$. We have

$$d(c(T), p_1) + d(p_1, q) \geq d(c(T), q) \tag{2.3.3}$$

by using triangle inequation.

On the other hand, think of any curve γ from $c(T)$ to q. There exists t, s.t. $\gamma(t) \in \partial B(c(T), \rho_1)$.

$$\begin{aligned} Length(\gamma) \geq & d(c(T), \gamma(t)) + d(\gamma(t), q) \\ \geq & d(c(T), p_1) + d(\gamma(t), q) \\ \geq & d(c(T), p_1) + d(p_1, q) \end{aligned}$$

$\Rightarrow d(c(T), q) \geq d(c(T), p_1) + d(p_1, q)$.

Combining with inequation (2.3.3), we get euqality:

$$\begin{aligned} d(c(T), q) &= d(c(T), p_1) + d(p, q) \\ \Rightarrow d(p_1, q) &= d(c(T), q) - d(c(T), p_1) \\ &= r - T - \rho_1 \\ &= r - (T + \rho_1). \end{aligned}$$

Now if we show

$$p_1 = exp_p(T + \rho_1)V = c(T + \rho_1), \tag{2.3.4}$$

we have $T + \rho_1 \in I$, which contradicts to the definition of T.

It remains to show (2.3.4). We use Proposition 2.1 to prove it. That is, we show the curve

$$c|_{[0,T]} \text{ and the radical geodesic from } c(T) \text{ to } p_1 \tag{2.3.5}$$

is the shortest curve from p to p_1. Note the length of the curve is $T + \rho_1$.

Consider the three points p, p_1 and q to figure out $d(p, p_1)$.

$$\begin{aligned} d(p, q) \leq & d(p, p_1) + d(p_1, q) \\ \Rightarrow T + \rho_1 \leq & d(p, p_1) \end{aligned}$$

Hence the "broken" curve in (2.3.5) is the shortest curve from p to q. Then proposition tells that it is a smooth geodesic when parametrized with arc length.

By uniqueness of geodesics with given initial values, it has to coincide with c. Therefore

$$p_1 = exp_p(T + \rho_1)V = c(T + \rho_1).$$

Then we finish the proof of (iv)⇒(v). ■

Next, **we prove the euivalence of (i)-(iv).**

(iv)⇒(iii) is trivial.

(iii)⇒(ii): Let $K \subset M$ be closed and bounded.

"boundness"⇒$K \subset B(p, r)$ for some $r > 0$.

Since exp_p is defined on all of T_pM, from the proof for (iv)⇒(v), we know any $q \in \overline{B(p,r)}$ can be connected to p via $c(t) = exp_p(tV)$, with $c(d(p, q)) = exp_p(d(p, q)V)$ for some V.

Hence $\overline{B(p,r)}$ is the image of the compact ball in T_pM of radius r under the continuous map exp_p. Hence, $\overline{B(p,r)}$ is compact. Since K is assumed to be closed, and shown to be contained in a compact set, it must to be compact itself. ■

(ii)⇒(i): Let $\{p_n\}_{n\in\mathbb{N}} \subset M$ be a cauthy sequence and $p_0 \in M$. Since $\{p_n\}_{n\in\mathbb{M}}$ is a cauthy sequence, we have $\forall\ \epsilon > 0,\ \exists\ N$, when $n, l > N$,

$$|d(p_n, p_0) - d(p_l, p_0)| \leq d(p_n, p_l) < \epsilon.$$

That is $\{d(p_n, p_0)\}_{n\in\mathbb{N}}$ is a cauthy sequence in $\mathbb{R}$, then $\lim_{n\to\infty} d(p_n, p_0)$ exists. If $\{p_n\}_n$ has an accumulation point a_0, i.e. $\exists$ subsequence $\{p_{n_k}\}$ s.t. $p_{i_k} \to a_0$ as $k \to \infty$.

Pick $p_0 = a_0$, we have

$$\lim_{n\to\infty} d(p_n, p_0) = \lim_{k\to\infty} d(p_{n_k}, p_0) = 0.$$

That is, $p_n \to p_0$ as $n \to \infty$.

Otherwise, if $\{p_n\}$ has no accumulate point, then $\{p_n\}$ is closed. Note $\{p_n\}$ is bounded since it is cauthy.

By assumeption (ii), $\{p_n\}$ is conpact. But each p_n is not an accumulate point, we have $p_n \in U_n,\ p_i \notin U_n,\ \forall i \neq n$. Hence $\{U_n\}$ is an open cover of $\{p_n\}$ without any finite subcover. This contradicts to the conpactness of $\{p_n\}$. ■

(i)⇒(iv): Let γ be a geodesic in M, parametrized by arc length, and being defined on a maximal interval I. Then I is not empty. Moreover, by the "local existence and uniqueness of geodesics"(ODE theory), we know I is an open interval. Next we show I is closed. Then I has to be $(-\infty. + \infty)$.

Let $\{t_n\}_{n\in\mathbb{N}} \subset I$ be converging to t. Notice that

$$d(\gamma(t_n), \gamma(t_m)) \leq |t_n - t_m| = length\ of\ \widehat{\gamma(t_n)\gamma(t_m)}.$$

We know $\{\gamma(t_n)\}_{n\in\mathbb{N}}$ is a cauthy sequence in M.

M is compact⇒ $\exists\ p_0 \in M,\ \gamma(t_n) \to p_0$ as $n \to \infty$. And, $\exists\ \delta > 0, \exists\ W$ which is a totally normal neighborhood of p_0, s.t. $\forall q \in W,\ exp_q(B(0, \delta)) \supset W$.

There exists N, s.t. when $n, m \geq N$, we have

$$|t_n - t_m| < \delta \tag{a}$$

$$and\ \ \gamma(t_n), \gamma(t_m) \in W \tag{b}$$

By (a) and the property of W, there exist a unique geodesic c from $\gamma(t_n)$ to $\gamma(t_m)$ less than δ. Therefore c has to be a subarc of γ. Since $exp_{\gamma(t_n)}$ is a diffeomorphic on $B(0, \delta) \subset T_{\gamma(t_n)}M$ and $exp_{\gamma(t_n)}(B(0, \delta)) \supset W$, c extends γ to p_0. ■ □

Corollary 2.6. *Compact Riemannian manifold is complete.*

Proof. closed subset of a compact space is compact. ■ □

Corollary 2.7. *A closed submanifold of a complete Riemannian manifold is complete in the induced metric. In particular, the closed submanifold of Euclidean space are complete.*

2.4 Existence of geodesics in given homotopy class

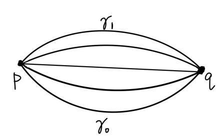
In complete Riemannian manifold, any two points p, q can be connected by a shortest geodesic. In this part, we discuss the existence of such geodesics in given homotopy class.

Definition 2.5. *Two curves γ_0, γ_1 on a manifold M with commen initial and end points p and q, i.e. two continuous maps*

$$\gamma_0, \gamma_1 : I = [0, 1] \to M$$

with $\gamma_0(0) = \gamma_1(0) = p$, $\gamma_0(1) = \gamma_(1) = q$, are called homotopic if there exists a continuous map $\Gamma : I \times I \to M$ with

$$\Gamma(0, s) = p, \quad \Gamma(1, s) = q \qquad \forall s \in I$$
$$\Gamma(t, 0 = \gamma_0(t), \Gamma(t, 1) = \gamma_1(t) \;\; \forall t \in I.$$



Two closed curves c_0, c_1 in M, i.e. two continuous maps $c_0, c_1 : S^1 \to M$ are called homotopic if there exists a continuous map $c : S^1 \times I \to M$ with $c(t, 0) = c_0(t)$, $c(t, 1) = c_1(t)$ for all $t \in S^1$.(S^1, as usual, is the unit circle parametrized by $[0, 2\pi)$)

Remark 2.12. *The concept of homotopy defines an equivalence relation on the set of all curves in M with fixed initial and each points as well as on the set of all closed curves in M.*

With the examples of torus in mind, let's first consider the existence of closed geodesic on a compact Riemannian manifold.

Theorem 2.8. *Let M be a compact Riemannian manifold. Then every homotopy class of closed curves in M contains a curve which is a shortest curve in its homotopy class and a geodesic.*

As a preparation, we first show

Lemma 2.4. *Let M be a compact Riemannian manifold. Let $\rho_0 > 0$ be the constants with the following property: any two points $p, q \in M$ with $d(p,q) \leq \rho_0$ can be connected by precisely one geodesic of shortest path. Let $\gamma_0, \gamma_1 : S^1 \to M$ be closed curves with*

$$d(\gamma_0(t), \gamma_1(t)) \leq \rho_0 \quad \forall\, t \in S^1.$$

Then γ_0 and γ_1 are homotopic.

Remark 2.13. *The existence of such ρ_0 on a compact Riemannian manifold has been proven in corollary 2.2.3. In fact, we know moreover, this geodesic depends continuously on (p,q).*

Proof of lemma 2.4.1:

$\forall t \in S^1$, let $c_t(s) : I \to M$ be the unique shortest curve(which is therefore, a geodesic) from $\gamma_0(t)$ to $\gamma_1(t)$, as usual parametrized proportionally to arc length. Recall that c_t depends continuously on its end points, hence on t, $\Gamma(t,s) = c_t(s)$ is continuously and yields the desired homotopy. ■

Proof of Theorem 2.4.1:

Consider the lengths of the curvesin a given homotopy class: they are numbers in $[0, +\infty)$. Let $\{\gamma_n\}_{n\in\mathbb{N}}$ be minimizing sequence for arc length in this homotopy class. All $\{\gamma_n\}_{n\in\mathbb{N}}$ are parametrized proportionally to arc length.

We may assume each γ_n is piecewise geodesic: for each γ_n, we may find $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = 2\pi$ with the property that

$$L(\gamma_n|_{[t_{j-1},t_j]}) < \frac{\rho_0}{2}, \; j = 1, \cdots, m+1.$$

Replacing $\gamma_n|_{[t_{j-1},t_j]}$ by the shortest geodesic arc between $\gamma_n(t_{j-1})$ and $\gamma_n(t_j)$, we obtain a new closed curve $\widetilde{\gamma_n}$.

By using triangle inequality, we have $d(\gamma_n(t), \widetilde{\gamma_n}(y)) \leq \rho_0$. As a consequence of Theorem 2.4.1, $\widetilde{\gamma_n}$ is homotopic to γ_n and the lengths of $\widetilde{\gamma_n}$ is no longer than γ_n.

We may thus assume that for any γ_n, $\exists$ points $p_{0,n}, \cdots, p_{m,n}$ for which $d(p_{j-1,n}, p_{j,n}) \leq \frac{\rho_0}{2}(p_{m+1,n} := p_{0,n}, \; j = 1, \cdots, m+1)$.

Observe that the lengths of γ_n are bounded as they constitute a minimizing sequence. Therefore, we may assume that m is independent of n.

$$\begin{array}{ccccc}
p_{0,1} & p_{1,1} & p_{2,1} & \cdots & p_{m,1} \\
p_{0,2} & p_{1,2} & p_{2,2} & \cdots & p_{m,2} \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
p_{0,n} & p_{1,n} & p_{2,n} & \cdots & p_{m,n} \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
\downarrow & \downarrow & \downarrow & & \downarrow \\
p_0 & p_1 & p_2 & & p_m
\end{array}$$

Recall that the geodesic between $p_{j-1,n}$ and $p_{j,n}$ depends continuously on its endpoints, and hence converges to the shortest arc between p_{j-1} and p_j.(shortest arc is geodesic)

These shortest geodesic arcs yields a closed curve γ. By Lemma 2.4.1, γ is homotopic to γ_n, and $Length(\gamma) = \lim\limits_{n\to\infty} Length(\gamma_n)$, Recall $\{\gamma_n\}$ is a minimizing sequence *for the length in their homotopy class. Therefore, γ is shortest curve in this homotopy class.*

Claim: γ is a geodesic.

Otherwise, there would exist points p and q on γ for which one of the two arcs of γ between p and q would have length at most $\frac{\rho_0}{2}$, but would not be geodesic.

Then this arc $\widehat{pq}$ can be shortened by replacing it by the shortest geodesic between p and q. Denote this new curve as $\widetilde{\gamma}$. We have

$$d(\gamma(t), \widetilde{\gamma}(t)) \leq \rho_0, \ \forall\, t \in S^1.$$

However, γ and $\widetilde{\gamma}$ is homotopic as a consequence of Lemma 2.4.1. This contradicts to the minimizing property of γ. Therefore, γ is desired closed geodesic. ∎

Remark 2.14. *If the compact Riemannian manifold M is simply-connected, the above arguement leads to the trivial closed geodesic: a point.*

Now, we discuss the existence of shortest geodesics in a given homotopic class of curves with fixed initial values and end points in a complete Riemannian manifold.

Theorem 2.9. *Let (M, g_M) be a complete and connected Riemannian manifold, $p, q \in M$. Every homotopy class of paths from p to q contains a geodesic γ that minimized length among all admissible curves in the same homotopy class.*

The idea to prove Theorem 2.4.2 is first going to the universal covering manifold of M. In $\widetilde{M}$, curves connecting corresponding points $\widetilde{p}$ and $\widetilde{q}$ only have one honotopy class. For that purpose, we need first show that M is complete $\Rightarrow \widetilde{M}$ is complete.

Recall: A covering map is a surjective continuous map $\pi : \widetilde{M} \to M$ between connected and locally-path-connected topological spaces, for which each points of M has connected neighborhood U that is evenly covered, meaning that each connected component of $\pi^{-1}(U)$ is mapped homeomorphically onto U by π.

It is called a smooth covering map if $\widetilde{M}$ and M are smooth manifolds and each component $\pi^{-1}(U)$ is mapped diffeomorphically onto U.

Any Riemannian metric on M induceds a Riemannian metric on $\widetilde{M}$. This makes π into a Riemannian covering. In particular, π is a local isometry.

Lemma 2.5. *Suppose $\widetilde{M}$ and M are connected Riemannian manifolds, and $\pi : \widetilde{M} \to M$ is a Riemannian covering map. If M is complete, then $\widetilde{M}$ is also complete.*

Proof. Let $\widetilde{p} \in \widetilde{M}$ and $\widetilde{v} \in T_{\widetilde{p}}\widetilde{M}$ be arbitrary, and let $p = \pi(\widetilde{p})$, and $v = d\pi(\widetilde{p})(\widetilde{v})$.

Completeness of M implies that the geodesic γ with $\gamma(0) = p$ and $\dot{\gamma} = v$ is defined for all $t \in \mathbb{R}$.(Recall a fundamental property of covering map is the path-lifting property: If $\pi : \widetilde{M} \to M$ is covering map, then every continuous map $\gamma : I \to M$ lifts to a path $\widetilde{\gamma}$ in $\widetilde{M}$ s.t. $\pi \circ \widetilde{\gamma} = \gamma$.)

Here the lifts $\widetilde{\gamma}$ of γ starting at $\widetilde{p}$ with the initial tangent vector $\widetilde{v}$. Since π is a local isometry, we know $\widetilde{\gamma}$ is a geodesic. Since γ is defined for all $t \in \mathbb{R}$, so does $\widetilde{\gamma}$. This proves the completeness of $\widetilde{M}$. ∎

Prood of Theorem 2.4.2: Consider the universal convering $\pi : \widetilde{M} \to M$ cof M, endowed with the induced metric $\widetilde{g} = \pi^* g$. Given $p, q \in M$ and a path $\sigma : [0, 1] \to M$

from p to q. Choose a $\widetilde{p} \in \pi^{-1}(p)$, and let $\widetilde{\sigma} : [0,1] \to \widetilde{M}$ be the lift of a starting with $\widetilde{p}$, and set $\widetilde{q} = \widetilde{\sigma}(1)$.

By Hopf-Rinow, and the fact $\widetilde{M}$ is complete, there exists a minimizing $\widetilde{g}$-geodesic $\widetilde{\gamma}$ from $\widetilde{p}$ to $\widetilde{q}$.

Because π is a local isometry, $\gamma = \pi \circ \widetilde{\gamma}$ is a geodesic from p to q in M.

Since $\widetilde{M}$ is simply connected, we have $\widetilde{\sigma}$ and $\widetilde{\gamma}$ are homotopic. Hence σ and γ are also homotopic. That is γ is a geodesic in the homotopy class $[\sigma]$. If σ_1 is any other admissible curve from p to q in the homotopy class $[\sigma]$, then by the monodromy theorem, its lifts $\widetilde{\gamma_1}$ starting at $\widetilde{p}$ also ends at $\widetilde{q_1}$,($\widetilde{\gamma_1}$ and $\widetilde{\sigma}$ are homotopic, which is trivial in a simply connected space.) In $\widetilde{M}$, we know $Length(\widetilde{\gamma}) \leq Length(\widetilde{\gamma_1})$. Therefore, $Length(\gamma) \leq Length(\gamma_1)$ ■ □

We'd like take this chance to discuss further about Riemannian covering map.

Theorem 2.10. *Let $(\widetilde{M}, \widetilde{g})$ and (M, g) are connected Riemannian manifolds with $\widetilde{M}$ complete, and $\pi : \widetilde{M} \to M$ is a local isometry. Then M is complete and π is a Riemannian covering map.*

Corollary 2.8. *Suppose $\widetilde{M}$ and M are connected Riemannian manifolds and $\pi : \widetilde{M} \to M$ is a Riemannian covering map. Then M is complete iff $\widetilde{M}$ is complete.*

Proof. Combination of Theorem 2.4.3 and Lemma 2.4.2.

Proof of Theorem 2.4.3:

. Path-lifting property for geodesic of a local isometry π.

Let $p \in \pi(\widetilde{M})$, and $\widetilde{p} \in \pi^{-1}(p)$. Let $\gamma : I \to M$ be a geodesic with $p = \gamma(0), v = \dot{\gamma}(0)$. Let $\widetilde{v} := (d\pi(\widetilde{p}))^{-1}(v) \in T_{\widetilde{p}}\widetilde{M}$.($\pi$ induces $d\pi(\widetilde{p})$ is a linear isometry.) Let $\widetilde{\gamma}$ be the geodesic in $\widetilde{M}$ with initial point $\widetilde{p}$ and initial tangent vector $\widetilde{v}$. Since $\widetilde{M}$ is complete, $\pi \circ \widetilde{\gamma}$ is a geodesic with initial point p and initial tangent vector v. Hence, $\pi \circ \widetilde{\gamma} = \gamma$ *on* I. So $\widetilde{\gamma}|_I$ is a lift of γ starting at $\widetilde{p}$.

. M is complete: Let $p \in \pi(M)$, $\gamma : I \to M$ be any geodesic starting at p, then γ has a lift $\widetilde{\gamma} : I \to \widetilde{M}$. Since $\widetilde{M}$ is complete, $\pi \circ \widetilde{\gamma}$ is a geodesic defined on all of $\mathbb{R}$ and coincides with γ on I. That is γ extends to all of $\mathbb{R}$. Thus M is complete by Hopf-Rinow Theorem.

. π is suijective.

$\forall \widetilde{p} \in \widetilde{M}$, write $p = \pi(\widetilde{p})$. Let $q \in M$ be arbitrary. M is complete$\overset{\Rightarrow}{H-R}$ $\exists$ a minimizing geodesic from p to q.

Let $\widetilde{\gamma}$ be the lift of γ starting at $\widetilde{p}$, and $r = d(p,q)$, we have $\pi(\widetilde{\gamma}(r)) = \gamma(r) = q$. So $q \in \pi(\widetilde{M})$.

. Every point of M has a neighborhood U that is evenly covered.

Let $p \in M$, let $U = B_\epsilon(p)$ be a geodesic ball(normal ball) centered at p, $\epsilon < inj(p)$.

Write $\pi^{-1}(p) = \{\widetilde{p_\alpha}\}_{\alpha \in A}$. For each α write $\widetilde{U_\alpha}$ be the metric ball of radius ϵ around $\widetilde{p_\alpha}$.

Claim: $\widetilde{U_\alpha} \bigcap \widetilde{U_\beta} = \emptyset$, $\forall \alpha \neq \beta$.

Proof: $\forall \alpha \neq \beta$, there exists a minimizing geodesic $\widetilde{\gamma}$ from $\widetilde{p_\alpha}$ to $\widetilde{p_\beta}$ because $\widetilde{M}$ is complete. The projective curve $\gamma := \pi \circ \widetilde{\gamma}$ is a geodesic that starts and end at p, whose length is the same as that of $\widetilde{\gamma}$. Such a geodesic must leave U and reenter it.

Since all geodesics passing through p are radial geodesics, we have $Length(\widetilde{\gamma}) = Length(\gamma) \geq 2\epsilon \Rightarrow d_{\widetilde{g}}(\widetilde{p_\alpha}, \widetilde{p_\beta}) \geq 2\epsilon \Rightarrow U_\alpha \bigcap U_\beta = \emptyset$ ■

Claim:$\pi^{-1}(U) = \bigcup\limits_\alpha \widetilde{U_\alpha}$.

Proof: $\forall \widetilde{q} \in \widetilde{U_\alpha}$ for some α, there is a geodesic $\widetilde{\gamma}$ of length¡ϵ from $\widetilde{p_alpha}$ to $\widetilde{q}$. Then $\pi \circ \widetilde{\gamma}$ is a geodesic of the same length from p to $\pi(\widetilde{q})$, showing that $\pi(\widetilde{q}) \in U = B_\epsilon(p)$. *i.e.* $\bigcup\limits_\alpha \widetilde{U_\alpha} \subseteq \pi^{-1}(U)$. $\forall \widetilde{q} \in \pi^{-1}(U)$, we get $q = \pi(\widetilde{q})$. That is, $q \in U$. So that is a minimizing radial geodesic. γ in U from p to q, and $r = d_g(p, q) < \epsilon$. Let $\widetilde{\gamma}$ be the lift of γ starting at $\widetilde{q}$.

It follows that $\pi(\widetilde{\gamma}(r)) = \gamma(r) = p$. Therefore $\widetilde{\gamma}(r) = \widetilde{p_\alpha}$ for some α, and $d_{\widetilde{g}}(\widetilde{q}, \widetilde{p_\alpha}) \leq Length(\widetilde{\gamma}) = r < \epsilon$. So $\widetilde{q} \in \widetilde{U_\alpha}$.

. It remains to show that $\pi : \widetilde{U_\alpha} \to U$ is a diffeomorphism for each α.

It is certaining a local diffeomorphism. It is bijective: we can construct the inverse explicitly. It sends each radial geodesic starting at p to its lift starting at $\widetilde{p_\alpha}$.

This completes the proof. ■ □

Chapter 3

Connections, Parallelism, and covariant Derivatives.

Consider the geodesic equation again. In (U, x),

$$\ddot{x}^i(t) + \Gamma^i_{jk}(x(t))\dot{x}^j(t)\dot{x}^k(t) = 0, \ i = 1, \cdots, n. \tag{*}$$

Recall under coordinate change $(x^i) \to (y^\alpha)$, the christoffel symbols behave as

$$\Gamma^i_{jk} = \widetilde{Gamma}^\alpha_{\eta\gamma}(y(x))\frac{\partial y^\eta}{\partial x^j}\frac{\partial y^\gamma}{\partial x^k}\frac{\partial x^i}{\partial y^\alpha} + \frac{\partial^2 y^\alpha}{\partial x^j \partial x^k}\frac{\partial x^i}{\partial y^\alpha}.$$

Therefore Γ^i_{jk} is not coefficients of a tensor!! This fact suggest in particular that we should pay more attention to taking derivetive in local coordinates. It would be nice if we have "the derivetive of a tensor is again a tensor". This will be solved by so-called "covariant derivatives".

On the other hand, the LHS of (*) behaves under coordinate change $(x^i) \to (y^\alpha)$ as

$$(\ddot{x}^i(t) + \Gamma^i_{jk}(x)\dot{x}^j\dot{x}^k) = (\ddot{y}^\alpha + \widetilde{\Gamma}^\alpha_{\eta\gamma}(y(x))\dot{y}^\eta\dot{y}^\gamma)\frac{\partial x^i}{\partial y^\alpha}.$$

That is, it behaves like a (1,0)-tensor(i.e. vector field). Recall, in local coordinatesm if $X = X^i\frac{\partial}{\partial x^i} = Y^\alpha\frac{\partial}{\partial y^\alpha}$. Then $X^i = Y^\alpha\frac{x^i}{y^\alpha}$. This suggest that $\ddot{x}^i(t) + \Gamma^i_{jk}(x)\dot{x}^j\dot{x}^k$ is coefficients of a (1,0)-tensor. This leads to the concepts of connections, and parallelism.

3.1 Affine Connections.

Refenrence:[WSY Chap.1][do Carmo, 2.2]

On R^n, let v be a vector at $p \in \mathbb{R}^n$, $f \in C^\infty(U)$, $p \in U \subset \mathbb{R}^n$, we hava the following "directional derivative" of f at p along v:

$$D_v f = \lim_{t\to 0}\frac{f(p + tv) - f(p)}{t}.$$

Let X be a C^∞ vector field. In local coordinates, (U, X), we have

$$X = (X^1, \cdots, X^n),\ X^i \in C^\infty(U)$$

where $X = X^i \frac{\partial}{\partial x^i}$. Then the directional derivative of X along v is defined as $D_v X = (D_v X^1, \cdots, D_v X^n)$. That is $D_v X = \sum_i (D_v X^i) \frac{\partial}{\partial x^i}$.

It is direct to check the following properties:

(a) $D_{\alpha v} X = \alpha D_v X,\ \forall \alpha \in \mathbb{R}$

(b) $D_v(fX) = (D_v f)X + f D_v X,\ \forall f$

(c) $D_v(X_1 + X_2) = D_v X^1 + D_v X^2,\ \forall X^1, X^2$

(d) $D_{v_1+v_2} X = D_{v_1} X + D_{v_2} X,\ \forall v_1, v_2$.

In fact, we also have $D_v \frac{\partial}{\partial x^i} = 0$. But this property can not be extended to manifold case. In general, we can define the following concept:

Definition 3.1. *(Affine connection). An affine connection ∇ on a smooth manifold M is a map*

$$\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM).$$

($\Gamma(TM)$ is the set of all smooth vector fields on M.)

This map is denoted by $(X, Y) \xrightarrow{\nabla} \nabla_X Y$, and which satisfies the following properties:

(i) $\nabla_{fX+gY} Z = f \nabla_X Z + g \nabla_Y Z$ (linear over the C^∞ functions in the arguement X.)

(ii) $\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z$

(iii) $\nabla_X f Y = f \nabla_X Y + X(f) Y$

in which $X, Y, Z \in \Gamma(TM)$ anf f,g are any real-valued C^∞ functions on M. The vector field $\nabla_X Y$ is called the covariant derivative of Y along X(with respect to the connection ∇).

Digest:

① On $\mathbb{R}^n$, the "directional derivatives" provides an affine connection. For $X, Y \in \Gamma(T\mathbb{R}^n)$, define $(\nabla_X Y)(p) = D_{X(p)} Y,\ p \in \mathbb{R}$. Then one can check $\triangle$ satisfies (i)-(iii).

② Let $X, Y \in \Gamma(TM)$. In a local coordinates $(U, x^1, \cdots, x^n)$, X, Y can be considered as vector fields on $x(U) \subset \mathbb{R}^n$. In (U, x), we can define $\nabla_X Y$ as the directional derivetive $D_X Y$. A natural question is: can we obtain an affine connection by defining it as directional derivatives in every local coordinates?

The answer is No! Suppose we have two coordinates $(U, x^1, \cdots, x^n)$ and $(V, y^1, \cdots, y^n)$. When $U \bigcup V \neq \emptyset$, we have

$$\begin{aligned} D_X Y &= \sum_i (D_X f^i) \frac{\partial}{\partial x^i} \ \ \textit{where } Y = f^i \frac{\partial}{\partial x^i} \textit{ in } U \\ &= \sum_i (D_X g^i) \frac{\partial}{\partial y^i} \ \ \textit{where } Y = g^i \frac{\partial}{\partial y^i} \textit{ in } V \\ &= \sum_i (D_X g^i) \frac{\partial x^k}{\partial y^i} \frac{\partial}{\partial x^k}. \end{aligned}$$

Need

$$\begin{aligned}
D_X f^i &= D_X G^j \frac{\partial x^i}{\partial y^j} \\
&= D_X (f^k \frac{\partial y^j}{\partial x^k}) \frac{\partial x^i}{\partial y^j} \\
&= D_X f^k \delta^i_k + f^k D_X (\frac{\partial y^j}{\partial x^k}) \frac{\partial x^i}{\partial y^j} \\
&= D_X f^i + f^k [D_X(\delta^i_k) - \frac{\partial y^j}{\partial x^k} D_X \frac{\partial x^i}{\partial y^k}] \\
&= D_X f^i - f^k \frac{\partial y^i}{\partial x^k} \frac{\partial x^i}{\partial y^j} \\
&= D_X f^i - g^j D_X \frac{\partial x^i}{\partial y^j}
\end{aligned}$$

That is we need

$$g_j D_X \frac{\partial x^i}{\partial y^j} = 0, \ \forall\, i. \tag{*}$$

We can find examples that (*) does not hold.

③ Existence: Many "trivial" connections: Fix a coordinate neighborhood U, define a "local" connection ∇^U on U via directional derivatives on $\mathbb{R}^n$. This can be extended "trivially" to a connection on M.

Lemma 3.1. *The set of all affine connections on M form a convex set. Namely, if $\nabla^{(1)}, \cdots, \nabla^{(k)}$ are affine connection on M, and $f_1, \cdots, f_k \in C^\infty(M)$, s.t. $\sum_i f_i = 1$. Then $\sum_i f_i \nabla^{(i)}$ is also an affine connection on M.*

Proof. Properties (i),(ii) of an affine connection can be checked directly.

For (iii), we check for $X, Y \in \Gamma(TM)$, $f \in C^\infty(M)$.

$$\begin{aligned}
(\sum_i f_i \nabla^{(i)})_X(gY) &= \sum_i f^i(\nabla^{(i)}_X(gY)) \\
&= \sum_i f_i(X(g)Y + g\nabla^{(i)}_X Y) \\
&= (\sum_i f_i)X(g)Y + g(\sum_i f_i \nabla^{(i)})_X Y.
\end{aligned}$$

Here we need the property that $\sum_i f^i = 1$. ■ □

Exercise 3.1. *Find a nontrivial connection on M via "partition of unity".*

④ Locality: "$\nabla_X Y$ depends only on local information of X and Y"

Proposition 3.1. *For any open subset $U \subset M$, if*

$$X|_U = \widetilde{X}|_U \ \ and \ \ Y|_U = \widetilde{Y}|_U,$$

then $\nabla_X Y|_U = \nabla_{\widetilde{X}} \widetilde{Y}|_{\widetilde{U}}$.

Proof. We will show $\nabla_X Y|_U \overset{(1)}{=} \nabla_{\widetilde{X}} Y|_U \overset{(2)}{=} \nabla_{\widetilde{X}} \widetilde{Y}|_{\widetilde{U}}$.

For (1), it's enough to show $X|_U = 0 \Rightarrow \nabla_X Y|_U$ (a).

For (2), it's enough to show $Y|_U = 0 \Rightarrow \nabla_X Y|_U$ (b).

Proof of (a): $\forall p \in U$, $\exists$ open $V \subset U$ and a function $f \in C_0^\infty(U)$ s.t. $f = 1$ *on* V. We check $(1-f)X = X$ since $X|_U = 0$. Then $\nabla_X Y = \nabla_{(1-f)X} Y \overset{i}{=} (1-f)\nabla_X Y$. In particular, $\nabla_X Y(p) = (1 - f(p))\nabla_X Y = 0$. Therefore $\nabla_X Y|_U = 0$. □

Exercise 3.2. *Show that* $Y|_U = 0$ *implies* $\nabla_X Y|_U = 0$.

Proposition 3.2. *If* $X(p) = \widetilde{X}(p)$, *then* $\nabla_X Y(p) = \nabla_{\widetilde{X}} Y(p)$.

Proof. Again, it's enough to show $X(p) = 0 \Rightarrow \nabla_X Y(p) = 0$ $(*)$.

By proposition 3.1, we only need to show (*) for X supported in an coordinate neighborhood (U, x), with $x(p) = 0$(the origin of $\mathbb{R}^{\varkappa}$). Now we can write $X = X^I \frac{\partial}{\partial x^i}$ with $X^i(0) = 0$. By taylor's theorem, $\exists$ functions X_k^i s.t.

$$X^i(x^1, \cdots, c^n) = X^i(0) + x^k X_k^i = x^k X_k^i.$$

So $\nabla_X Y = \nabla_{x^k X_k^i \frac{\partial}{\partial x^i}} Y = x^k \nabla_{X_k^i \frac{\partial}{\partial x^i}} Y$. In particular at p,

$$\nabla_X Y(p) = x^k(p) \nabla_{X_k^i \frac{\partial}{\partial x^i}} Y(p) = 0.$$

■ □

Consequently, for $v \in T_p M$, and $Y \in \Gamma(TM)$, we can define $\nabla_v Y(p) := \nabla_X Y(p)$, where X is any vector field with $X(p) = v$.(This is like a "directional derivative"of Y at p along v.)

But, it is not true that $Y(p) = \widetilde{Y}(p) \Rightarrow \nabla_X Y(p) = \nabla_X \widetilde{Y}(p)$. It is not hard to construct counterexamples.

Proposition 3.3. *Let* $\gamma : (-\epsilon, \epsilon) \to M$ *be a smooth curve on* M, *with* $\gamma(0) = p$ *and* $\dot{\gamma}(0) = v$. *Suppose* $X, Y, \widetilde{Y}$ *are vector fields on* M *s.t.*

$$X(p) = v, \ \ Y(\gamma(t)) = \widetilde{Y}(\gamma(t)), \ \ -\epsilon < t\epsilon,$$

then $\nabla_X Y(p) = \nabla_X \widetilde{Y}(p)$.

Proof. It's enough to show $Y = 0$ along $\gamma \Rightarrow \nabla_v Y(p) = 0$.

Let $(U, x^1, \cdots, x^n)$, $p \in U$ be a coordinate neighborhood around p with $x(p) = 0$. Let $Y = f^i \frac{\partial}{\partial x^i}$. Then

$$\begin{aligned} \nabla_v Y(p) &= \nabla_v (f^i \frac{\partial}{\partial x^i})(p) = (v(f^i)\frac{\partial}{\partial x^i} + f^i \nabla_v \frac{\partial}{\partial x^i})(p) \\ &= \frac{d}{dt}|_{t=0} f^i \circ \gamma(t) \frac{\partial}{\partial x^i} + f^i(p) \nabla_v \frac{\partial}{\partial x^i} \end{aligned}$$

Since $f^i \circ \gamma(t) = 0$, $t \in (-\epsilon, \epsilon)$, we have $\nabla_v Y(p) = 0$. ■ □

3.2 Parallelism

What is going an geometrically? [Spivak II, chapter 6]

Consider a curve $c : [a, b] \to M$. By a vector field V along c, we mean

$$t \in [a, b] \mapsto V(t) \in T_{\gamma(t)}M.$$

In a coordinate neighborhood $(U, x^1, \cdots, x^n)$, we can write

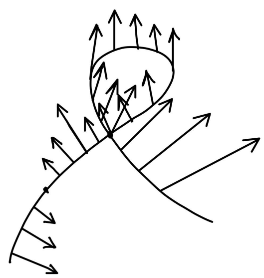
$$V(t) = \sum_{i=1}^{n} v_i(t) \frac{\partial}{\partial x^i}|_{c(t)}.$$

We call V a C^∞ vector field along c if the functions v^i are C^∞ on $[a, b]$. This is equivalent to saying that

$$t \mapsto V(t)f$$

is C^∞ for every C^∞ function f on M.

Notice that a vector field V along c may not be extended to a vector field on M.



When c is an embedding, $V(t)$ can be extended to a vector field $\widetilde{V}$ on M. We have

$$\widetilde{V}(c(t)) = V(t), \ \forall t \in [a, b].$$

Then $\nabla_{\frac{dc}{dt}} \widetilde{V}$ is a C^∞ vector field along c.

By locality, we know $\nabla_{\frac{dc}{dt}} \widetilde{V}$ does not depend on the extension $\widetilde{V}$. We call $\nabla_{\frac{dc}{dt}} \widetilde{V}$ the covariant derivative of V along c, we denote it by the convenient symbolism $\frac{DV}{dt}$.

We would like to generalize this covariant derivative along c to any curve c.(This is actually the concept of "induced connections" for which we will discuss later.)

Proposition 3.4. *Let M be a differential manifold with an affine connection ∇. There exists a unique correspondence from C^∞ vector fields V along the smooth curve $c : [a, b] \to M$ to C^∞ vector fields along $c : V \to \frac{DV}{dt}$, called the covariant derivative of V along c, such that*

(a) $\frac{D}{dt}(V + W) = \frac{DV}{dt} + \frac{DW}{dt}$.

(b) $\frac{D}{dt}(fV) = \frac{df}{dt}V + f\frac{DV}{dt}$, *for* $F \in C^\infty([a, b])$.

(c) If $V(s) = Y(c(s))$ *for some* C^∞ *vector field* Y *defined in a neighborhood of* $c(t)$, *then* $\frac{DV}{dt} = \nabla_{\frac{dc}{dt}} Y$.

Proof. Let us suppose initially that there exists a correspondence satisfying (a),(b) and (c).

Let $p = c(t_0) \in M$, and $(U, x^1, \cdots, x^n)$ is a coordinate neighborhood of p. For t sufficiently to t_0, we can express V locally as $V(t) = \sum_{j=1}^{n} v^j(t) \frac{\partial}{\partial x^j}|_{c(t)}$. By (a),(b),(c), we have

$$\begin{aligned}
\frac{DV}{dt} &\overset{(a)}{=} \sum_{j=1}^{n} \frac{D}{dt}(v^j(t)\frac{\partial}{\partial x^j}|_{c(t)}) \\
&\overset{(b)}{==} \sum_{j=1}^{n} [\frac{dv^j(t)}{dt}\frac{\partial}{\partial x^j}|_{c(t)} + v^j(t)\frac{D}{dt}(\frac{\partial}{\partial x^j}|_{c(t)})] \\
&\overset{(c)}{=} \sum_{j=1}^{n} [\frac{dv^j(t)}{dt}\frac{\partial}{\partial x^j}|_{c(t)} + v^j(t)\nabla_{\frac{dc}{dt}}\frac{\partial}{\partial x^j}] \quad (\frac{dc}{dt} = dc(\frac{\partial}{\partial t}) = \frac{dc^i}{dt}\frac{\partial}{\partial x^i}|_{c(t)}) \\
&= \sum_{j=1}^{n} [\frac{dv^j(t)}{dt}\frac{\partial}{\partial x^j}|_{c(t)} + v^j(t)\frac{dc^i}{dt}\nabla_{\frac{\partial}{\partial x^i}|_{c(t)}}\frac{\partial}{\partial x^j}]
\end{aligned}$$

Note $\nabla_{\frac{\partial}{\partial x^i}|_{c(t)}}\frac{\partial}{\partial x^j}$ is a C^∞ vector field along c. Hence $\exists \left\{ {k \atop ij} \right\}$ s.t. $\nabla_{\frac{\partial}{\partial x^i}|_{c(t)}}\frac{\partial}{\partial x^j} = \left\{ {k \atop ij} \right\}(c(t))\frac{\partial}{\partial x^k}|_{c(t)}$
$\Rightarrow \frac{DV}{dt} = \sum_{k=1}^{n}(\frac{dv^k}{dt} + \sum_{i,j}\left\{ {k \atop ij} \right\}(c(t))\frac{dc^i}{dt}v^j(t))\frac{\partial}{\partial x^k}|_{c(t)} \quad (*)$

The expression (*) show us that if there is a correspondence satisfying (a),(b),(c), then such a correspondence is unique.

To show existence, define $\frac{DV}{dt}$ in (U, x) by (*). We can verify that (*) possesses the desired properties. If (V, y) is another coordinate neighborhood with $U \bigcap V \neq \emptyset$, then we define $\frac{DV}{dt}$ in (V, y) by (*), the definition agree in $U \bigcap V$ by the uniqueness of $\frac{DV}{dt}$ in U. Therefore, the definition can be extended over all of M. ■ □

Remark 3.1. *Even at points where $\frac{dc}{dt} = 0$, $\frac{DV}{dt}$ is not necessarily 0!! If c is a constant curve, $c(t) = p \in M$, $\forall t$. Then a vector field V along c is just a curve in T_pM, and $\frac{DV}{dt}$ is just the ordinary derivative of this curve.*

Definition 3.2. *(Parallelism) Let M be a differentiable manifold with an affine connection ∇. A vector field V along a curve $c : [a, b] \to M$ is called parallelism when $\frac{DV}{dt} = 0$, $\forall t \in [a, b]$. When $M = \mathbb{R}^n$, ∇ be the directional derivative, we obtain the standard picture of a parallel vector field.*

Proposition 3.5. *[do Carmo, Prop 2.6] Let M be a differentiable manifold with an affine connection ∇. Let $c : I \to M$ be a smooth curve in M, and let $V_0 \in T_{c(t_0)}M$, $t_0 \in I$. Then there exists a unique parallel vector field V along c, such that $V(t_0) = V_0$.*

Remark 3.2. *$V(t)$ is called the parallel transport of $V(t_0)$ along c.*

Proof. First consider the case when $c(I)$ is certained in a coordinate neighborhood $(U, x^1, \cdots, x^n)$. Then V_0 can be expressed as:$V_0 = \sum_j v_0^j \frac{\partial}{\partial x_j}|_{c(t_0)}$.

Suppose there exists a vector field V in U which is parallel along c, with $V(t_0) = V_0$. Then $V = \sum v^j(t)\frac{\partial}{\partial x^j}|_{c(t)}$ satisfies

$$0 = \frac{DV}{dt} = \sum_k \{\frac{dv^k}{dt} + \sum_{i,j} \left\{ {k \atop ij} \right\} (c(t)) \frac{dc^i}{dt} v^j(t)\} \frac{\partial}{\partial x^k}|_{c(t)}.$$

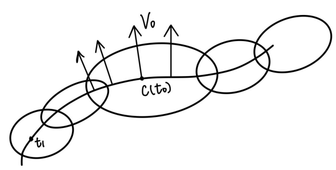
The equations

$$\frac{dv^k}{dt} + \sum_{i,j} \left\{ {k \atop ij} \right\} (c(t)) \frac{dc^i}{dt} v^j(t) = 0, \ k = 1, \cdots, n$$

are linear differential equations. So there is a unique solution satisfying the initial condition

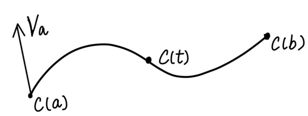
$$v^k(t_0) = v_0^k, \ \ k = 1, \cdots, n.$$

Due it's linearity, the solution is defined for all $t \in I$, this proves the existence and uniqueness of V in this case. In general, for any $t_1 \in I$, there is a finite cover of $c([t_0, t_1])$ by coordinate neighborhood.



In each of those coordinate neighborhood , V is defined. By uniqueness, the definitions coincide when the intersections are not empty, there allowing the definition of V along all of $[t_0, t_1]$ ■ □

Now consider $c : [a, b] \to M, \ V_a \in T_{c(0)}M$. Then there is a unique $V(t) \in T_{c(t)}M$ s.t. V_t is the parallel transport of V_a along c.



It's clear from the definition that

$$(V + W)_t = V_t + W_t, \ (\lambda V)_t = \lambda V_t.$$

That is, we have a linear transformation $P_{c,a,t} = P_t : T_{c(a)}M \to T_{c(t)}M, V_a \mapsto V_t$. Moreover, P_t is one-to-one. Its inverse is given by the parallel transport along the reversed portion of c from t to a.

$$\varphi : [a, b] \to M, t \mapsto c(a + b - t).$$

$$\varphi(a) = c(b), \varphi(b) = c(a). \ \frac{d\varphi}{dt}(t) = -\frac{dc}{dt}(a + b - t).$$

Therefore, $V_b \in T_{c(b)}M$ is the parallel transportation of $V_a \in T_{c(a)}M$ along c iff V_a is the parallel transportation of V_b along φ.(When c is embedding, this is seen from $\nabla_{\frac{d\varphi}{dt}(t)}\widetilde{V} = \nabla_{-\frac{dc}{dt}(a+b-t)}\widetilde{V} = -\nabla_{\frac{dc}{dt}(a+b-t)}\widetilde{V}$.)

Hence P_t is an isomorphism between two vector space $T_{c(0)}M$ and $T_{c(t)}M$.

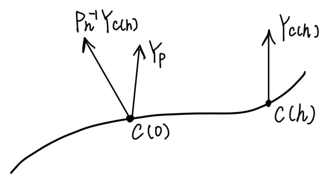
Remark 3.3. *(justification of the term "connection") A connection ∇ gives the possibility of comparing, or "connecting", tangent spaces at different points.*

Note the isomorphism between two tangent spaces given by the parallel transport depends on the choice of curves connecting the two points.

The parallel transport P_t is defined in terms of ∇, but we can also reverse the process.

Proposition 3.6. *[spivak II, Chapter 6.Prop 3] Let c be a curve with $c(0) = p$ and $\dot{c}(0) = X_p$. Let $Y \in \Gamma(TM)$, then*

$$\nabla_{X_p} Y = \lim_{h\to 0} \frac{1}{h}(P_h^{-1} Y_{c(h)} - Y_p)$$



Remark 3.4. *Parallel transport enables us to use the idea of "directional derivative" to define $\nabla_{X_p} Y$.*

Proof. Let $V_1, c \ldots, V_n$ be parallel vector fields along c which are linearly independent at $c(0)$, and(since parallel transports are isomorphisms), hence at all points of c. Set $Y(c(t)) = \sum_{i=1}^{n} f^i(t) V_i(t)$. Then

$$\begin{aligned}
&\lim_{h\to 0} \frac{1}{h}(P_h^{-1} Y_{c(h)} - Y_p)\\
=&\lim_{h\to 0} \frac{1}{h}(\sum_i f^i(h) P_h^{-1} V_i(h) - \sum_i f^i(0) V_i(0))\\
=&\lim_{h\to 0} \frac{1}{h} \sum_i (f^i(t) V^i(0) - f^i(0) V_i(0))\\
=&\sum_i \lim_{h\to 0} (f^i(t) - f^i(0)) V_i(0) = \sum_i \frac{df^i}{dt}|_{t=0} V_i(0)\\
=&\frac{D}{dt}|_{t=0} \sum_i f^i(t) V_i(t)\\
=&\nabla_{X_p} Y.
\end{aligned}$$

■ □

Remark 3.5. *Recall (*) on page 73, and geodesic equation in last Chapter. If $\gamma : [a,b] \to M$ is a geodesic, then we have $\frac{D\dot{\gamma}(t)}{dt} = 0$ where $\frac{D}{dt}$ is determined by a connection ∇ on M, for which in (U, x), $\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma_{ij}^k \frac{\partial}{\partial x^k}$.*

3.3 Covariant derivatives of a tensor field

In this section, we extend the covariant derivative of a vector field Y along X to that of a tensor field along X. Similar as previous cases, we can do this via pure algebraic discussions, or via parallel transport.

For (0,0)-tensor(=functions), we have a nice derivative:

$$\nabla_X : C^\infty(M) \to C^\infty(M), f \mapsto c\nabla_X f = X(f) = df(X).$$

We can check that this derivative satisfies (i)-(iii) in Definition 3.1.1. The following property enables us to define the covariant derivative $\triangle_X A$ of (r,s)-tensor A(via an algebraic discussions).

In fact we can define a connection on (r,s)-tensor fields

$$\nabla : \Gamma(TM) \times \Gamma(\otimes^{r,s} TM) \to \Gamma(\otimes^{r,s} TM), (X, A) \mapsto \nabla_X A.$$

Proposition 3.7. *Let M be a diffentiable manifold with an affine connection ∇. There is a unique connection on all tensor fields $\nabla : \Gamma(TM) \times \Gamma(\otimes^{r,s} TM) \to \Gamma(\otimes^{r,s} TM)$ that satisfies*

(i) $\nabla_{fX+gY} A = f\nabla_X A + g\nabla_X A$.

(ii) $\nabla_X(A_1 + A_2) = \nabla_X A_1 + \nabla_X A_2$.

(iii) $\nabla_X(fA) = X(f)A + f\nabla_X A$.

and

(iv) ∇ coincide with the given connections on $\Gamma(TM)$ and $C^\infty(M)$.

(v) $\nabla_X(T_1 \otimes T_2) = (\nabla_X T_1) \otimes T_2 + T_1 \otimes (\nabla_X T_2)$

(vi) $C(\nabla_X T) = \nabla_X C(T)$, where $C : \Gamma(\otimes^{r,s} TM) \to \Gamma(\otimes^{r-1,s-1} TM)$ is the contraction map that pairs the first vector with the first covector.

Remark 3.6. *(i)-(iii) is the properties for a connection, (iv)-(vi) provides a unique extension to all tensor fields.*

Proof. First, we derive the formula of ∇ on 1-forms.

Let $\omega \in \Omega^1(M) = \Gamma(T^*M)$ be any 1-form, then

$$\begin{aligned} X(\omega(Y)) &\overset{(iv)}{=} \nabla_X(\omega(Y)) = \nabla_X(C(\omega \otimes Y)) \\ &\overset{(vi)}{=} C\nabla_X(\omega \otimes Y) \overset{(v)}{==} C(\nabla_X \omega \otimes Y + \omega \otimes \nabla_X Y) \\ &= (\nabla_X \omega)(Y) + \omega(\nabla_X Y) \end{aligned}$$

So we conclude

$$① \ (\nabla_X \omega)(Y) = X(\omega(Y)) - \omega(\nabla_X Y)$$

Next, we can use (v) iteratively to show that for any (r,s)-tensor field A,

$$② \ (\nabla_X A)(\omega_1, \cdots, \omega_r, Y_1, \cdots, Y_s)$$
$$= X(A(\omega_1, \cdots, \omega_r, Y_1, \cdots, Y_s)) - \sum_i A(\omega_1, \cdots, \nabla_X \omega_i, \cdots, \omega_r, Y_1, \cdots, Y_s) - \sum_j A(\omega_1, \cdots, \omega_r, Y_1, \cdots, \nabla_X Y_j, \cdots, Y_s)$$

This shows the uniqueness.

For the existence, one need to check that the connections defined by ① and ② satisfies all conditions (i)-(vi). ■ □

Remark 3.7. *$\nabla_X A$ is called the covariant derivative of the (r,s)-tensor fields A along X.*

The properties (iv)-(vi) are very natural. To elaborate this point, we briefly discuss another way of defining $\nabla_X A$, via parallel transport.

Recall for an isomorphism $\varphi : V \to W$ between two vector spaces $V$and W, there is an induced isomorphism $\varphi^* : W^* \to V^*$ between their dual spaces W^*, V^* defined by

$$for\ \alpha \in W^* :\ \varphi^*(\alpha)(v) := \alpha(\varphi(v)), \forall v \in V.$$

Then for any $v_i \in V, \alpha^j \in V^*$, define

$$\begin{aligned}&\widetilde{\varphi}(v_1 \otimes \cdots \otimes v_r \otimes \alpha^1 \otimes \cdots \otimes \alpha^s)\\ :=&\varphi(v_1) \otimes \cdots \otimes \varphi(v_r) \otimes (\varphi^*)^{-1}(\alpha^1) \otimes \cdots \otimes (\varphi^*)^{-1}(\alpha^s)\end{aligned}$$

Using linearity, we can extend $\widetilde{\varphi}$ to $\otimes^{r,s}V$ all (r,s)-tensor over V! This defines an isomorphism between $\otimes^{r,s}V \to \otimes^{r,s}W$.

Recall the parallel transport along c. $P_{c,t} : T_{c(0)}M \to T_{c(t)}M$ is an isomorphism. We can extend it to be an isomorphism $\widetilde{P}_{c,t} : \otimes^{r,s}T_{c(0)}M \to \otimes^{r,s}T_{c(t)}M$. As in proposition 3.4, we define

$$\nabla_{X_p} A := \lim_{h\to 0} \frac{1}{h}(\widetilde{P}^{-1}_{c,h}A_{c(h)} - A_p) \qquad (**)$$

where c is a curve with $c(0) = p, \dot{c}(0) = X_p$.

Clearly if $A \in \Gamma(\otimes^{r,s}TM)$, then $\nabla_{X_p} \in \Gamma(\otimes^{r,s}TM)$. We also check that $\nabla_{X_p}A$ given in (**) satisfies prop.(iv)-(vi).

Exercise 3.3. *Let $Y \in \Gamma(TM)$, $\omega, \eta \in \Gamma(T^*M)$. Consider the tensor field $K = Y\otimes\omega\otimes\eta$. Let $X_p \in T_pM$, and $\nabla_{X_p}K$ be defined in (**).*

(i) Show $\nabla_{X_p}K = \nabla_{X_p}Y \otimes \omega \otimes \eta + Y \otimes \nabla_{X_p}\omega \otimes \eta + Y \otimes \omega \otimes \nabla_{X_p}\eta$.

(ii) Let $CK = \omega(Y)\eta$. Show $\nabla_{X_p}(CK) = C(\nabla_{X_p}K)$.

Remark 3.8. *The definition (**) leads to be dependent on X_p and the curve c. However, (**) does not depend on choice of c. Recall $\nabla_{X_p}Y$ depends only on X_p. We only need to show for any $\eta \in \Gamma(T^*M)$, $\nabla_{X_p}\eta$ also depends only on X_p. We need show $(\nabla_{X_p}\eta)(Y)$, $\forall Y \in \Gamma(TM)$, depends only on X_p, not on c.*

Consider $Y \otimes \eta$, we have

$$\begin{aligned}&\nabla_{X_p}(Y \otimes \eta) = (\nabla_{X_p}Y) \otimes \eta + Y \otimes \nabla_{X_p}\eta\\ \overset{exchange with contraction}{\Rightarrow}&X_p(\eta(Y)) = \eta(\nabla_{X_p}Y) + (\nabla_{X_p}\eta)Y\\ \Leftrightarrow&(\nabla_{X_p}\eta)(Y) = X_p(\eta(Y)) - \eta(\nabla_{X_p}Y)\end{aligned}$$

RHS only depends on X_p, not on c. ■

Now, for any tensor field A, and a field X, we can define $(\nabla_X A)(p) = \nabla_{X_p}A$, $\forall p \in M$.

3.4 Levi-Civita Riemannian Connections

There are too many connections on a given smooth manifold. Let $X, Y \in \Gamma(TM)$. In a coordinate neighborhood (U, x), write $X = X^i(x)\frac{\partial}{\partial x^i}$, $Y = Y^j(x)$
$frac\partial\partial x^j$. By definition we have

$$\nabla_X Y = \nabla_{X^i \frac{\partial}{\partial x^i}}(Y^j \frac{\partial}{\partial x^j}) = X^i \frac{\partial Y^j}{\partial x^i}\frac{\partial}{\partial x^j} + X^i Y^j \nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}.$$

Since $\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j} \in \Gamma(TM)$, there exists functions $\left\{ {k \atop ij} \right\}$ s.t. $\nabla_{\frac{\partial}{\partial x^i}|_{c(t)}}\frac{\partial}{\partial x^j} = \left\{ {k \atop ij} \right\}, k = 1, 2, \cdots, n$ s.t.

$$\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j} = \left\{ {k \atop ij} \right\}.$$

$$\Rightarrow \nabla_X Y = X(Y^j)\frac{\partial}{\partial x^j} + X^i Y^j \left\{ {k \atop ij} \right\}\frac{\partial}{\partial x^k} = (X(Y^k) + X^i Y^j \left\{ {k \atop ij} \right\})\frac{\partial}{\partial x^k}.$$

That is, the connection ∇ is determined by the n^s smooth functions $\left\{ {k \atop ij} \right\}$.

Let $c : [a, b] \to M$ be a curve such that the relocity vector field $\dot{c}(t)$(along c) is parallel. Then locally, we can write $c(t) = (x^1(t), \cdots, x^n(t))$ and

$$0 = \frac{D\dot{c}(t)}{dt} = \frac{d}{dt}\dot{x}^k(t)\frac{\partial}{\partial x^k}|_{c(t)} + \dot{x}^j(t)\frac{dx^i}{dt}\left\{ {k \atop ij} \right\}(x(t))\frac{\partial}{\partial x^k}|_{c(t)}$$
$$= (\ddot{x}^k(t) + \dot{x}^i(t)\dot{x}^j(t)\left\{ {k \atop ij} \right\}(c(t)))\frac{\partial}{\partial x^k}|_{c(t)}.$$

$\Rightarrow \ddot{x}^k(t) + \left\{ {k \atop ij} \right\}(c(t))\dot{x}^i(t)\dot{x}^j(t) = 0,\ k = 1, \cdots, n.$

Recall the geodesic equation of a Riemannian manifold (M, g) are

$$\ddot{x}^k(t) + \Gamma^k_{ij}(c(t))\dot{x}^i(t)\dot{x}^j(t),\ k = 1, \cdots, n.$$

We hope to find a connection, under which a geodesic is a curve whose velocity vector field is parallel along it. That is, we are looking for a connection ∇, s.t.

$$\left\{ {k \atop ij} \right\} = \frac{1}{2}g^{kl}(g_{lj,i} + g_{il,j} - g_{ij,l}).$$

From this aim, we see the connection has to be "compatible" with the Riemannian metric.

Recall that along a geodesic γ, we have $\langle\dot{\gamma}, \dot{gamma}\rangle_g \equiv const$. It is natural to require g, as a (0,2)-tensor, is parallel w.r.t ∇. i.e.

$$\nabla_X g(Y, Z) = X(g(Y, Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) = 0,\ \forall X, Y, Z \in \Gamma(TM).$$

Definition 3.3. *We say ∇ is compatible with g if the Riemannian metric g is paralell. In other words, ∇ is conpatible with g iff for all $X, Y, Z \in \Gamma(TM)$,*

$$X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z).$$

Let us calculate $\left\{ {k \atop ij} \right\}$ of such connections.

$$\begin{aligned} g_{ij,k} = \frac{\partial}{\partial x^k}(g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})) &= g(\nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) + g(\frac{\partial}{\partial x^k}, \nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^j}) \\ &= g(\left\{ {l \atop ki} \right\} \frac{\partial}{\partial x^l}, \frac{\partial}{\partial x^j}) + g(\frac{\partial}{\partial x^i}, \left\{ {l \atop kj} \right\} \frac{\partial}{\partial x^l}). \end{aligned}$$

That is

$$g_{ij,k} = g_{lj} \left\{ {l \atop ki} \right\} + g_{il} \left\{ {l \atop kj} \right\} \tag{1}$$

Permutation indius, we obtain

$$g_{ki,j} = g_{li} \left\{ {l \atop jk} \right\} + g_{kl} \left\{ {l \atop ji} \right\} \tag{2}$$

$$g_{jk,i} = g_{lk} \left\{ {l \atop jk} \right\} + g_{jl} \left\{ {l \atop ik} \right\} \tag{3}$$

(1),(2),(3) give us

$$g_{ij,k} + g_{ki,j} - g_{jk,i} = g_{jl}(\left\{ {l \atop ki} \right\} - \left\{ {l \atop ik} \right\}) + g_{kl}(\left\{ {l \atop ji} \right\} - \left\{ {l \atop ij} \right\}) + g_{il}(\left\{ {l \atop kj} \right\} + \left\{ {l \atop jk} \right\})$$

Now if we further have the symmetry

$$\left\{ {l \atop ki} \right\} = \left\{ {l \atop ik} \right\}, \ \forall i, l, k \tag{3.4.1}$$

then

$$\begin{aligned} g_{ij,k} + g_{ki,j} - g_{jk,i} &= 2g_{il} \left\{ {l \atop kj} \right\} \\ \Rightarrow \frac{1}{2} g^{pi}(g_{ij,k} + g_{ki,j} - g_{jk,i}) &= 2g^{pi} g_{il} \left\{ {l \atop kj} \right\} \\ \left\{ {p \atop kj} \right\} &= \frac{1}{2} g^{pi}(g_{ij,k} + g_{ki,j} - g_{jk,i} = \Gamma^p_{kj}.) \end{aligned}$$

Then we obtain the christoffel symbols!!(That is, under such connections, a geodesic is a curve whose velocity v.f. is parallel)

Express the condition (3.4.1) in global terms:

$$\nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^i} - \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^k}. \tag{3.4.2}$$

For $X, Y \in \Gamma(TM)$, only $\nabla_X Y - \nabla_Y X$ is not a tensor. The global expression of LHS of (3.4.2) is as follows. For $X, Y \in \Gamma(TM)$

$$T(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y].$$

Proposition 3.8. *T is a (1,2)-tensor.*

Proof. T gives the multilinear map

$$T : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM), (X, Y) \mapsto T(X, Y).$$

Moreover $T(fX, Y) = T(X, fY) = fT(X, Y)$.

Example 3.1.

$$\begin{aligned} T(fX, Y) &= \nabla_{fX}Y - \nabla_Y(fX) - [fX, Y] \\ &= f\nabla_X Y - Y(f)X - f\nabla_Y X - fXY + Y(f)X + fYX \\ &= fT(X, Y) \end{aligned}$$

Hence T is a tensor. It is a (1,2)-tensor in the sense. $T(\omega, X, Y) = \omega(T(X, Y))$. □

Definition 3.4. *(Torsion free) We call T the torsion tensor of ∇. If $T = 0$, we call ∇ torsion free(or symmetric) connection.*

So, our calculations tell us: A torsion free connection ∇ which is compatible with g has in each coordinate neighborhood.

$$\left\{ {l \atop jk} \right\} = \Gamma^i_{jk}.$$

Definition 3.5. *A connection ∇ on (M, g) is called a Levi-Civita connection(also called a Riemannian connection), if it is torsion free, and it is compatible with g.*

In this language, our previous calculations tell that if a Levi-Civita connection exists on (M, g), it is uniquely determined by the Christoffel symbols.

Conversly, we can define a connection ∇ as follows: in each coordinate neighborhood (U, x),

$$\nabla_X Y := \nabla_{X^i \frac{\partial}{\partial x^i}}(Y^j \frac{\partial}{\partial x^j}) := (X^i \frac{\partial Y^k}{\partial X^i} + X^i Y^j \Gamma^k_{ij})\frac{\partial}{\partial x^k}.$$

We can check this is well-defined, and ∇ is torsion free and is compatible with g. This shows the existence of a Levi-Civita connections on (M, g).

Actually, we prove the following important result.

Theorem 3.1. *(The fundamental theorem of Riemannian geometry) On any Riemannian manifold (M, g), there exists a unique Levi-Civita connection,*

Remark 3.9. *This is a remarkable point to note the following observation: On a smooth manifold, once we fix a Riemannian metric g, then we get:*

- *a canonical distance function*
- *a canonical measure*
- *a canonical affine connection*

We in fact already show a proof via local coordinate calculations for Theorem 3.4.1. We provide a coordinate free proof below.

Proof of Theorem 3.4.1:

Assume the Levi-Civita connection ∇ exist, then we calculate for all $X, Y, Z \in$

$\Gamma(TM)$,

$$\begin{aligned} g(\nabla_X Y, Z) :&= \langle \nabla_X Y, Z\rangle \overset{\nabla g=0}{=} X(\langle Y, Z\rangle) - \langle Y, \nabla_X Z\rangle \\ &\overset{torsionfree}{=} X(\langle Y, Z\rangle) - \langle Y, \nabla_Z X + [X, Z]\rangle \\ &= X(\langle Y, Z\rangle) - \langle Y, \nabla_Z X\rangle - \langle Y, [X, Z]\rangle \\ &\overset{\nabla g=0}{=} X(\langle Y, Z\rangle) - (Z\langle Y, Z\rangle - \langle \nabla_Z Y, X\rangle) - \langle Y, [X, Z]\rangle \\ &= X(\langle Y, Z\rangle) - Z(\langle Y, X\rangle) + \langle \nabla_Z Y, X\rangle - \langle Y, [X, Z]\rangle \\ &\overset{torsion-free}{=} X(\langle Y, Z\rangle) - Z(\langle Y, X\rangle) + \langle \nabla_Y Z, X\rangle + \langle [Z, Y], X\rangle - \langle Y, [X, Z]\rangle \\ &\overset{\nabla g=0}{=} X(\langle Y, Z\rangle) - Z(\langle Y, X\rangle) + Y(\langle Z, X\rangle) - \langle Z, \nabla_Y X\rangle + \langle [Z, Y], X\rangle - \langle Y, [X, Z]\rangle \\ &\overset{torsionfree}{=} X(\langle Y, Z\rangle) - Z(\langle Y, X\rangle) + Y(\langle Z, X\rangle) - \langle Z, \nabla_X Y\rangle - \langle Z, [Y, X]\rangle + \langle [Z, Y], X\rangle - \langle Y, [X, Z]\rangle. \end{aligned} \tag{3.4.3}$$

$$\Rightarrow 2\langle \nabla_X Y, Z\rangle = X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle - \langle rX, [Y, Z]\rangle + \langle Y, [Z, X]\rangle + \langle Z, [X, Y]\rangle \tag{3.4.4}$$

$\Rightarrow$

$$2\langle \nabla_X Y, Z\rangle = X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle - \langle X, [Y, Z]\rangle + \langle Y, [Z, X]\rangle + \langle Z, [X, Y]\rangle$$

The RHS is determined by the metric g. So the uniqueness is proved.

For existence, check the $\nabla_X Y$ defined by (3.4.3) satisfies all conditions of Levi-Civita connections. ∎

Remark 3.10. *The formula (3.4.3) is called the Koszul formula. In local coordinate (U, x), let X, Y, Z be $\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}$, we will derive the formula for Christoffel symbols Γ^i_{jk}.*

Sometimes, using (3.4.3) is more important than using Γ^k_{ij}. If in an open subset $U \subset M$, there exists an orthonormal frame field $E_1, \cdots, E_n$,(i.e. $\langle E_i, E_j\rangle(p) = \delta_{ij}, \forall\ p \in U$), (3.4.3) gives

$$2\langle \nabla_{E_i} E_j, E_k\rangle = -\langle E_i, [E_j, E_k]\rangle + \langle E_j, [E_k, E_i]\rangle + \langle E_k, [E_i, E_j]\rangle.$$

Exercise 3.4. *Suppose we know the following fact: There exists three vector field on $S^3 \subset \mathbb{R}^4$, i, j, k, which are linearly independent at any point of S^3, such that*

$$[i, j] = k,\ [j, k] = i,\ [k, i] = j.$$

Assign to S^3 a Riemannian metric g s.t. i, j, k are orthonormal at any point of S^3.

Calculate the Levi-Civita connection ∇ of g on S^3.

Next, we give more geometric interpretations for the two properties of the Levi-Civita connection.

(a) ∇ is compatible with the metric.

(b) ∇ is torsion free.

Proposition 3.9. *(geometric meaning of (a)) Let M be a smooth manifold with an affine connection ∇. Then ∇ is conpatible with the g iff any parallel transport is an isometry.*

Proof. $\Rightarrow$: Let $c : [a, b] \to M$ be a smooth curve with $p = c(a)$. The parallel transport

$$P_{c,a,t} : T_{c(a)}M \to T_{c(t)}M,\ t \in [a, b]$$

is an isomorphism.

$\Leftarrow$: i.e. any parallel transport is an isometry$\Rightarrow \nabla$ is compatible with g.

For any $X, Y, Z \in \Gamma(TM)$. Look at $X(p)$, $X(\langle Y, Z\rangle) = X(p)\langle Y, Z\rangle$. Let $c : [0, 1] \to M$ with $c(0) = p, \dot{c}(0) = X(p)$. We have $X\langle Y, Z\rangle = \frac{d}{dt}|_{t=0}\langle Y(c(t)), Z(c(t))\rangle$.

Let $\{E_1, \cdots, E_n\}$ is an orthonormal basis of T_pM, and $\{E_1(t), \cdots, E_n(t)\}$ is given by $E_i(t) = P_{c,t}E_i$. Since $P_{c,t}$ is isomotry, $\{E_i(t)\}$ is orthonormal($\forall t$).

$\Rightarrow \langle Y(c(t)), Z(c(t))\rangle = \langle Y^i(t)E_i(t), Z^i(t)E_i(t)\rangle = Y^i(t)Z^j(t)\delta_{ij} = \sum_i Y^i(t)Z^i(t)$.

$$\begin{aligned}\Rightarrow X\langle Y, Z\rangle = \sum_i \frac{d}{dt}|_{t=0}(Y^i(t)Z^i(t)) &= \sum_i \frac{dY^i}{dt}(0)Z^i(0) + \sum_i Y^i(0)\frac{dZ^i}{dt}(0)\\ =&\langle \frac{DY}{dt}|_{t=0}, Z\rangle + \langle Y, \frac{DZ}{dt}|_{t=0}\rangle\\ =&\langle \nabla_{X(p)}Y, Z\rangle + \langle Y, \nabla_{X(p)}Z\rangle\end{aligned}$$

This shows ∇ is compatible with g ■ □

Let $V_a, W_a \in T_{c(a)}M$, and $V_t := P_{c,a,t}V_a, W_t := P_{c,a,t}W_a$. Then V_t, W_t are two C^∞ vector fields along c.

If V_t, W_t can be extended to two C^∞ vector fields on M, we have

$$\nabla_{\frac{dc}{dt}}\langle V_t, W_t\rangle \overset{\text{metric compatibility}}{=} \langle \nabla_{\frac{dc}{dt}}V_t, W_t\rangle + \langle V_t, \nabla_{\frac{dc}{dt}}W_t\rangle \overset{\frac{dc}{dt}=0}{=} 0.$$

That is $P_{c,a,t}$ preserves the norms of vectors and angles between vectors. $\Rightarrow P_{c,a,t}$ is an isometry.

In general, we have to use the following property of <u>induced connection:</u>

$$\frac{d}{dt}\langle V_t, W_t\rangle = \langle \frac{DV_t}{dt}, W_t\rangle + \langle V_t, \frac{DW_t}{dt}\rangle. \tag{3.4.5}$$

Proof. In a coordinate neighborhood $(U, x^1, \cdots, x^n)$. $c(t) := (x^1(t), \cdots, x^n(t))$, $V_t := V^i(t)\frac{\partial}{\partial x^i}|_{c(t)}$, $W_t = W^i(t)\frac{\partial}{\partial x^i}|_{c(t)}$.

We calculate

$$
\begin{aligned}
&\frac{d}{dt}\langle V_t, W_t\rangle \\
&= \frac{d}{dt}(V^i(t)W^j(t)\langle \frac{\partial}{\partial x^i}|_{c(t)}, \frac{\partial}{\partial x^j}|_{c(t)}\rangle) \\
&= \frac{d}{dt}(V^i(t)W^j(t))\langle \frac{\partial}{\partial x^i}|_{c(t)}, \frac{\partial}{\partial x^j}|_{c(t)}\rangle + V^i(t)W^j(t)\frac{d}{dt}\langle \frac{\partial}{\partial x^i}|_{c(t)}, \frac{\partial}{\partial x^j}|_{c(t)}\rangle \\
&= (\dot{V}^i(t)W^j(t) + V^i(t)\dot{W}^j(t))\langle \frac{\partial}{\partial x^i}|_{c(t)}, \frac{\partial}{\partial x^j}|_{c(t)}\rangle + V^i(t)W^j(t)dc(\frac{d}{dt})\langle \frac{\partial}{\partial x^i}|_{c(t)}, \frac{\partial}{\partial x^j}|_{c(t)}\rangle \\
&= (\dot{V}^i(t)W^j(t) + V^i(t)\dot{W}^j(t))\langle \frac{\partial}{\partial x^i}|_{c(t)}, \frac{\partial}{\partial x^j}|_{c(t)}\rangle \\
&+ V^i(t)W^j(t)(\langle \nabla_{dc(\frac{d}{dt})}\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\rangle + \langle \frac{\partial}{\partial x^i}|_{c(t)}, \nabla_{dc(\frac{d}{dt})}\frac{\partial}{\partial x^j}|_{c(t)}\rangle) \\
&= \langle \dot{V}^i(t)\frac{\partial}{\partial x^i}|_{c(t)} + V^i(t)\nabla_{dc(\frac{d}{dt})}\frac{\partial}{\partial x^i}, W^j(t)\frac{\partial}{\partial x^j}|_{c(t)}\rangle \\
&+ \langle V^i(t)\frac{\partial}{\partial x^i}|_{c(t)}, \dot{W}^j(t)\frac{\partial}{\partial x^j}|_{c(t)} + W^j(t)\nabla_{dc(\frac{d}{dt})}\frac{\partial}{\partial x^j}\rangle \\
&= \langle \frac{DV_t}{dt}, W_t\rangle + \langle V_t, \frac{DW_t}{dt}\rangle.
\end{aligned}
$$

■

Remark 1: In fact (3.4.5) is a general property of the induced connection $\widetilde{\nabla}$ of ∇ compatible with g. Let $\varphi : N \to M$ be a $C^\infty map$, $u \in T_xN$, V, W are two smooth vector fielda along φ, then

$$\langle \widetilde{\nabla}_u V.W\rangle + \langle V, \widetilde{\nabla}_u W\rangle = u\langle V, W\rangle.$$

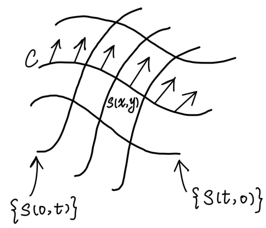
□

Proposition 3.10. *(geometric meaning of (b)) Let ∇ be a torsion-free connection of M. Let $s : \mathbb{R}^2 \to M$ be a C^∞ map.(a "parametrized surface" in M. Let $V(x,y) \in T_{s(x,y)}M$ be a vector field along s. For convinience, let us denote $ds(\frac{\partial}{\partial x}) := \frac{\partial s}{\partial x}, ds(\frac{\partial}{\partial y}) := \frac{\partial s}{\partial y}$. Then for the induced connection $\widetilde{\nabla}$,*

$$\widetilde{\nabla}_{\frac{\partial}{\partial x}} V(x,y) = (\frac{DV}{\partial x})_{(x,y)}$$

can be considered as the <u>*covariant derivative*</u> *along $c(t) := s(t,y)$ of the vector field $t \mapsto V(t,y)$ along c, evalued at $t = x$. Similarly, we have $\widetilde{\nabla}_{\frac{\partial}{\partial y}} V = \frac{DV}{\partial y}$. Then, we have*

$$\frac{D}{\partial x}\frac{\partial s}{\partial y} = \frac{D}{\partial y}\frac{\partial s}{\partial x} \tag{3.4.6}$$



Remark 3.11. *In symbols of induced connection, (3.4.5) can be written as* $\widetilde{\nabla}_{\frac{\partial}{\partial x}} ds(\frac{\partial}{\partial y}) = \widetilde{\nabla}_{\frac{\partial}{\partial y}} ds(\frac{\partial}{\partial x})$. *In case* $ds(\frac{\partial}{\partial x}), ds(\frac{\partial}{\partial y})$ *are both vector fields on* M, *(e.g. when* s *is an embedding), is equivalent to say*

$$\nabla_{ds(\frac{\partial}{\partial x})} ds(\frac{\partial}{\partial y}) = \nabla_{ds(\frac{\partial}{\partial y})} ds(\frac{\partial}{\partial x}).$$

This equivalent to the torsion free property since

$$[ds(\frac{\partial}{\partial x}), ds(\frac{\partial}{\partial y})] = ds([\frac{\partial}{\partial x}, \frac{\partial}{\partial y}]) = 0.$$

Proof. Express both sides in a coordinate neighborhood as $(U, x^1, \cdots, x^n)$, $s = (s^1, \cdots, s^n)$

$$\frac{\partial s}{\partial y} = ds(\frac{\partial}{\partial y})\frac{\partial s^i}{\partial y}\frac{\partial}{\partial s^i}, \quad \frac{\partial s}{\partial x} = ds(\frac{\partial}{\partial x}) = \frac{\partial s^i}{\partial x}\frac{\partial}{\partial s^i}.$$

$\Rightarrow$

$$\begin{aligned} \frac{D}{\partial x}\frac{\partial s}{\partial y} &= \frac{\partial^2 s^i}{\partial x \partial y}\frac{\partial}{\partial s^i} + \frac{\partial s^i}{\partial y}\nabla_{ds(\frac{\partial}{\partial x})}\frac{\partial}{\partial s^i} \\ &= \frac{\partial s^i}{\partial x \partial y}\frac{\partial}{\partial s^i} + \frac{\partial s^i}{\partial y}\frac{\partial s^j}{\partial x}\nabla_{\frac{\partial}{\partial s^j}}\frac{\partial}{\partial s^i} \end{aligned}$$

Similarly,

$$\frac{D}{\partial y}\frac{\partial s}{\partial x} = \frac{\partial^2 s^i}{\partial y \partial x} + \frac{\partial s^i}{\partial y}\frac{\partial s^j}{\partial x}\nabla_{\frac{\partial}{\partial s^i}}\frac{\partial}{\partial s^j}.$$

Then the proposition follows from the fact that

$$\frac{\partial^2 s^i}{\partial y \partial x} \text{ and } \nabla_{\frac{\partial}{\partial s^j}}\frac{\partial}{\partial s^i} - \nabla_{\frac{\partial}{\partial s^i}}\frac{\partial}{\partial s^j} = [\frac{\partial}{\partial s^i}, \frac{\partial}{\partial s^j}] = 0$$

■

Remark 2: (3.4.6) is also a general property of an induced connection $\widetilde{\nabla}$ of a torsion free connection ∇. Let $\varphi : N \to M$ be a C^∞ map, X, Y be two C^∞ vector fields on N. Then $d\varphi(X), d\varphi(Y)$ are C^∞ vector fields along φ, then $\widetilde{\nabla}_X d\varphi(Y) - \widetilde{\nabla}_Y d\varphi(X) = d\varphi([X, Y])$.

By Remark 1 and the above Remark 2, when doing calculations, we can assume the notation $\widetilde{\nabla}$ and proceed formally as if vector fields along φ were actually defined on M. □

Exercise 3.5. *(Variation of the energy functional: A coordinate free calculation).*
Let $\gamma : [a, b] \to M$ be a C^∞ curve, and $\alpha : (-\epsilon, \epsilon) \times [a, b] \to M$ be a variation. Recall

$$E(\gamma) := \frac{1}{2}\int_a^b \langle \dot\gamma(t), \dot\gamma(t)\rangle dt$$
$$= \frac{1}{2}\int_a^b \langle d\gamma(\frac{d}{dt}), d\gamma(\frac{d}{dt})\rangle dt.$$

Show that

$$\frac{dE(\alpha(s))}{ds}|_{s=0} = -\int_a^b \langle d\alpha(\frac{\partial}{\partial s})(0,t), \frac{D\dot\gamma(t)}{dt}\rangle dt - \langle \frac{\partial\alpha}{\partial s}(0,a), \frac{D\dot\gamma}{dt}(a)\rangle + \langle \frac{\partial\alpha}{\partial s}(0,b), \frac{D\dot\gamma}{dt}(b)\rangle$$

Hint: using Proposition 3.10.

Recall a calculation in coordinates has be carried out in our disscussions in Chapter 2, Geodesics.

Remark 3.12. *Recall that any Riemannian manifold M can be embedded to the standard Euclidean space E isometrically. In the Euclidean space the Levi-Civita Connections $\widetilde{\overline{\nabla}}$ is given by the directional derivatives. So for any $X, Y \in \Gamma(TM)$, X, Y can also be extended to vecter fields $\overline{X}, \overline{Y}$ on E(at least locally around M.) But usually $\overline{\nabla}_{\overline{X}}\overline{Y}(p) \in T_pE$ is not lie in T_pM any more, The orthogonal projection $\pi : T_pE \to T_pM$ gives $\pi(\overline{\nabla}_{\overline{X}}\overline{Y}(p)) \in T_pM$. One can check that $\pi(\overline{\nabla}_{\overline{X}}\overline{Y}(p))$ gives a Levi-Civita connection on M w.r.t. the induced metric from E.*

3.5 The First variation of Arc Length and Energy

In chapter 1, we derive the geodesic equations as the Euler-Langrange equations of the Length and Energy functionals .via local coordinates computations. Now, with the convenient notion of (Levi-Civita) Connections, we can carry out an easier computation(intrinstic).

Recall for any smooth curve $c : [a, b] \to M$, we have

$$L(c) : = \int_a^b \sqrt{\langle \dot c(t), \dot c(t)\rangle} dt$$
$$= \int_a^b \sqrt{\langle dc(\frac{\partial}{\partial t}).dc(\frac{\partial}{\partial t})\rangle} dt$$
$$E(c) = \frac{1}{2}\int_a^b \langle dc(\frac{\partial}{\partial t}), dc(\frac{\partial}{\partial t})\rangle dt.$$

Definition 3.6. *Let $c : [a, b] \to M$ be a smooth curve, $\forall\epsilon\rangle 0$. A variation of c is a map $F : [a, b] \times (-\epsilon, \epsilon) \to M$ with $F(t, 0) = c(t), \forall t \in [a, b]$. The variation is called proper if the endpoints stay fixed, i.e. $F(a, s) = c(a).F(b, s) = c(b), \forall s \in (-\epsilon, \epsilon)$.*

For simplicity, we will denote

$$\frac{\partial F}{\partial s} = dF(\frac{\partial}{\partial s}),\ \frac{\partial F}{\partial t} = dF(\frac{\partial}{\partial t}).$$

We also denote $c_s(t) = F(t, s)$.

Definition 3.7. *We call $V(t) = \frac{\partial F}{\partial s}(t, 0) = \frac{\partial F}{\partial s}(c(t))$ the variation field of f along c.(It is a vector field along c)*

Theorem 3.2. *(The First Variation Formula) Let $F(t, s)$ be a variation of a smooth curve c. Let us write $L(s) := L(c_s), E(s) := E(c_s)$ for simplicity. Then*

$$\frac{d}{ds}|_{s=0}L(s) := L'(0) = \int_a^b \frac{1}{|\dot{c}(t)|}(\frac{d}{dt}\langle V(t), \dot{c}(t)\rangle - \langle V(t), \nabla_{\frac{\partial}{\partial t}}\dot{c}(t)\rangle)dt$$

($\nabla_{\frac{\partial}{\partial t}}\dot{c}(t)\rangle$ is oftenly written as $\nabla_{dotc}\dot{c}$.)

$$\frac{d}{ds}|_{s=0}E(s) := E'(0) = \langle V(b), \dot{c}(b)\rangle - \langle V(a), \dot{c}(a)\rangle - \int_a^b -\langle V(t), \nabla_{\frac{\partial}{\partial t}}\dot{c}(t)\rangle dt.$$

Proof.

$$\begin{aligned}
\frac{d}{ds}L(s) &= \int_a^b \frac{d}{ds}\langle\frac{\partial F}{\partial t}(t, s), \frac{\partial F}{\partial t}(t, s)\rangle^{\frac{1}{2}}dt \\
&= \int_a^b \frac{1}{2\langle\frac{\partial F}{\partial t}(t, s), \frac{\partial F}{\partial t}(t, s)\rangle^{\frac{1}{2}}}\frac{d}{ds}\langle\frac{\partial F}{\partial t}(t, s), \frac{\partial F}{\partial t}(t, s)\rangle dt \\
&\overset{(3.4.5)}{=} \int_a^b \frac{1}{\langle\frac{\partial F}{\partial t}(t, s), \frac{\partial F}{\partial t}(t, s)\rangle^{\frac{1}{2}}}\langle\widetilde{\nabla}_{\frac{\partial}{\partial s}}\frac{\partial F}{\partial t}(t, s), \frac{\partial F}{\partial t}(t, s)\rangle dt \\
&\overset{(3.4.6)}{=} \int_a^b \frac{1}{\langle\frac{\partial F}{\partial t}(t, s), \frac{\partial F}{\partial t}(t, s)\rangle^{\frac{1}{2}}}\langle\widetilde{\nabla}_{\frac{\partial}{\partial t}}\frac{\partial F}{\partial s}(t, s), \frac{\partial F}{\partial t}(t, s)\rangle dt \\
&\overset{(3.4.5)}{=} \int_a^b \frac{1}{\langle\frac{\partial F}{\partial t}(t, s), \frac{\partial F}{\partial t}(t, s)\rangle^{\frac{1}{2}}}(\frac{d}{dt}\langle\frac{\partial F}{\partial t}(t, s), \frac{\partial F}{\partial t}(t, s)\rangle - \langle\frac{\partial F}{\partial t}(t, s), \widetilde{\nabla}_{\frac{\partial}{\partial t}}\frac{\partial F}{\partial t}(t, s)\rangle) \\
\Rightarrow \frac{d}{ds}|_{s=0}L(s) &= \int_a^b \frac{1}{|\dot{c}(t)|}(\frac{d}{dt}\langle V(t), \dot{c}(t)\rangle - \langle V(t), \widetilde{\nabla}_{\frac{\partial}{\partial t}}\dot{c}(t)\rangle)dt.
\end{aligned}$$

Similarly, we obtain the formula for $E'(0)$. ■ □

Observe that when c is parametrized proportionally to arc length i.e. $|\dot{c}(t)| \equiv const$, the variations of L and E leads to the same critical point.(We observed this fact using Holder inequality in Chapter 2.)

A smooth curve $c : [a, b] \to M$ is a critical point of the energy E for all proper variations iff $\widetilde{\nabla}_{\frac{\partial}{\partial t}}\dot{c}(t)\rangle = 0$.(i.e. c is a geodesic.)

Note, by property of parallel transport $\widetilde{\nabla}_{\frac{\partial}{\partial t}}\dot{c}(t)\rangle = 0 \Rightarrow |\dot{c}(t)| \equiv const \Rightarrow c$ is parametrized proportionally to arc length.

More generally, we consider piecewise smooth curve $c : [a, b] \to M$. That is, we have a subdivision $a = t_0 < t_1 < t_2 < \cdots < t_k < t_{k+1} = b$ s.t. c is smooth on each interval $[t_i, t_{i+1}]$.

Correspondingly, we consider "piecewise smooth variations" of c, which are continuous functions $F : [a, b] \times (-\epsilon, \epsilon) \to M$ such that F is smooth on each $[t_i, t_{i+1}] \times \epsilon \times \epsilon$, and $\frac{\partial F}{\partial s}$ is well defined even at $t_i' s$.

Then, as a direct consequence of Theorem 3.5.1, we have

Corollary 3.1. *Let c be a piecewise smooth curve and F be a correponding piecewise smooth variation. Then*

$E'(0)$

$$= \frac{d}{ds}|_{s=0} E(c_s) = \langle V(b), \dot{c}(b) \rangle - \langle V(a), \dot{c}(a) \rangle - \sum_{i=1}^{k} \langle V(t_i), \dot{\gamma}(t_i^+) - \dot{\gamma}(t_i^-) \rangle - \int_a^b \langle V(t), \nabla_{\frac{\partial}{\partial t}} \dot{c}(t) \rangle dt.$$

It turns out the first variation formulas are also very useful for non-proper variations. We discuss here Gauss' lemma. Recall in a normal neighborhood U_p of a point $p \in M$, we can introduce a polar coordinate such that $g = dr \otimes dr + g_{\varphi\varphi}(r, \varphi) d\varphi \otimes d\varphi$. Here the fact $g_{r,\varphi} \equiv 0$ on the whole U_p is also called Gauss' lemma.

Lemma 3.2. *(Gauss' Lemma). In U_p, the geodesics through p are perpendicular to the hypersurfaces $\{\exp_p(v) : \|v\| = const < \delta\}$.(Piecely, let $v \in T_pM$, $\rho(t) = tv$ is a ray through $0 \in T_pM$. Let $\omega \in T_{\rho(r)}(T_pM)$ is perpendicular to $\rho'(r)$. Then*

$$\langle (dexp_p)(\rho(r))(\omega), (dexp_p)(\rho(r))(\rho'(r)) \rangle \tag{3.5.1}$$

)

Proof. Let $v(s) : (-\epsilon, \epsilon) \to T_pM$ be a curve with $v(0) = rv = \rho(r), \dot{v}(0) = \omega$, and $\|v(s)\| = r$.

Then we have a variation $F(t, s) = exp_p(tv(s)), t \in [0, r], s \in (-\epsilon, \epsilon)$. with $F(t, 0) = exp_p(tv) = c(t)$.

Notice that $E(c_s) = \frac{1}{2} \int_0^r \langle v(s), v(s) \rangle dt = \frac{1}{2} r^3 \equiv const.$

Theorem 2$\Rightarrow$

$$0 = E'(0) = \langle V(r), \dot{c}(r) \rangle - \langle V(0), \dot{c}(0) \rangle - \int_0^r \langle V(t), \widetilde{\nabla}_{\frac{\partial}{\partial t}} \dot{c}(t) \rangle dt.$$

Since $\widetilde{\nabla}_{\frac{\partial}{\partial t}} \dot{c}(t) = 0$, and $V(0) = \frac{\partial F}{\partial s}|_{t=0,s=0} = 0$, we conclude

$$\langle V(r), \dot{c}(r) \rangle = 0. \tag{3.5.2}$$

Recall

$$V(r) = \frac{\partial F}{\partial s}|_{t=r,s=0} = \frac{\partial}{\partial s}|_{s=0} exp_p(rv(s)) = (dexp_p)(\rho(r))(\omega).$$
$$\dot{c}(r) = (dexp_p)(\rho(r))(\rho'(r)).$$

We see (3.4.2) implies (3.4.1). ■ □

3.6 Covariant differentiation, Hessian, and Laplacian

Recall the covariant derivative of a (r,s)-tensor A satisfies $\nabla_{fX+gY}A = f\nabla_X A + g\nabla_Y A$. That is $\nabla_X A$ is linear over C^∞ functions for the arguement X. Therefore we can define a (r,s+1)-tensor ∇A for each (r,s)-tensor A as

$$\nabla A(\omega^1, \cdots, \omega^r, X_1, \cdots, X_s, X) := \nabla_X A(\omega^1, \cdots, \omega^r, X_1, \cdots, X_s),\ \forall \omega^i \in \Gamma(T^*M),\ X^j, X \in \Gamma(TM).$$

We call A the covariant differention of A.

Portionally, consider a (0,0)-tensor, i.e. a function f. The covariant differention ∇f of f is given as

$$\forall X \in \Gamma(TM) : \nabla f(X) := \nabla_X f = X(f) = df(X)$$

$\Rightarrow \nabla f = df$ is a (0,1)-tensor.

We can then discuss iteratively

$$\nabla^2 f := \nabla(\nabla f), \nabla^3 f = \nabla(\nabla^2 f), \cdots$$

generally, $\nabla^2 A, \nabla^3 A, \cdots$

Warning: $\nabla^2 A(\cdots, X, Y) \neq \nabla_Y \nabla_X A(\cdots)$**!!**

For $X, Y \in \Gamma(TM)$, we have=

$$\begin{aligned}\nabla^f(X, Y) &= \nabla(\nabla f)(X, Y) = \nabla_Y(\nabla f)(X) \\ &= Y(\nabla f(X)) - (\nabla f)(\nabla_Y X) \\ &= Y(Xf) - \nabla_Y X(f).\end{aligned}$$

Proposition 3.11. **??**$\nabla^2 f(X, Y) = \nabla^2 f(X, Y) = T(X, Y)(f)$.

Proof.

$$\begin{aligned}&\nabla^2 f(X, Y) - \nabla^2 f(Y, X) \\ =&YXf - (\nabla_Y X)f - XYf + (\nabla_X Y)f \\ =&[Y, X]f - (\nabla_Y X - \nabla_X Y)f \\ =&T(X, Y)f.\end{aligned}$$

■ □

That is, when the connection ∇ is torsion-free, we have

$$\nabla^2 f(X, Y) = \nabla^2 f(Y, X),\ \forall X, Y \in \Gamma(TM)$$

i.e. $\nabla^2 f$ is a symmetric (0,2)-tensor field.

We call $\nabla^2 f$ the Hessian of f.

Example 3.2. *On $\mathbb{R}^n$, given the canonical connection, we have*

$$\nabla^2 f(X, Y) = (Y^1, \cdots, Y^n)(\frac{\partial^2 f}{\partial x^i \partial x^j})\begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix}$$

where $X = X^i \frac{\partial}{\partial x^i}, Y = Y^j \frac{\partial}{\partial x^j}$.

Since $\nabla^2 f(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) = \frac{\partial}{\partial x^i}(\frac{\partial}{\partial x^j} f) = \frac{\partial^2}{\partial x^i \partial x^j}$.

The trace of the Hessian is the Laplacian

For any $X \in \Gamma(TM)$, we have a linear map $\nabla X : \Gamma(TM) \to \Gamma(TM), Y \mapsto \nabla_Y X$.

At a point p, $\nabla X = T_pM \to T_pM$ is a linear transformation between two vector space. Hence, it make sense to talk about the trace of ∇X at each p, which gives us a function on M.

Lemma 3.3. $\forall X \in \Gamma(TM)$, *we have* $div(X) = tr(\nabla X)$.

Recall from Chapter 1, $divX = \frac{1}{\sqrt{G}} \frac{\partial}{\partial X^i}(X^i \sqrt{G}),\ G = det(g_{ij})$ in local coordinate.

Proof. We only need to prove it at one point $p \in M$. Pick a coordinate neighborhood $p \in U,\ (U, x)$, we have

$$\begin{aligned} \nabla X &= (\nabla_{\frac{\partial}{\partial x^i}} X) dx^i \\ &= \frac{\partial X^k}{\partial x^i} \frac{\partial}{\partial x^k} \otimes dx^i + X^j \Gamma^k_{ij} \frac{\partial}{\partial x^k} \otimes dx^i \end{aligned}$$

Therefore $tr(\nabla X) = \frac{\partial X^i}{\partial x^i} + X^j \gamma^i_{ij}$.

proposition: Let ∇ be the Levi-Civita connection on (M, g). Then $\Gamma^j_{ji} = \frac{1}{\sqrt{G}} \frac{\partial}{\partial x^i} \sqrt{G}$. (see Appendix)

$\frac{\partial X^i}{\partial x^i} + X^j \gamma^i_{ij} = \frac{\partial X^i}{\partial x^i} + X^i \frac{1}{\sqrt{G}} \frac{\partial}{\partial x^i} \sqrt{G} = \frac{1}{\sqrt{G}} \frac{\partial}{\partial x^i}(X^i \sqrt{G}) = div(X)$. ■ □

Recall since g is non-degenerate and bilinear on T_pM, we have isomorphisms between TM and T^*M:

$$b : TM \to T^*M, X \mapsto \phi(X).\ \ b(X)(Y) := g(X, Y)$$

and

$$\# : T^*M \to TM, \omega \mapsto \#(\omega).\ \ g(\#(\omega), Y) = \omega(Y).$$

In local coordinate, $b(X^i \partial_i) = g_{ij} X^i dx^j, \#(w_i dx^i) = g^{ij} \omega_i \frac{\partial}{\partial x^j}$.

Now we define the trace of S as the trace of the linear map: $X \mapsto \#S(X, .)$. Note $g(\#S(X, .), Y) = S(X, Y)$.

In local coordinate, $S = S_{ij} dx^i dx^j$

$$\#S(X, .) = \#(S_{ij} X^i dx^j) = S_{ij} X^i g^{jk} \frac{\partial}{\partial x^k}.$$

Hence $tr(S) := tr(X \mapsto \#S(X, .)) = g^{ij} S_{ij} = g^{ij} S(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$.

Let us come back to $Hess\, f$.

Lemma 3.4. $\forall X, Y \in \Gamma(TM),\ Hess\, f(X, Y) = g(\nabla_X (grad\, f), Y)$

Proof.

$$\begin{aligned} RHS &= \nabla_X(g((gard)f, Y)) - g(gard\, f, \nabla_X Y) \\ &= \nabla_X(Yf) - (\nabla_X Y)f \\ &= X(Yf) - (\nabla_X Y)f = Hess\, f(X, Y). \end{aligned}$$

■ □

Hence we have

$$\begin{aligned} tr(Hess\, f) &= tr(X \mapsto \nabla_X(grad\, f)) \\ &= tr(\nabla]\, grad\, f) \\ &= div(grad\, f) \\ &= \Delta f \ (Laplace - Beltrami\ operator) \end{aligned}$$

Recall from Chapter 1. that

$$\Delta f = \frac{1}{\sqrt{G}} \frac{\partial}{\partial x^k}(g^{ik} \sqrt{G}) \frac{\partial f}{\partial x^i}.$$

3.7 Appendix: The technical lemma.

Proposition 3.12. *Let ∇ be the Levi-Civita connection on (M, g). Then $\Gamma^i_{ji} = \frac{1}{\sqrt{G}} \frac{\partial}{\partial x^i}(\sqrt{G})$, $G := det(g_{ij})$.*

Proof. Recall

$$\begin{aligned} \Gamma^j_{ji} &= \frac{1}{2} g^{jk}(g_{jk,i} + g_{ki,j} - g_{ji,k}) \\ &= \frac{1}{2} g^{jk} \frac{\partial}{\partial x^i}(g_{ik}) \\ &= \frac{1}{2}\, tr[(g^{rs})_{n\times n} \cdot \frac{\partial}{\partial x^i} \cdot (g_{jk})_{n\times n}] \end{aligned}$$

Note moreover, $(g^{rs})_{n\times n}$ is the inverse matrix of $(g_{jk})_{n\times n}$.

We need the following result:

Lemma: Let $A = A(t)$ be a family of nonsingular matrices that depends smoothly on t, then

$$tr(A^{-1} \frac{d}{dt} A^{-1}) = \frac{d}{dt}\, ln\, det\, A.$$

Sketch of proof: Observe the Lemma is obvious, when A is 1×1. For a diagonal matrix

A.

$$= tr[\begin{pmatrix} A_1^{-1}(t) & & \\ & \ddots & \\ & & A_n^{-1}(t) \end{pmatrix} \begin{pmatrix} A_1'(t) & & \\ & \ddots & \\ & & A_n'(t) \end{pmatrix}]$$
$$= tr \begin{pmatrix} A_1^{-1}(t)A_1'(t) & & \\ & \ddots & \\ & & A_n^{-1}(t)A_n'(t) \end{pmatrix} = \sum_{i=1}^{n} A_i^{-1}(t)A_i'(t)$$
$$= \frac{d}{dt}\sum_{i=1}^{n} ln\, A_i(t) = \frac{d}{dt} ln \prod_{i=1}^{n} A_i(t) = \frac{d}{dt} ln\, det \begin{pmatrix} A_1(t) & & \\ & \ddots & \\ & & A_n(t) \end{pmatrix}$$

Recall both *trace* and *det* are invariants under similar transformation. For diagonalizable A, we have $A = P^{-1}DP^{-1}$. Then $det\, A = det\, D$ and $tr(A^{-1}\frac{d}{dt}A) = tr(D^{-1}\frac{d}{dt}D)$ since

$$tr(P^{-1}DP(P^{-1}DP)') = tr(P^{-1}D^{-1}P(P^{-1})'DP + P^{-1}D^{-1}PP^{-1}D'P + P^{-1}D^{-1}PP^{-1}DP')$$
$$= tr(P(P^{-1})' + P^{-1}P') + tr(D^{-1}D') = tr(D^{-1}D')$$

Hence Lemma is true for diagonalizable metrices.

By standard permutation trick, one can prove Lemma in its full generality. ■

Let us continue:

$$\Gamma^i_{ji} = \frac{1}{2} tr[(g^{rs})_{n\times n} \cdot \frac{\partial}{\partial x^i} \cdot (g_{jk})_{n\times n}]$$
$$= \frac{1}{2}\frac{\partial}{\partial x^i} ln\, det(g_{jk})_{n\times n} = \frac{1}{2}\frac{\partial}{\partial x^i}(ln\, G)$$
$$= \frac{\partial}{\partial x^i}(ln\, \sqrt{G}) = \frac{1}{\sqrt{G}}\frac{\partial}{\partial x^i}(\sqrt{G})$$

■. □

Chapter 4

Curvatures

The Riemannian curvature tensor was introduced by Riemann in his 1854 lecture as a natural invariant for what is called the equivalence problem in Riemannian geometry. This problem, comes out of the problem one faces when writing the same metric in two different coordinates. Namely, how is one to know that they are the same or equivalent. The idea is to find invariants of the metric that can be computed in coordinates and then try to show that two metrics are equivalent if their invariant expressions are equal. This problem was furthur classified by Christoffel.

Our previous discussions on geodesics and connections follow roughly the historical development. However, we will have a discussion on the curvature tensor different from it's historical development.(Notice that the idea of a connection postdates Riemann's introduction of the curvature tensor).

On retrospection of our previous discussions, roughly speaking ,"the first variation (of length or energy) gives the connection". In this chapter, we will see, roughly speaking, "the second variation gives the curvation"!

4.1 The Second Variation

We already know that a geodesic is not necessarily minimizing. Is a geodesic a "local minima"? One way to explore this properly is to calculate the second variation of length or energy. (Recall from 3.5 and Exercise 6.2 ,among curves $\in C_{p,q}$(piecewise smooth curves from p to q), a geodesic is characterized as the critical point of the energy functional).

Let $\gamma : [a,b] \to M$ be a normal geodesic, i.e. $\dot{\gamma}(t) \equiv 1$. We consider a 2-parameter variation F of γ . That is , a smooth map

$$F : [a,b] \times (-\epsilon,\epsilon) \times (-\delta,\delta) \to M$$

such that $F(t,0,0) = \gamma(t)$

Let $E(v,w)$ be the energy of the curve $\gamma_{v,w}(t) := F(t,v,w)$. And

$$V(t) = \frac{\partial F}{\partial v}(t,0,0), W(t) = \frac{\partial F}{\partial w}(t,0,0)$$

are the two corresponding variation fields.

Recall

$$\begin{aligned}\frac{\partial E}{\partial w}(v,w) &= \frac{1}{2}\int_a^b \frac{\partial}{\partial w}\langle \frac{\partial F}{\partial t}(t,v,w), \frac{\partial F}{\partial t}(t,v,w)\rangle dt \\ &= \int_a^b \langle \widetilde{\nabla}_{\frac{\partial}{\partial v}} \frac{\partial F}{\partial t}, \frac{\partial F}{\partial t}\rangle dt \ (compatibility) \\ &= \int_a^b \langle \widetilde{\nabla}_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial v}, \frac{\partial F}{\partial t}\rangle dt \ (torsion-free)\end{aligned}$$

Then

$$\begin{aligned}\frac{\partial^2}{\partial w \partial v}E(v,w) &= \int_a^b \frac{\partial}{\partial w}\langle \widetilde{\nabla}_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial v}, \frac{\partial F}{\partial t}\rangle dt \\ &= \int_a^b \left(\langle \nabla_{\frac{\partial}{\partial w}} \nabla_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial v}, \frac{\partial F}{\partial t}\rangle + \langle \nabla_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial v}, \nabla_{\frac{\partial}{\partial w}} \frac{\partial F}{\partial t}\rangle\right) dt \ (compatibility) \\ &= \int_a^b \left(\langle \nabla_{\frac{\partial}{\partial w}} \nabla_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial v}, \frac{\partial F}{\partial t}\rangle + \langle \nabla_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial v}, \nabla_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial w}\rangle\right) dt \ (torsion-free)\end{aligned}$$

Restrcting the above equation to the curve γ ,i.e. to the case where $v = w = 0$:

$$\frac{\partial^2}{\partial w \partial v}\Big|_{v=w=0} E(v,w) = \int_a^b \left(\langle \nabla_{\frac{\partial}{\partial w}} \nabla_{\frac{\partial}{\partial t}} V(t), \dot{\gamma}(t)\rangle + \langle \nabla_{\frac{\partial}{\partial t}} V(t), \nabla_{\frac{\partial}{\partial t}} W(t)\rangle\right) dt$$

Now ,we hope to make use of the fact that γ is geodesic ,i.e.,$\nabla_{\frac{\partial}{\partial t}}\dot{\gamma}(t) = 0$. For this purpose,we hope to interchange the order of the covariant derivative $\nabla_{\frac{\partial}{\partial w}}, \nabla_{\frac{\partial}{\partial t}}$. Hence we proceed:

$$\begin{aligned}\frac{\partial^2}{\partial w \partial v}\Big|_{v=w=0} E(v,w) &= \int_a^b \langle \nabla_{\frac{\partial}{\partial w}} \nabla_{\frac{\partial}{\partial t}} V(t) - \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial w}} V(t), \dot{\gamma}(t)\rangle dt \\ &+ \int_a^b (\langle \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial w}} V(t), \dot{\gamma}(t)\rangle + \langle \nabla_{\frac{\partial}{\partial t}} V(t), \nabla_{\frac{\partial}{\partial t}} W(t)\rangle) dt \\ &=: I + II\end{aligned}$$

Then the second term becomes

$$\begin{aligned}II &= \int_a^b \left(\frac{\partial}{\partial t}\langle \nabla_{\frac{\partial}{\partial w}} V(t), \dot{\gamma}(t)\rangle - \langle \nabla_{\frac{\partial}{\partial w}} V(t), \nabla_{\frac{\partial}{\partial t}} \dot{\gamma}(t)\rangle + \langle \nabla_{\frac{\partial}{\partial t}} V, \nabla_{\frac{\partial}{\partial t}} W\rangle\right) dt \\ &= \langle \nabla_{\frac{\partial}{\partial w}} V(t), \dot{\gamma}(t)\rangle\Big|_a^b + \int_a^b \langle \nabla_{\frac{\partial}{\partial t}} V(t), \nabla_{\frac{\partial}{\partial t}} W(t)\rangle dt.\end{aligned}$$

Therefore,we obtain the following Second Variation Formula:

$$\frac{\partial^2}{\partial w \partial v}\Big|_{v=w=0} E(v,w) = \langle \nabla_{\frac{\partial}{\partial w}} V(t), \dot{\gamma}(t)\rangle\Big|_a^b + \int_a^b \left(\langle \nabla_{\frac{\partial}{\partial t}} V(t), \nabla_{\frac{\partial}{\partial t}} W(t)\rangle + \langle \nabla_{\frac{\partial}{\partial w}} \nabla_{\frac{\partial}{\partial t}} V(t) - \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial w}} V(t), \dot{\gamma}(t)\rangle\right) dt \quad \text{(SVF)}$$

Remark 4.1. *Usually ,we will suppose the notation* $\widetilde{\nabla}$ *and proceed formula , as if vectors along* γ *were actually defined on* M *and write* $\frac{\partial^2}{\partial w \partial v}|_{v=w=0} E(v,w) = \langle \nabla_w V, \dot{\gamma} \rangle \Big|_a^b + \int_a^b \left(\langle \nabla_j V, \nabla_j W \rangle + \langle \nabla_w \nabla_j V - \nabla_j \nabla_w V, \dot{\gamma} \rangle \right) dt$

Remark 4.2. *In particular ,(SVF) tells*

$$\frac{d^2}{dv^2}\Big|_{v=0} E(v) =: E''(0) = \langle \nabla_{\frac{\partial}{\partial v}} V(t), \dot{\gamma}(t) \rangle \Big|_a^b + \int_a^b \langle \nabla_{\frac{\partial}{\partial t}} V(t), \nabla_{\frac{\partial}{\partial t}} V(t) \rangle + \langle \nabla_{\frac{\partial}{\partial v}} \nabla_{\frac{\partial}{\partial t}} V(t) - \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial v}} V(t), \dot{\gamma}(t) \rangle dt$$

For proper variations,(i.e. $V(a) = V(b) = 0$*), or generally,when*$\nabla_{\frac{\partial}{\partial v}} V(t) = 0$ *at* $t = a$ *and* $t = b$ *,we have*

$$E"(0) = \int_a^b \left(\langle \nabla_{\frac{\partial}{\partial t}} V(t), \nabla_{\frac{\partial}{\partial t}} V(t) \rangle + \langle \nabla_{\frac{\partial}{\partial v}} \nabla_{\frac{\partial}{\partial t}} V(t) - \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial v}} V(t), \dot{\gamma}(t) \rangle \right) dt$$

Now we see the sign of the second term is very importan to decide the sign of $E"(0)$, which is useful to decide whther a geodesic has a locally minimal energy functional (for curves parametrized proportionally to arclength, equicalent to a locally minimal arc length).

In particular ,if the second term vanishes,(or ≥ 0), we have $E"(0) \geq 0$ and the local minimum is guaranted.

In $\mathbb{R}^n$ (a flat case), any geodesic is minimizing. From that sense , te term

$$\langle \nabla_{\frac{\partial}{\partial v}} \nabla_{\frac{\partial}{\partial t}} V(t) - \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial v}} V(t), \dot{\gamma}(t) \rangle$$

play the role of "curvature".

Consider variations with the property $\langle \nabla_{\frac{\partial}{\partial w}} V(t), \dot{\gamma}(t) \rangle \Big|_a^b = 0$ (e.g., proper variations), we have

$$\begin{aligned} \frac{\partial^2}{\partial w \partial v}\Big|_{v=w=0} E(v,w) &= \int_a^b \left(\langle \nabla_{\frac{\partial}{\partial t}} V(t), \nabla_{\frac{\partial}{\partial t}} W \rangle + \langle \nabla_{\frac{\partial}{\partial w}} \nabla_{\frac{\partial}{\partial t}} V(t) - \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial w}} V(t), \dot{\gamma}(t) \rangle \right) dt \\ &:= I(V(t), W(t)) \end{aligned}$$

We will see this quality plays a central role in our subsequent discussions about curvature-related geometries.

The second term in $I(V,W)$ suggests to define for $X, Y, Z \in \Gamma(TM)$,

$$\bar{R}(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z \in \Gamma(TM)$$

But

$$\begin{aligned} \bar{R}(X, fY)Z &= \nabla_X (f \nabla_Y Z) - f \nabla_Y \nabla_X Z \\ &= X(f) \nabla_Y Z + f[\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z] \end{aligned}$$

i.e. $\bar{R}$ is not a tensor!!

We define for $X, Y, Z \in \Gamma(TM), R(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]Z}$ It gives a multilinear map

$$R : \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$$

Proposition 4.1. *R is a (1,3) tensor.*

Proof. Notice that this time

$$\begin{aligned} R(X, fY)Z &= \nabla_X(f(\nabla_Y Z)) - f\nabla_Y\nabla_X Z - \nabla_{[X,fY]}Z \\ &= X(f)\nabla_Y Z + f(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z) - \nabla_{X(f)Y+f[X,Y]}Z \\ &= X(f)\nabla_Y Z + f(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z) - X(f)\nabla_Y Z - f\nabla_{[X,Y]}Z \\ &= fR(X,Y)Z \end{aligned}$$

One can furthur check that

$$R(fX, Y)Z = R(X, fY)Z = R(X,Y)(fZ) = fR(X,Y)Z$$

Hence R is a tensor . We say it is a (1,3) tensor . We actually mean $R(W, X, Y, Z) := W(R(X,Y)Z)$.

□

We will call R the curvature tensor.

Remark 4.3. *(1) The curvature tensor is well-defined for any affine connection on M*

(2) Notice that X,Y appear skew-symmetrically in $R(X,Y)Z$ while Z plays its own role on top of the line ,hence we use the usual notation $R(X,Y)Z$ instead of $R(X,Y,Z)$.

(3) Some textbooks adopt a different sign in the definition of R. One should always first check the author's notation for curvature tensor when reading works on Riemannian geometry , unfortunately.

(4)(Locality): At $p \in M, R(X,Y)Z(p)$ only depends on $X(p), Y(p), Z(p) \in T_pM$. This is due to the tensorial property

$$R(X,Y)Z = X^i Y^j Z^k R(\frac{\partial}{\partial X^i}, \frac{\partial}{\partial X^j})\frac{\partial}{\partial X^k}$$

Now let's come back to the SVF:

$$I(V,W) == \int_a^b (\langle \nabla_{\frac{\partial}{\partial t}} V(t), \nabla_{\frac{\partial}{\partial t}} W\rangle + \langle \nabla_{\frac{\partial}{\partial w}}\nabla_{\frac{\partial}{\partial t}} V(t) - \nabla_{\frac{\partial}{\partial t}}\nabla_{\frac{\partial}{\partial w}} V(t), \dot\gamma(t)\rangle)dt$$

Proposition 4.2. *Let $s : \mathbb{R}^2 \to M$ be a paramatrized surface , and V a C^∞ vector field along s .Then*

$$\frac{D}{\partial x}\frac{D}{\partial y}V - \frac{D}{\partial y}\frac{D}{\partial x}V = R\left(\frac{\partial s}{\partial x}, \frac{\partial s}{\partial y}\right)V \qquad (*)$$

(or,in another notation ,$\nabla_{\frac{\partial}{\partial x}}\nabla_{\frac{\partial}{\partial y}}V - \nabla_{\frac{\partial}{\partial y}}\nabla_{\frac{\partial}{\partial x}}V = R(\frac{\partial s}{\partial x}, \frac{\partial s}{\partial y})V$)

Sketch of proof: First by the locality remark above ,at each point $s(x,y) \in M$, the RHS is well-defined since

$$\frac{\partial s}{\partial x} = ds\left(\frac{\partial}{\partial x}\right), \frac{\partial s}{\partial y}, V \in T_{s(x,y)}M.$$

Then (*) can be proved by computing in a coordinate neighborhood.

Definition of both sides are clear: pick $(u, x^1, x^2....x^n)$, $s(x, y) = (s^1(x, y), ..., s^n(x, y))$

$$R\left(\frac{\partial s}{\partial x}, \frac{\partial s}{\partial y}\right) V = \frac{\partial s^i}{\partial x}\frac{\partial s^j}{\partial y} V^k R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^k}$$

while in LHS

$$\begin{aligned}
\frac{D}{\partial x}\frac{D}{\partial y} V &= \frac{D}{\partial x}\left(\frac{\partial V^i(x,y)}{\partial y}\frac{\partial}{\partial x^k} + V^i \nabla_{ds(\frac{\partial}{\partial y})}\frac{\partial}{\partial x^k}\right) \\
&= \frac{D}{\partial x}\left(\frac{\partial V^i}{\partial y}\frac{\partial}{\partial x^i} + V^i \frac{\partial s^j}{\partial y}\nabla_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^k}\right) \\
&= \frac{\partial^2 V^i}{\partial x \partial y}\frac{\partial}{\partial x^i} + \frac{\partial V^i}{\partial y}\frac{\partial s^j}{\partial x}\nabla_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^i} + \frac{\partial}{\partial x}\left(V^i \frac{\partial s^j}{\partial y}\right)\nabla_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^k} + V^i \frac{\partial s^j}{\partial y}\frac{\partial s^l}{\partial x}\nabla_{\frac{\partial}{\partial x^l}}\nabla_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^k}
\end{aligned}$$

So the meaning of each term is clear. The equality (*) then just follows from direct computation

In particular , 6.4.2 tells

$$\nabla_{\frac{\partial}{\partial w}}\nabla_{\frac{\partial}{\partial t}} V(t) - \nabla_{\frac{\partial}{\partial t}}\nabla_{\frac{\partial}{\partial w}} V(t) = R(w(t), \dot{\gamma}(t))V(t)$$

and hence

$$\begin{aligned}
I(V, W) &= \int_a^b \langle \nabla_{\frac{\partial}{\partial t}} V(t), \nabla_{\frac{\partial}{\partial t}} W(t)\rangle + \langle R(W(t), \dot{\gamma}(t))V(t), \dot{\gamma}(t)\rangle dt \\
&= \int_a^b \langle \nabla_{\dot{\gamma}}(t) V(t), \nabla_{\dot{\gamma}(t)} W(t)\rangle + \langle R(W, \dot{\gamma}), \dot{\gamma}\rangle(t) dt
\end{aligned}$$

If we furthur denote $T = \dot{\gamma}$, then we have

$$I(V, W) = \int_a^b \left(\langle \nabla_T V, \nabla_T W\rangle + \langle R(W, T)V, T\rangle\right) dt$$

In particular ,$I(V, V) = \int_a^b (\langle \nabla_T V, \nabla_T V\rangle + \langle R(V, T)V, T\rangle)dt$

4.2 Properties of Curvature tensor : Geometric meaning and Symmetries

Curvature tensor measures the non-commutatively of the convariant derivatives.

4.2.1 Ricci Identity

Recall for $f \in C^\infty(M)$, its Hessian $\nabla^2 f$ is symmetric (for torsion free connection)

$$\nabla^2 f(X, Y) = \nabla^2 f(Y, X)$$

For any tensor field $\Phi \in \Gamma(\bigotimes^{r,s} TM)$, we can define

$$R(X, Y)\Phi = \nabla_X \nabla_Y \Phi - \nabla_Y \nabla_X \Phi - \nabla_{[X,Y]}\Phi$$

It is obvious that

$$R(X,Y)f = X(Yf) - Y(Xf) - [X,Y]f = 0$$

So we can write (for torsion-free connection ∇)

$$\nabla^2 f(X,Y) - \nabla^2 f(Y,X) = R(Y,X)f = -R(X,Y)f$$

We can furthur check the case $\Phi = Z \in \Gamma(TM)$

$$\nabla^2 Z(X,Y) = \nabla_Y(\nabla Z)X = \nabla_Y(\nabla_X Z) - \nabla Z(\nabla_Y X) = \nabla_Y \nabla_X Z - \nabla_{\nabla_Y X} Z$$

Hence

$$\begin{aligned}\nabla^2 Z(X,Y) - \nabla^2 Z(Y,X) &= \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z - \nabla_{\nabla_Y X} Z + \nabla_{\nabla_X Y} Z \\ &= \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z - \nabla_{[Y,X]} Z \\ &= R(Y,X)Z = -R(X,Y)Z\end{aligned}$$

It is direct to check the gneral case.

Proposition 4.3. *(Ricci Identity)* $\forall X, Y \in \Gamma(TM), \Phi \in \Gamma(\bigotimes^{r,s} TM)$ *we have*

$$\nabla^2 \Phi(..., X, Y) - \nabla^2 \Phi(..., Y, X) = R(Y,X)\Phi(...) = -R(X,Y)\Phi(...)$$

Remark 4.4. *In Euclidean space $\mathbb{R}^n$, pick the directional derivative as the covariant derivative .One easily check that $R(X,Y)$ vanishes. In $\mathbb{R}^n$, we can interchange the order of taking derivatives freely . However this is not true anymore when R is nontrivial.*

4.2.2 Geometric meaning :A test case [Spivak 2.Chap6.Thm 10]

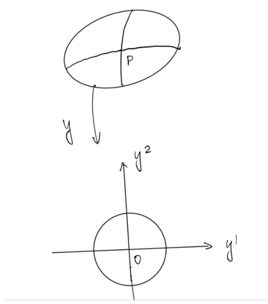
The Ricci identity from last subsection provides an explanation of the curvature tensor from a viewpoint of analysis : it is a term measuring the non-commutativity of taking cavariant derivatives.

We persue for a geometric meaning of the curvature tensor. Back to Riemann's "equicalence problem": If we know a Riemannian metric $g = g_{ij} dx^i \otimes dx^j$ gives $R = 0$, is there a coordinate change $x \to y$ s.t. $g = \sum_i dy^i \otimes dy^i$? (Or ,does $R = 0$ implies locally isometric to $(\mathbb{R}^n, \langle , \rangle)$?)

The answer is yes!

Theorem 4.1. *Let (M, g) be an n-dim Riemannian manifold for which the curvature tensor R (for the Levi-Civita connection) is 0. Then M is locally isometric to $\mathbb{R}^n$ with its canonical Riemannian metric .*

Proof. Let $p \in M$, pick a coordinate neighborhood $(U, y^1 ...., y^n)$ Let $g = g_{ij} dy^i \otimes dy^j$



To prove this theorem, it is equivalent to show there exist open set$V \subset U$, and a coordinate change $x : V \to \mathbb{R}^n$, s.t. $g = \sum_i dx^i \otimes dx^i$

So without loss of generality, we can assume we are in $\mathbb{R}^n$,with $y^1 ..., y^n$ the standard coordinate system ,with a metric $g = g_{ij} dy^i \otimes dy^j$ and ∇ be the corresponding Levi-Civita connection

Step 1:We claim that we can find vector fields X, with arbitrary initial values $X(0) \in T_0\mathbb{R}^n$, satisfying

$$\nabla_{\frac{\partial}{\partial y^i}} X = 0, for\ all\ i$$

and hence $\nabla_Z X = 0$ for all Z.

To do this ,we first choose the curve $y \mapsto (y, 0, ...0)$

Then for each fixed y_1,we choose the curve $y \mapsto (y_1, y, 0..., 0)$ with $X(y_1, 0..., 0)$ as the initial value, we obtain $X(y_1, y, 0..., 0)$ via parallel transport along $y \mapsto (y_1, y, 0..., 0)$

Now we have a vector field X defined on the surface

$$s(y^1, y^2) = (y^1, y^2, 0, ..., 0)$$

By construction ,we have $\widetilde{\nabla}_{\frac{\partial}{\partial y^2}} X = 0$ along s

while $\widetilde{\nabla}_{\frac{\partial}{\partial y^1}} X = 0$ along $\{s(y, 0)\}$

Question: Does $\widetilde{\nabla}_{\frac{\partial}{\partial y^1}} X$ vanish along s?

Now we use

$$\nabla_{\frac{\partial}{\partial y^1}} \nabla_{\frac{\partial}{\partial y^2}} X - \nabla_{\frac{\partial}{\partial y^2}} \nabla_{\frac{\partial}{\partial y^1}} X = R\left(\frac{\partial s}{\partial y^1}, \frac{\partial s}{\partial y^2}\right) X = 0 \Longleftrightarrow \widetilde{\nabla}_{\frac{\partial}{\partial y^2}} (\widetilde{\nabla}_{\frac{\partial}{\partial y^1}} X) = 0 \qquad (*)$$

Since $\widetilde{\nabla}_{\frac{\partial}{\partial y^1}} X\Big|_{y_2=0} = 0$,i.e., $\widetilde{\nabla}_{\frac{\partial}{\partial y^1}} X$ is parallel along $\{s(y, 0)\}$,

We have by (*) $\widetilde{\nabla}_{\frac{\partial}{\partial y^1}} X = 0$ along s .

We can continue in this way to obtain the desired X . This proves the claim.

Now at 0,we can choose $X_1^{(0)}, ..., X_n^{(0)}$ as orthonormal w.r.t the metric g . And construct $X_1, ..., X_n$ in the above way. By property of parallel transport, they are orthonormal everywhere.

Step 2: Since ∇ is torsion free , we have

$$0 = \nabla_{X_i} X_j - \nabla_{X_j} X_i - [X_i, X_j]$$

By construction ,$\nabla_{X_i} X_j = \nabla_{X_j} X_i = 0$, therefore , we obtain $[X_i, X_j] = 0, \forall i, j$

This means that thereis a coordinate system $x^1, ..., x^n$ with $X_i = \frac{\partial}{\partial x^i}$.(Frobenius theorem in "differential manifold" course . $[x_i, x_j] = 0$ means intergrability.)

Step 3: Since $\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^n}$ are orthonormal everywhere , we have

$$g = \sum_i dx^i \otimes dx^i$$

□

Remark 4.5. *In some sense , the flatness $R = 0$ is a kind of integrability condition.*

It is not true that $R = 0$ implies M is globally isometric to $\mathbb{R}^n$.

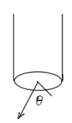
Example 4.1. *$S^1 \times \mathbb{R}^1$ cylinder is the product of a unit circle S^1 and $\mathbb{R}^1$.*

$\forall p = (x, y, z) \in S^1 \times \mathbb{R}^1$, *we can write* $x = \cos\theta, y = \sin\theta, z = z\ 0 \leq \theta \langle 2\pi$

Therefore in the coordinate neighborhood $\{(\theta, z) | 0 < \theta < 2\pi, z \in \mathbb{R}^1\}$

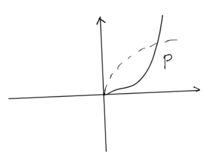
We have the induced metric $g = d\theta \otimes d\theta + dz \otimes dz$

Hence cylinder has $R = 0$



Corollary 4.1. *If we find n everywhere linearly independent vector fields $X_1, ..., X_n$,which are parallel (i.e. $\nabla_Z X_i = 0, \forall Z$)*

then the manifold is flat.



Parallel translation of a vector along a closed curve generally bring it back to a different vector.

4.2.3 Bianchi Identities

Before continuing the duscussions of the geometric aspect of the curvature tensor , we prepare symmetry properties of the curvature tensor in this section . We will work on a smooth manifold with a symmetric (i.e. torsion-free) connection ∇.

Proposition 4.4. *The curvature tensor satisfies the following identities : For any $X, Y, Z, W \in \Gamma(TM)$,*

(1)$R(X, Y)Z = -R(Y, X)Z$

(2)$R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$ *(The first Bianchi identity)*

(3)$(\nabla_X R)(Y, Z)W + (\nabla_Y R)(Z, X)W + (\nabla_Z R)(X, Y)W = 0$ *(The Second Bianchi identity)*

Remark 4.6. *Let T be any mapping with 3 vector field variables and values that canbe added. Summing over cyclic permutations (denote by symbol "S")of the cariables gives us a new map.*

$$ST(X, Y, Z) = T(X, Y, Z) + T(Y, Z, X) + T(Z, X, Y)$$

For example , the Jacobi identity for vector fields can be written as $S[X, [Y, Z]] = 0$

In this way, the First Bianchi identity is $SR(X, Y)Z = 0$ *while the second Bianchi identity is* $S(\nabla_X R)(Y, Z)W = 0$

Proof. (1) is obvious from the definition

(2):

$$\begin{aligned}
SR(X, Y)Z &= S\,(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z) \\
&= S\,(\nabla_X\nabla_Y Z) - S\,(\nabla_Y\nabla_X Z) - S\,(\nabla_{[X,Y]}Z) \\
&= S\,(\nabla_Z\nabla_X Y) - S\,(\nabla_Z\nabla_Y X) - S\,(\nabla_{[X,Y]}Z) \\
&= S\,(\nabla_Z\,(\nabla_X Y - \nabla_Y X)) - S\,(\nabla_{[X,Y]}Z) \\
&= S\,(\nabla_Z[X, Y]) - S\,(\nabla_{[X,Y]}Z) \\
&= S([Z, [X, Y]]) \\
&= 0
\end{aligned}$$

(3)Denote

$$\begin{aligned}
R(X, Y)Z &= \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z \\
&= [\nabla_X, \nabla_Y]\,Z - \nabla_{[X,Y]}Z
\end{aligned}$$

then

$$\begin{aligned}
(\nabla_Z R)\,(X, Y)W &= \nabla_Z(R(X, Y)W) - R\,(\nabla_Z X, Y)\,W - R(X, \nabla_Z Y)W - R(X, Y)\,(\nabla_Z W) \\
&= [\nabla_Z, R(X, Y)]\,W - R\,(\nabla_Z X, Y)\,W - R(X, \nabla_Z Y)W
\end{aligned}$$

Keeping in mind that we only do cyclic sums over X, Y, Z and that we have the Jacobi identity for oprators:

$$S\,[\nabla_X, [\nabla_Y, \nabla_Z]] = 0$$

We have

$$\begin{aligned}
S\,(\nabla_X R)\,(Y, Z)\,W &= S\,[\nabla_X, R\,(Y, Z)]\,W - SR\,(\nabla_X Y, Z)\,W - SR\,(Y, \nabla_X Z)\,W \\
&= S\,[\nabla_X, [\nabla_Y, \nabla_Z]]\,W - S\,[\nabla_X, \nabla_{[Y,Z]}]\,W - SR\,(\nabla_X Y, Z)\,W - SR\,(Y, \nabla_X Z)\,W \\
&= -S\,[\nabla_X, \nabla_{[Y,Z]}]\,W - SR\,(\nabla_X Y, Z)\,W + SR\,(\nabla_X Z, Y)\,W \\
&= -S\,[\nabla_X, \nabla_{[Y,Z]}]\,W - SR\,(\nabla_X Y, Z)\,W + SR\,(\nabla_Y X, Z)\,W \\
&= -S\,[\nabla_X, \nabla_{[Y,Z]}]\,W - SR([X, Y], Z)W \\
&= -S\,[\nabla_X, \nabla_{[Y,Z]}]\,W - S[\nabla_{[X,Y]}, \nabla_Z]W + S\nabla_{[[X,Y],Z]}W \\
&= S[\nabla_{[Y,Z]}, \nabla_X]W - S[\nabla_{[X,Y]}, \nabla_Z]W \\
&= 0
\end{aligned}$$

□

In local coordinates , we write

$$
\begin{aligned}
&R\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^l} = R^k_{lij}\frac{\partial}{\partial x^k}.\\
=&\nabla_{\frac{\partial}{\partial x^i}}\nabla_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^l} - \nabla_{\frac{\partial}{\partial x^j}}\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^l}\\
=&\nabla_{\frac{\partial}{\partial x^i}}\left(\Gamma^\gamma_{jl}\frac{\partial}{\partial x^\gamma}\right) - \nabla_{\frac{\partial}{\partial x^j}}\left(\Gamma^\gamma_{il}\frac{\partial}{\partial x^\gamma}\right)\\
=&\frac{\partial\Gamma^\gamma_{jl}}{\partial x^i}\frac{\partial}{\partial x^\gamma} + \Gamma^\gamma_{jl}\Gamma^\mu_{i\gamma}\frac{\partial}{\partial x^\mu}\\
&-\frac{\partial\Gamma^\gamma_{il}}{\partial x^j}\frac{\partial}{\partial x^\gamma} - \Gamma^\gamma_{il}\Gamma^\mu_{j\gamma}\frac{\partial}{\partial x^\mu}\\
=&\left(\frac{\partial\Gamma^k_{jl}}{\partial x^i} - \frac{\partial\Gamma^k_{il}}{\partial x^j} + \Gamma^\gamma_{jl}\Gamma^k_{i\gamma} - \Gamma^\gamma_{il}\Gamma^k_{j\gamma}\right)\frac{\partial}{\partial x^k}
\end{aligned}
$$

That is ,

$$
R^k_{lij} = \frac{\partial\Gamma^k_{jl}}{\partial x^i} - \frac{\partial\Gamma^k_{il}}{\partial x^j} + \Gamma^\gamma_{jl}\Gamma^k_{i\gamma} - \Gamma^\gamma_{il}\Gamma^k_{j\gamma}
$$

We see $R^k_{lij} = -R^k_{lji}$, and $R^k_{lij} + R^k_{ijl} + R^k_{jli} = 0$

4.2.4 Riemannian curvature tensor

Now we consider a Riemannian manifold (M, g) with a Levi-Civita connection ∇ . We can use g to convert the (1,3)-tensor R to be a (0,4)-tensor:

$$
\langle R(X,Y)Z, W\rangle_g := R(W,Z,X,Y)
$$

In local coordinates

$$
\begin{aligned}
R_{klij} &= R\left(\frac{\partial}{\partial x^k},\frac{\partial}{\partial x^l},\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right) = \left\langle R\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^l},\frac{\partial}{\partial x^k}\right\rangle\\
&= g_{km}R^m_{lij}\\
&= g_{km}\left(\frac{\partial\Gamma^m_{jl}}{\partial x^i} - \frac{\partial\Gamma^m_{il}}{\partial x^j} + \Gamma^\gamma_{jl}\Gamma^m_{i\gamma} - \Gamma^\gamma_{il}\Gamma^m_{j\gamma}\right)
\end{aligned}
$$

$$
\begin{aligned}
g_{km}\frac{\partial\Gamma^m_{jl}}{\partial x^i} &= \frac{\partial}{\partial x^i}\left(g_{km}\Gamma^m_{jl}\right) - \Gamma^m_{jl}\frac{\partial g_{km}}{\partial x^i}\\
&= \frac{1}{2}\frac{\partial}{\partial x^i}\left(g_{jk,l} + g_{kl,j} - g_{jl,k}\right) - \Gamma^m_{jl}\left(g_{mp}\Gamma^p_{ik} + g_{kp}\Gamma^p_{im}\right)\\
&= \frac{1}{2}\left(\frac{\partial^2 g_{jk}}{\partial x^i\partial x^l} + \frac{\partial^2 g_{kl}}{\partial x^i\partial x^j} - \frac{\partial^2 g_{jl}}{\partial x^i\partial x^k}\right) - g_{mp}\Gamma^m_{jl}\Gamma^p_{ik} - g_{kp}\Gamma^m_{jl}\Gamma^p_{im}
\end{aligned}
$$

$$
g_{km}\frac{\partial\Gamma^m_{il}}{\partial x^j} = \frac{1}{2}\left(\frac{\partial^2 g_{ik}}{\partial x^j\partial x^l} + \frac{\partial^2 g_{kl}}{\partial x^j\partial x^i} - \frac{\partial^2 g_{il}}{\partial x^j\partial x^k}\right) - g_{mp}\Gamma^m_{il}\Gamma^p_{jk} - g_{kp}\Gamma^m_{il}\Gamma^p_{jm}
$$

$$\Rightarrow R_{klij} = \frac{1}{2}\left(\frac{\partial^2 g_{jk}}{\partial x^i \partial x^l} - \frac{\partial^2 g_{ik}}{\partial x^j \partial x^l} - \frac{\partial^2 g_{jl}}{\partial x^i \partial x^k} + \frac{\partial^2 g_{il}}{\partial x^j \partial x^k}\right) + g_{mp}\left(\Gamma^m_{il}\Gamma^p_{jk} - \Gamma^m_{jl}\Gamma^p_{ik}\right)$$

Proposition 4.5. *We have the following identities*
(1) $\langle R(X,Y)Z,W\rangle = -\langle R(Y,X)Z,W\rangle$*, i.e.* $R_{klij} = -R_{klji}$
(2)$\langle R(X,Y)Z,W\rangle = -\langle R(X,Y)W,Z\rangle$*,i,e,* $R_{lkij} = -R_{lkij}$
(3)$\langle R(X,Y)Z,W\rangle + \langle R(Y,Z)X,W\rangle + \langle R(Z,X)Y,W\rangle$
(4)$\langle R(X,Y)Z,W\rangle = \langle R(Z,W)X,Y\rangle$*, i,e,* $R_{klij} = R_{ijkl}$
(5)$\nabla R(W,Z,X,Y,V) + \nabla R(W,Z,Y,V,X) + \nabla R(W,Z,V,X,Y) = 0$

Proof. (1) is obvious.

(2)can be seen from its expression in local coordinates. One can also use the compatibility of ∇ with g ,to derive directly

$$\langle R(X,Y)Z,W\rangle = -\langle Z, R(X\langle Y)W\rangle$$

(3) follows directly from the First Bianchi Identity.

(5) follows from the second Bianchi Identity once we observe

$$\nabla_V R(W,Z,X,Y) = \langle \nabla_V R(X,Y)Z,W\rangle$$

(4) is a consequence of properties (1)–(3).

Although one can also see (4) directly from its expressions in local coordinats, it is deserved to have a look at the proof in [Spivak 2 ,Chap 4D,Proposition 11]. A clever diagram proof taken from Milnor's Morse Theory book is presented there.

□

There are interesting consequence derived from these symmetries.

The Proposition 4.5 (1) (2) , that is ,$\langle R(X,Y)Z,W\rangle$ is skew-symmetric in both (X,Y) and (Z,W) , tells

Corollary 4.2. *For two vector fields*

$$(aX+bY, \quad cX+dY) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix}$$

We have

$$\langle R(aX+bY, cX+dY)(cX+dY), cX+dY\rangle$$
$$= \left[\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right]^2 \langle R(X,Y)Y,X\rangle$$

Proof. Exercise

□

Proposition 4.5 (1)(2)(3) tells the curvature tensor R is completely determined by the values of $\langle R(X,Y)Y,X\rangle$

Corollary 4.3. *Suppose* $\langle R_1(X,Y)Y,X\rangle = R_2(X,Y)Y,X\rangle, \forall X,Y$ *, Then* $\langle R_1(X,Y)Z,W\rangle = R_2(X,Y)Z,W\rangle, \forall X,Y,Z,W$

Proof. It is clearly suffice to prove that if

$$\langle R(X,Y)Y,X\rangle = 0, \forall X,Y \Rightarrow \langle R(X,Y)Z,W\rangle = 0$$

Now we have

$$\begin{aligned} 0 = \langle R(X,Y+W)(Y+W),X\rangle &= \langle R(X,Y)Y,X\rangle + \langle R(X,Y)W,X\rangle + \langle R(X,W)Y,X\rangle + \langle R(X,W)W,X\rangle \\ &= 2\langle R(X,Y)W,X\rangle, \forall X,Y,W(Use(1)(2)(4)) \end{aligned}$$

Moreover

$$\begin{aligned} 0 &= \langle R(X+Z,Y)W,X+Z\rangle \\ &= \langle R(X,Y)W,X\rangle + \langle R(X,Y)W,Z\rangle + \langle R(Z,Y)W,X\rangle + \langle R(Z,Y)W,X\rangle \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} 0 &= \langle R(X,Y)W,Z\rangle + \langle R(Z,Y)W,X\rangle \\ &= -\langle R(Y,W)X,Z\rangle - \langle R(W,X)Y,Z\rangle + \langle R(Z,Y)W,X\rangle (using\ First\ Bianchi\ Identity) \end{aligned}$$

$$\Rightarrow 2\langle R(Z,Y)W,X\rangle = \langle R(Y,W)X,Z\rangle \tag{1}$$

By a similiar argument starting from

$$0 = \langle R(X+W,Y)Y,X+W\rangle$$

We will obtain

$$2\langle R(X,Z)Y,W\rangle = \langle R(Y,Z)X,W\rangle \tag{2}$$

Using symmetries , we can rewrite (1) and (2) as

$$2\langle R(Y,Z)X,W\rangle = \langle R(X,Z)Y,W\rangle$$

and $2\langle R(X,Z)Y,W\rangle = \langle R(Y,Z)X,W\rangle$
which implies $\langle R(X,Z)Y,W\rangle = 0, \forall X,Y,Z,W$

□

4.3 Sectional Curvature

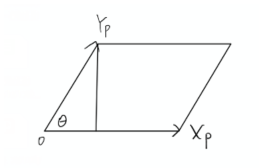
Consider another (0,4)-tensor : for $X,Y,Z,W \in \Gamma TM$

$$G(X,Y,Z,W) = \langle X,Z\rangle_g \langle Y,W\rangle_g - \langle X,W\rangle_g - \langle Y,Z\rangle_g$$

It is not hard to check G satisfies the following properties

(1)$G(X, Y, Z, W) = -G(Y, X, Z, W)$
(2)$G(X, Y, W, Z) = -G(X, Y, Z, W)$
(3)$G(X, Y, Z, W) + G(Y, Z, X, W) + G(Z, X, Y, W) = 0$
(4)$G(X, Y, Z, W) = G(Z, W, X, Y)$
Recall from last section that (4) is actually a consequence of properties (1)–(3)
Hence G behaves very similar to the Riemannian curvature tensor $\langle R(X, Y)Z, W\rangle$.
In particular ,for the linearly independent vector $X_p, Y_p \in T_pM$,

$$\begin{aligned} G(X_p, Y_p, X_p, Y_p) &= \langle X_p, X_p\rangle\langle Y_p, Y_p\rangle - \langle X_p, Y_p\rangle^2 \\ &= \langle X_p, X_p\rangle\langle Y_p, Y_p\rangle - \langle X_p, X_p\rangle\langle Y_p, Y_p\rangle \cos^2\theta \\ &= \langle X_p, X_p\rangle\langle Y_p, Y_p\rangle \sin^2\theta \end{aligned}$$



equals the area of the parallelogram spanned by X_p, Y_p ,
By the proof of Corollary 4.2, we have

$$\begin{aligned} &G\left(aX_p + bY_p, cX_p + dY_p, aX_p + bY_p, cX_p + dY_p\right) \\ =&\left[\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right]^2 G\left(X_p, Y_p, X_p, Y_p\right) \end{aligned}$$

Therefore, we have

Proposition 4.6. *The quantity*

$$\begin{aligned} K\left(X_p, Y_p\right) := \frac{\left\langle R\left(X_p, Y_p\right) Y_p, X_p\right\rangle}{G\left(X_p, Y_p, X_p, Y_p\right)} &= \frac{R\left(X_p, Y_p, X_p, Y_p\right)}{G\left(X_p, Y_p, X_p, Y_p\right)} \\ &= \frac{R\left(X_p, Y_p, X_p, Y_p\right)}{\left\langle X_p, X_p\right\rangle\left\langle Y_p, Y_p\right\rangle - \left\langle X_p, Y_p\right\rangle^2} \end{aligned}$$

depends only on the two dimensional subspace

$$\pi_p = span(X_p, Y_p) \subset T_pM$$

That is , it is independent of the choice of basis $\{X_p, Y_p\}$ *of* π_p

Definition 4.1. *(sectional curvature) We will call* $K(\pi_p) = K(X_p, Y_p)$
the <u>*sectional curvature*</u> *of* (M, g) *at* p *with respect to the plane* $\pi_p = span(X_p, Y_p)$

Remark 4.7. *Note that Proposition*

(1) The sectional curvature K is Not a function on M except when dimM=2.

(2) $K(Ag) = \frac{1}{A}K(g)$

Proposition 4.7. *Let* (M, g) *be a 2-dim Riemannian manifold , and let* $X_p, Y_p \in T_pM$ *be linearly independent . Then*

$$K(p) = K\left(X_p, Y_p\right) = \frac{\left\langle R\left(X_p, Y_p\right)Y_p, X_p\right\rangle}{\left\langle X_p, X_p\right\rangle\left\langle Y_p, Y_p\right\rangle - \left\langle X_p, Y_p\right\rangle^2}$$

is the same as the Gaussian curvature at p.

Sketch of proof :Let (u, x^1, x^2) be a coordinate neighborhood of $p \in M$
By Proposition 4.6 ,it suffices to verify the proposition when

$$X_p = \frac{\partial}{\partial x^1}\Big|_p, Y_p = \frac{\partial}{\partial x^2}\Big|_p$$

In this case

$$\left\langle R\left(\frac{\partial}{\partial x^1}\Big|_p, \frac{\partial}{\partial x^2}\Big|_p\right)\frac{\partial}{\partial x^2}\Big|_p, \frac{\partial}{\partial x^1}\Big|_p\right\rangle = R_{1212}(p)$$

while $G(X_p, Y_p, X_p, Y_p) = g_{11}g_{12} - g_{12}^2$
Hence $K(p) = \frac{R_{1212}(p)}{g_{11}g_{12}-g_{12}^2}$
Recall that the Gaussian curvature can be expressed via the first fundamental form

$$Edx^1 \otimes dx^1 + Fdx^1 \otimes dx^2 + Fdx^2 \otimes dx^1 + Gdx^2 \otimes dx^2$$

where in our case $E = g_{11}, F = g_{12} = g_{21}, G = g_{22}$

Remark 4.8. *Note that Proposition 4.6 and Proposition 4.7 together implies that Gaussian curvature is indeed independent of the choice of coordinates.*

Or equivalently, if g_1, g_2 *are locally isometric , then they lead to the same Gauss curvature . This is the celebrated " Theorem Egregium"*

Remark 4.9. *We see in Exercise 5(2)(This exercise is in that isometries preserve Levi-Civita connctions. That is , given* $(M_1, g_1, \nabla^{(1)}), (M_2, g_2, \nabla^{(2)})$ *,and* $\varphi : M_1 \longrightarrow M_2$ *be an isometry. Then for any* $X, Y \in \Gamma(TM)$ *we habe*

$$d\varphi(\nabla^{(1)}_x Y) = \nabla^{(2)}_{d\varphi(x)} d\varphi(Y)$$

As direct consequences, we see if

$$R^{(i)}(X, Y)Z := \nabla^{(i)}_X\nabla^{(i)}_Y Z - \nabla^{(i)}_Y\nabla^{(i)}_X Z - \nabla^{(i)}_{[X,Y]}Z$$

then

$$d\varphi(R^{(1)}(X, Y)Z) = R^{(2)}(d\varphi(X), d\varphi(Y))d\varphi(Z) \text{ and } g_1(R^{(1)}(X, Y)Z, W) = g_2(R^{(2)}(d\varphi(X), d\varphi(Y))d\varphi(Z), d\varphi(W))\circ\varphi$$

In particular , if $\varphi : M_1 \longrightarrow M_2$ *is an isometry s.t.* $d\varphi(\pi_p) = \pi'_{\varphi(p)} \subset T_{\varphi(p)}M_2$
We have the sectional curvature of π_p *and that of* $\pi'_{\varphi(p)}$ *are the same .*

Proposition 4.8. *Let (m, g) be a Riemannian manifold , and let π_p be a 2-dim subspave of T_pM , spanned by $X_p, Y_p \in T_pM$. Let $\mathscr{O} \subset \pi_p$ be a neighborhood of $0 \in T_pM$ on which exp_p is a diffeomorphism , let $i : exp_p(\mathscr{O}) \hookrightarrow M$ be the inclusion , and let $\bar{R}$ be the curvature tensor for $exp_p(\mathscr{O})$ with the induced Riemannian metric i^*g. Then*

$$\left\langle \bar{R}(X_p, Y_p)Y_p, X_p \right\rangle = \left\langle R(X_p, Y_p)Y_p, X_p \right\rangle$$

Consequently ,$K(\pi_p) = \frac{\left\langle R(X_p,Y_p)Y_p,X_p \right\rangle}{G(X_p,Y_p,X_p,Y_p)}$ is the Gaussian curvature at p of the surface $exp_p(\mathscr{O})$

Proposition 4.9. *The Riemannian curvature tensor at p is determined by all the sectional curvature at p.*

Proof. By Corollary 4.3

□

Definition 4.2. *A Riemannian manifold (M, g) is said to have constant (sectional) curvature if its sectional curature $K(\pi_p)$ is a constant,i.e. is independent of p and is independent of $\pi_p \in T_pM$.*

Proposition 4.10. *A Riemannian manifold (M, g) has constant curvature k if and only if*

$$\langle R(X, Y)W, Z\rangle = kG(X, Y, Z, W), \forall X, Y, Z, W \in \Gamma(TM)$$

i.e.

$$R(Z, W, X, Y) = R(X, Y, Z, W) = kG(X, Y, Z, W), orR = kG$$

Proof. Recall both $\langle R(X, Y)W, Z\rangle$ and $G(X, Y, Z, W)$ satisfy the symmetries (1)–(3) .Hence

$$S(X, Y, Z, W) := \langle R(X, Y)W, Z\rangle - kG(X, Y, Z, W)$$

satisfies

(1) $S(X, Y, Z, W) = -S(Y, X, Z, W)$
(2) $S(X, Y, Z, W) = -S(X, Y, W, Z)$
(3) $S(X, Y, Z, W) + S(Y, Z, X, W) + S(Z, X, Y, W) = 0$
(4) $S(X, Y, Z, W) = S(Z, W, X, Y)$
Notice our assumption implies $S(X, Y, X, Y) = 0$
By the proof of Corollary 4.3 , we have $S(X, Y, Z, W) = 0$

□

Up to now ,we haven't made use of the second Bianchi identity . Proposition 4.5 (5)

$$\nabla R(W, Z, X, Y, V) + \nabla R(W, Z, Y, V, X) + \nabla R(W, Z, V, X, Y) = 0$$

In fact , it leads to the following Schur's theorem.

Theorem 4.2. *(Schur) Let (M, g) be a connected Riemannian manifold of dimension $n \geq 3$. If*

$$K(\pi_p) = f(p) \tag{*}$$

depends only on p ,then (M, g) is of constant curvature.

Remark 4.10. *(1) Thm 4.2 is obviously not true for (M, g) with dim =2. We know in that case (*) always holds, but M need not be of constant curvature.*

(2) Thm4.2 says that the isometry of a Riemannian manifold ,i.e., the property that at each point all directions are geometrically indistinguishable, implies the homogeneity , i.e., that all points are geometrically indistinguishable. In particular, a pointwise propert implies a global one .

Before proving Thm 4.2 , we prepare the following lemma.

Lemma 4.1. *The tensor G is parallel ,i.e. $\nabla G = 0$.*

Proof. For any $X, Y, Z, W, V \in \Gamma(TM)$, we have

$$\begin{aligned}(\nabla_V G)\,(X, Y, Z, W) =&\ V(\langle X, Z\rangle\langle Y, W\rangle - \langle X, W\rangle\langle Y, Z\rangle)\\ &- \langle\nabla_V X, Z\rangle\,\langle Y, W\rangle - \langle X, \nabla_V Z\rangle\,\langle Y, W\rangle\\ &-\langle X, Z\rangle\,\langle\nabla_V Y, W\rangle - \langle X, Z\rangle\,\langle Y, \nabla_V W\rangle\\ &- \langle\nabla_V X, W\rangle\,\langle Y, Z\rangle - \langle X, \nabla_V W\rangle\,\langle\, Y, Z\rangle\\ &- \langle X, W\rangle\,\langle\nabla_V Y, Z\rangle - \langle X, W\rangle\,\langle Y, \nabla_V Z\rangle\end{aligned}$$

By

$$V(\langle X, Z\rangle\,\langle Y, W\rangle) = V(\langle X, Z\rangle)\,\langle Y, W\rangle + \langle X, Z\rangle \cdot V\,\langle Y, W\rangle$$

and compatibility of ∇ with g ,we conclude

$$(\nabla_V G)(X, Y, Z, W) = 0$$

□

Proof. Proof of Thm 2:(An application of the second Bianchi Identity).

By assumption and Proposition 4.10 ,we have

$$R = fG,\ when\ f : M \longrightarrow \mathbb{R}$$

Lemma 4.1 above tells $\nabla G = 0$ Hence for all $V \in \Gamma(TM)$,
we have $\nabla_V R = \nabla_V(fG) = V(f)G$
By the second Bianchi Identity ,we have

$$\begin{aligned}0 &= \nabla_V R(W, Z, X, Y) + \nabla_X R(W, Z, Y, V) + \nabla_Y R(W, Z, V, X) \qquad (*)\\ &= V(f)G(W, Z, X, Y) + X(f)G(W, Z, Y, V) + Y(f)G(W, Z, V, X)\end{aligned}$$

for any $X, Y, Z, W, V \in \Gamma(TM)$

Since it is a tensor identity , the RHS only depends on $X_p, Y_p, Z_p, W_p, V_p \in T_pM$. Since $dim(M) \geq 3$, we can pick $X_p, Y_p, V_p \in T_pM$ such that .

$$\left\langle X_p, Y_p \right\rangle = \left\langle X_p, V_p \right\rangle = \left\langle Y_p, V_p \right\rangle = 0$$

and $X_p \neq 0, Y_p \neq 0, |V_p| = 1$
then (*) implies

$$\begin{aligned} O = V_p(f)\Big(\langle W_p, X_p\rangle\langle Z_p, Y_p\rangle - \langle W_p, Y_p\rangle\langle Z_p, X_p\rangle\Big) \\ + X_p(f)\Big(\langle W_p, Y_p\rangle\langle Z_p, V_p\rangle - \langle W_p, V_p\rangle\langle Z_p, Y_p\rangle\Big) \\ + Y_p(f)\Big(\langle W_p, V_p\rangle\langle Z_p, X_p\rangle - \langle W_p, X_p\rangle\langle Z_p, V_p\rangle\Big) \end{aligned}$$

Recall , we still have freedom for the choice of W_p, Z_p .
Let us set $Z_p = V_p$,then

$$0 = X_p(f)\left\langle W_p, Y_p \right\rangle - Y_p(f)\left\langle W_p, X_p \right\rangle$$

for $\forall W_p \in T_pM$
Hence $0 = X_p(f)Y_p - Y_p(f)X_p$
However ,$\left\langle X_p, Y_p \right\rangle = 0$. That is

$$X_p(f) = Y_p(f) = 0, \forall X_p \neq 0, Y_p \neq 0$$

So f must be a constant function on M.

□

4.4 Ricii Curvature and Scalar curvature

The Ricci curvature tensor is defined to be

$$Ric(Y, Z) := tr(X \mapsto R(X, Y)Z)$$

Notice that at p

$$R(\cdot, Y)Z : T_pM \longrightarrow T_pM$$

is a linear map between vector spaces.
In local coordinate $(u, x^1, x^2, ..., x^n)$,we have

$$Ric_{pq} := Ric\left(\frac{\partial}{\partial x^p}, \frac{\partial}{\partial x^q}\right) = tr\left(X \mapsto R\left(X, \frac{\partial}{\partial x^p}\right)\frac{\partial}{\partial x^q}\right) = \sum_j R^j_{qjp}$$

Moreover,we have

$$\begin{aligned}\sum_j R^j_{qjp} &= \sum_{i,j} g^{ij} g_{il} R^l_{qjp} \\ &= \sum_j g^{ij} R_{iqjp} \\ &= \sum_{i,j} g^{ij} \left\langle R\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^p}\right)\frac{\partial}{\partial x^q}, \frac{\partial}{\partial x^i}\right\rangle \\ &= tr\left\langle R(\cdot, \frac{\partial}{\partial x^p})\frac{\partial}{\partial x^q}, \cdot\right\rangle \\ &= trR\left(\cdot, \frac{\partial}{\partial x^q}, \cdot, \frac{\partial}{\partial x^p}\right)\end{aligned}$$

Therefore

$$Ric(Y,Z) = trR(\cdot, Y, \cdot, Z)$$

(as a (0,2)-tensor)

Recall the trace of a (0,2)-tensor from the section on Hessian .

In particular ,we observe that

$$Ric(Y,Z) = Ric(Z,Y)$$

That is ,Ric is a symmetric (0,2)-tensor fiels on M.

Definition 4.3. *(Ricci curvature) The Ricci curvature at p in the direction $X_p \in T_pM$ is defined as*

$$Ric(X_p) := Ric(X_p, X_p)$$

Remark 4.11. *Ricci curvature is again NOT a function on M . We can think of the Ricci curvature as a function defined on one-dimensional subspace of T_pM.*

We can ask similar questions about Ricci curvature as in the case of sectional curvature : What information do we lose when restricting the Ricci curvature tensor to $Ric(X, X$? The answer is again that we do not lose anything .

Lemma 4.2. *Let T be a symmetric 2-tensor ,then for any X, Y ,we have*

$$T(X,Y) = \frac{1}{2}(T(X+Y, X+Y) - T(X,X) - T(Y,Y))$$

Hence

$$Ric(X_p, Y_p) = \frac{1}{2}(Ric(X_p + Y_p) - Ric(X_p) - Ric(Y_p))$$

We actually should have normalized the length of the vector along which the Ricci curvature is calculated. After that , Ricci curvature is defined on "tangent directions"

$$Ric\left(\frac{X}{\|X\|}\right) := \frac{Ric(X)}{g(X,X)} = \frac{Ric(X,X)}{g(X,X)}$$

Definition 4.4. *The Ric manifold is called an Einstein manifold with Einstein constant k, if*

$$Ric(X) = kg(X, X), \forall X \in \Gamma(TM)$$

i,e., M has "constant Ricci curvature".

Remark 4.12. *Let $X_p \in T_pM$ be an unit tangent vector . Extend it to be an othornormal basis $\{X_p, e_2, ..., e_n\}$ of T_pM . Then*

$$\begin{aligned} Ric(X_p) &= trR(\cdot, X_p, \cdot, X_p) \\ &= \sum_{i=2}^{n} R(e_i, X_p, e_i, X_p) \\ &= \sum_{i=2}^{n} K(e_i, X_p) \end{aligned}$$

In particular , if (M, g) has constant curvaturek,then (M, g) is Einstein with Einstein constant $(n-1)k$

Proposition 4.11. *A Riemannian manifold is an Einstein constant k if and only if*

$$Ric = kg$$

Proof. Define $T(X, Y) = Ric(X, Y) - kg(X, Y)$. Hence T is symmetric. By assumption $T(X, X) = 0$ Lemma 4.2 tells $T(X, Y) = 0$ i.e. $Ric = kg$ □

We also have the following version of Schur's theorem.

Theorem 4.3. *(Schur) Let (M, g) be a connected Riemannian manifold of dimesion ≥ 3. If $Ric(X_p) = f(p)g(X_p, X_p), \forall X_p \in T_pM$, where $f(p)$ depends only on p ,then (M, g) is Einstein.*

Proof. Apply the second Bianchi identity in the same manner as in Theorem 4.2

Step 1: By Proposition 4.11 , the assumption implies

$$Ric = fg$$

Note for Levi-Civita connection, we have automatically

$$\nabla g = 0$$

Hence $\forall V \in \Gamma(TM)$, we have

$$\nabla_V Ric = V(f)g$$

Step 2: Apply 2nd Bianchi Identity . At $p \in M$,pick as normal coordinate $(u, x^1, ..., x^n)$, we have for $X_p, Y_p, V_p \in T_pM$

$$\begin{aligned}\nabla_V \operatorname{Ric}\left(X_p, Y_p\right) &= V\left(\operatorname{Ric}\left(X_p, Y_p\right)\right) - Ric\left(\nabla_{V_p} X_p, Y_p\right) - Ric\left(X_p, \nabla_X Y_p\right)\\ &= V_p\left(\sum_{i=1}^n R\left(\frac{\partial}{\partial x^i}, X_p, \frac{\partial}{\partial x^i}, Y_p\right)\right)\\ &- \sum_{i=1}^n R\left(\frac{\partial}{\partial x^i}, \nabla_{V_p} X_p, \frac{\partial}{\partial x^i}, Y_p\right)\\ &- \sum_{i=1}^n R\left(\frac{\partial}{\partial x^i}, X_p, \frac{\partial}{\partial x^i}, \nabla_{V_p} Y_p\right)\\ &= \sum_{i=1}^n \left(\left(\nabla_{V_p} R\right)\left(\frac{\partial}{\partial x^i}, X_p, \frac{\partial}{\partial x^i}, Y_p\right)\right)\end{aligned}$$

(We used $\nabla_{V_p} \frac{\partial}{\partial x^i} = 0$ since in normal coordinate)
Second Bianchi identity implies

$$\begin{aligned}0 &= \sum_{i=1}^n \left[\nabla_{V_p} R(\frac{\partial}{\partial x^i}, X_p, \frac{\partial}{\partial x^i}, Y_p) + \nabla_{\frac{\partial}{\partial x^i}} R(\frac{\partial}{\partial x^i}, X_p, Y_p, V_p) + \nabla_{Y_p}(\frac{\partial}{\partial x^i}, X_p, V_p, \frac{\partial}{\partial x^i})\right]\\ &= \nabla_{V_p} Ric(X_P, Y_p) - \nabla_{Y_p} Ric(X_p, V_p) + \sum_{i=1}^n \nabla_{\frac{\partial}{\partial x^i}} R(\frac{\partial}{\partial x^i}, X_p, Y_p, V_p)\\ &= V_p(f) g(X_p, Y_p) - Y_p(f) g(X_p, V_p) + \sum_{i=1}^n \nabla_{\frac{\partial}{\partial x^i}} R(\frac{\partial}{\partial x^i}, X_p, Y_p, V_p)\end{aligned}$$

Step 3: Pick special X_p, Y_p, V_p
Let $X_p = Y_p = \frac{\partial}{\partial x^j}$, $V_p = \frac{\partial}{\partial x^h}$
We have

$$0 = \frac{\partial f}{\partial x^h} - \frac{\partial f}{\partial x^i} \delta_{jh} + \sum_{i=1}^n \nabla_{\frac{\partial}{\partial x^i}} R(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^h})$$

Summing j from 1 to n,

$$\begin{aligned}0 &= n \cdot \frac{\partial f}{\partial x^h} - \frac{\partial f}{\partial x^h} - \sum_{i=1}^n \sum_{j=1}^n \nabla_{\frac{\partial}{\partial x^i}} R(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^h})\\ &= (n-1)\frac{\partial f}{\partial x^h} - \sum_{i=1}^n (\nabla_{\frac{\partial}{\partial x^i}} Ric)(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^h})\\ &= (n-1)\frac{\partial f}{\partial x^h} - \sum_{i=1}^n \frac{\partial f}{\partial x^i} g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^h})\\ &= (n-2)\frac{\partial f}{\partial x^h}\end{aligned}$$

Hence, when $n \geq 3$, we have $\frac{\partial f}{\partial x^h} = 0, \forall h = 1, 2..., n$
This implies that $f \equiv$ constant .

□

Remark 4.13. *(1): In fact ,Theorem 4.3 implies 4.2 . Notice that $K(\pi_p) = f(p)$ depends only on p implies*

$$\frac{Ric(X_p)}{g(X_p, X_p)} = f(p), depends\ only\ on\ p \Longrightarrow \frac{Ric(X_p)}{g(X_p, X_p)} \equiv constant(By\ Theorem\ 4.3) \quad \text{(a)}$$

$$\frac{Ric(X_p)}{g(X_p, X_p)} = f(p), depends\ only\ on\ p\ implies\ \frac{Ric(X_p)}{g(X_p, X_p)} = (n-1)f(p) \quad \text{(b)}$$

(a)+(b) $\Longrightarrow K(\pi_p) = f(p) \equiv constant$
(2) In [BSSG] ,(M, g) is called Einstein if

$$Ric(X_p) = f(p)g(X_p, X_p)$$

when $f(p)$ depends only on p . By Theorem 4.3 ,there is no difference from our definition in case dim ≥ 3

[BSSG] 's notation has the preperty that "any 2-dim Riemannian manifold in Einstein".

Definition 4.5. *(Scalar curvature) The scalar curvature S is defined as the trace of the Ricci curvature tensor (which is a symmetric (0,2)-tensor),i.e.*

$$S = g_{ij} Ric_{ij} = trRic(\cdot, \cdot)$$

Remark 4.14. *(1) S is indeed a function on M.*
(2) Let $\{e_1, ..., e_n\}$ be an orthonormal basis of T_pM , we have

$$\begin{aligned} S(p) = tr(Ric)(p) &= \sum_{i=1}^{n} Ric(e_i, e_i) = \sum_{i=1}^{n} Ric(e_i) \\ &= \sum_{i=1}^{n} trR(\cdot, e_i, \cdot, e_i) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} R(e_j, e_i, e_j, e_i) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} K(e_j, e_i) \\ &= 2 \sum_{i<j} K(e_i, e_j) \end{aligned}$$

(3) If (M, g) is of constant curvature k, we have

$$Ric = (n-1)kg, and\ S = n(n-1)k$$

If (M, g) is Einstein with Einstein constant k , we have

$$S = nk$$

Proposition 4.12. *An $n(\geq 3)$ -dimensional Riemannian manifold (M, g) is Einstein iff*

$$Ric = \frac{S}{n} g$$

Proof. $\Longrightarrow$ By definition .

$\Longleftarrow Ric = \frac{S}{n} g$ where $\frac{S}{n}(p)$ depends only on p. Schur's Theorem $(n \geq 3) \Longrightarrow \frac{S}{n} \equiv$ *constant*

□

In particular , Ricci curvature provides less information than sectional curvature, and scalar curvature provides even less information than Ricci curvature . But in dimension 2 or 3 , something special happens.

$n = 2$, $K(\pi_p) = \frac{Ric(X_p)}{g(X_p, X_p)} = 2S(p)$

There is no difference from an information point of view in knowing $K, Ric,\ or S$.

Let $\{e_1, e_2, e_3\}$ be an orthonormal basis for $T_p M$, then

$$\begin{cases} K(e_1, e_2) + K(e_1, e_3) = Ric(e_1) \\ K(e_1, e_2) + K(e_2, e_3) = Ric(e_2) \\ K(e_1, e_3) + K(e_2, e_3) = Ric(e_3) \end{cases}$$

In other words,

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} K(e_1, e_2) \\ K(e_2, e_3) \\ K(e_1, e_3) \end{pmatrix} = \begin{pmatrix} Ric(e_1) \\ Ric(e_2) \\ Ric(e_3) \end{pmatrix} \qquad (**)$$

Notice that

$$\det \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = 2 \neq 0$$

Therefore any sectional curvature can be computed from Ric .

Proposition 4.13. *(M^3, g) is <u>Einstein</u> iff (M^3, g) has <u>constant sectional curvature</u> .*

Proof. $\Longleftarrow$ By definition

$\Longrightarrow$ Solving (**) for the case $Ric(e_1) = Ric(e_2) = Ric(e_3)$ we obtain $K(e_1, e_2) = K(e_2, e_3) = K(e_1, e_3)$

□

But for scalar curvature , when $n = 3$, there are metrics with constant scalar curvature that are not Einstein.

We will see whether the (sectional, Ricci , scalar) curvatures of Riemannian manifolds are constant , or more generally although not constant but still bounded by some inequalities have much implications to the analysis, geometry and topology of (M, g).

In particular , we explain the terminology "Ric $\geq k$", (Ricci curvature is lower bounded), this means more precisely that

$$Ric(X) = Ric(X, X) \geq kg(X, X), \forall X$$

We have discussed several times that a symmetric (0,2)-tensor has a "corresponding" linear transformation ,since

$$\begin{aligned}
\operatorname{Ric}\left(X_p, X_p\right) &= \sum_i R\left(e_i, X_p, e_i, X_p\right) \\
&= \sum_i \left\langle R(e_i, X_p)X_p, e_i\right\rangle \\
&= \sum_i \left\langle R(X_p, e_i)e_i, X_p\right\rangle \\
&= \left\langle \#Ric\left(X_p, \cdot\right), X_p\right\rangle
\end{aligned}$$

$$\Longrightarrow \#Ric(X_p, \cdot) = \sum_i R(X_p, e_i)e_i$$

$$X_p \mapsto \sum_i R(X_p, e_i)e_i$$

is a linear transformation between T_pM and T_pM. The condition "$Ric \geq k$" is equivalent to say all eigenvalues of $X_p \mapsto \sum_i R(X_p, e_i)e_i$ are $\geq k$.

Let us mentions the following theorem of Lohkamp.

Theorem (Lohkamp , Annuls of Math . 140(1994),655-683) For each manifold M^n, $n \geq 3$, there is a complete metric g_M with

$$-a(n)g_M < Ric(g_M) < -b(n)g_M$$

with constants $a(n) > b(n) > 0$ depending only on the dimension n.

Theorem 4.4. *(Lohkamp)For each manifold M^n, $n \geq 3$, there is a complete metric g_M with negative Ricci curvature and finite volume. That is , there are NO topological obstructions for negative Ricci curvature metrics.*

4.5 The Second Variation : Revisited [JJ,4.1] [WSY,chap 6]

Recall from section 1 that the curvature tensor is closely related to the second variation of the energy functional (and the length functional) of a normal geodesic . In this section , we will discuss some geometric and topological implications when assuming curvature restrictions via applying SVF .

Let γ be a normal geodesic , i.e. ,$|\dot{\gamma}| = 1$. Consider a variation

$$F : [a,b] \times (-\epsilon, \epsilon) \longrightarrow M$$
$$(t, v) \mapsto F(t, v)$$

(i.e. F is smooth and $F(t,0) = \gamma(t)$)
The variational field $V(t) = \frac{\partial F}{\partial v}(t,0)$ is a vector field along γ.

Definition 4.6. *(geodesic variation) . The variation F is called a geodesic variation if each curve $\gamma_v(t) := F(t,v)$ is a geodesic*

Next , we recall briefly the second variation formula from section 1. For the one-parameter family of curvs $\{\gamma_v\}_{v\in(-\epsilon,\epsilon)}$, we have $E(v) := E(\gamma_v)$ be a function on $(-\epsilon, \epsilon)$ .Since $\gamma_0 = \gamma$ is a geodesic , we have $E'(0) = 0$.

$$\begin{aligned}
\frac{\partial^2}{\partial v^2} E(v) &= \int_a^b \left(\left\langle \nabla_{\frac{\partial}{\partial v}} \nabla_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial v}, \frac{\partial F}{\partial t} \right\rangle + \left\langle \nabla_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial v}, \nabla_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial v} \right\rangle \right) dt \\
&= \int_a^b \left\langle R\left(\frac{\partial F}{\partial v}, \frac{\partial F}{\partial t}\right)\frac{\partial F}{\partial v}, \frac{\partial F}{\partial t} \right\rangle + \left\langle \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial v}} \frac{\partial F}{\partial v}, \frac{\partial F}{\partial t} \right\rangle + \left\langle \nabla_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial v}, \nabla_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial v} \right\rangle dt \\
&= -\int_a^b \left\langle R\left(\frac{\partial F}{\partial v}, \frac{\partial F}{\partial t}\right)\frac{\partial F}{\partial t}, \frac{\partial F}{\partial v} \right\rangle dt + \int_a^b \frac{\partial}{\partial t} \left\langle \nabla_{\frac{\partial}{\partial v}} \frac{\partial F}{\partial v}, \frac{\partial F}{\partial t} \right\rangle - \left\langle \nabla_{\frac{\partial}{\partial v}} \frac{\partial F}{\partial v}, \nabla_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial t} \right\rangle dt \\
&\quad + \int_a^b \left\langle \nabla_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial v}, \nabla_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial v} \right\rangle dt
\end{aligned}$$

Proposition 4.14. *Let F be a geodesic variation of a curve $\gamma : [a,b] \longrightarrow M$ Then*

$$\frac{\partial^2}{\partial v^2} E(v) = \int_a^b \left(\left\langle \nabla_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial v}, \nabla_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial v} \right\rangle - \left\langle R\left(\frac{\partial F}{\partial v}, \frac{\partial F}{\partial t}\right)\frac{\partial F}{\partial t}, \frac{\partial F}{\partial v} \right\rangle \right) dt$$

Proof. Use the fact $\nabla_{\frac{\partial}{\partial v}} \frac{\partial F}{\partial v} = 0$ since F is a geodesic variation

□

In particular , for a geodesic variation of a normal geodesic $\gamma : [a,b] \longrightarrow M$,we have

$$\frac{\partial^2}{\partial v^2}\Big|_{v=0} E(v) := E''(0) = \int_a^b (\langle \nabla_T V, \nabla_T V \rangle - \langle R(V,T)T, V \rangle)\, dt$$

Observe that when M has nonpositive sectional curvature , we have

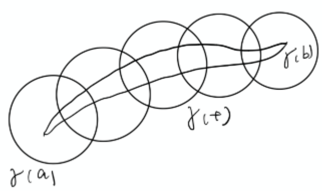
$$-\left\langle R\left(\frac{\partial F}{\partial v}, \frac{\partial F}{\partial t}\right)\frac{\partial F}{\partial t}, \frac{\partial F}{\partial v} \right\rangle = -K\left(\frac{\partial F}{\partial v}, \frac{\partial F}{\partial t}\right) G\left(\frac{\partial F}{\partial v}, \frac{\partial F}{\partial t}, \frac{\partial F}{\partial v}, \frac{\partial F}{\partial t}\right) \geq 0$$

Hence $\frac{\partial^2}{\partial v^2} E(v) \geq 0$,for $v \in (-\epsilon, \epsilon)$. This tells immediately:

Corollary 4.4. *On a Riemannian manifold with nonpositive sectional curvature , geodesics with fixed endpoints are always locally minimizing*

Remark 4.15. *Here a geodesic γ is locally minimizing means that for rhis $\gamma : [a,b] \longrightarrow M$ there exists some $\delta > 0$,such that for any smooth curve $c : [a,b] \longrightarrow M$ with $c(a) = \gamma(a), c(b) = \gamma(b)$, $d(\gamma(t), c(t)) \leq \delta, \forall t \in [a,b]$ we have $E(c) \geq E(\gamma)$*

Proof. For each $t \in [a,b]$,Let δ_t be the parameter of the totally normal neighborhood W_t of $\gamma(t)$ (That is ,$\forall q \in W_t$,exp_q) Since $\gamma([a,b])$ is compact , we can find a finite subcover of the cover $\{exp_{\gamma(t)} B(0,\delta_t)\}_{t\in[a,b]}$. Hence we can find a positive number $\delta > 0$ for $\gamma : [a,b] \longrightarrow M$ such that $\delta_t \geq \delta, \forall t \in [a,b]$



Let $c : [a,b] \longrightarrow M$ be another curve s.t. $d(\gamma(t), c(t)) \leq \delta$,$\forall t \in [a,b]$
Construct the variation as
$F(t,s) = exp_{\gamma(t)} s \cdot exp^{-1}_{\gamma(t)}(c(t)),\ t \in [a,b].s \in [-1,1]$
Notice that $F(t,0) = \gamma(t), F(t,1) = c(t)$
F is a geodesic variation (and proper)
F is proper and γ is a geodesic $\Longrightarrow E'(0) = 0$
F is a geodesic variation $\Longrightarrow E''(s) \geq 0, s \in [-1,1]$
Recall Taylor's expansion of an one-variablw smooth functional, we have

$$E(1) = E(0) + E'(0) + \int_0^1 (1-t)E''(t)dt \geq E(0)$$

That is $E(c) \geq E(\gamma)$

□

Remark 4.16. *(1) Note that the "locally minimizing energy " also implies "locally minimizing length" From the proof above , for any curve $c : [a,b] \longrightarrow M$ close to the normal geodesic $\gamma(t)$, we can reparametrize $c : [a,b] \longrightarrow M$,s.t.*

Exercise 4.1. *Let $\gamma : [a,b] \longrightarrow M$ be a smooth curve , and*

$$F : [a,b] \times (-\epsilon,\epsilon) \times (-\delta,\delta) \longrightarrow M$$
$$(t,v,w) \mapsto F(t,v,w)$$

be a 2-parameters variation of γ . Denote by

$$V(t) = \frac{\partial F}{\partial v}(t,0,0),\ W(t) = \frac{\partial F}{\partial w}(t,0,0)$$

the two corresponding variational field.

(1) Show that

$$\frac{\partial^2}{\partial w \partial v} L(v,w) = \int_a^b \frac{1}{\|\frac{\partial F}{\partial t}\|} \Big\{ \left\langle \nabla_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial v}, \nabla_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial w} \right\rangle - \left\langle R(\frac{\partial F}{\partial w}, \frac{\partial F}{\partial t}) \frac{\partial F}{\partial t}, \frac{\partial F}{\partial v} \right\rangle + \left\langle \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial w}} \frac{\partial F}{\partial v}, \frac{\partial F}{\partial t} \right\rangle - \frac{1}{\|\frac{\partial F}{\partial t}\|^2} \left\langle \nabla_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial v}, \frac{\partial F}{\partial t} \right\rangle \left\langle \nabla_{\frac{\partial}{\partial v}} \frac{\partial F}{\partial w}, \frac{\partial F}{\partial t} \right\rangle \Big\} dt$$

where $\|\frac{\partial F}{\partial t}\| = \left\langle \frac{\partial F}{\partial t}, \frac{\partial F}{\partial t} \right\rangle^{\frac{1}{2}}$
(2) Let γ *be a mpr,a; geodesic , i.e.,*$\|\dot{\gamma}\| = 1$. *Show that*

$$\frac{\partial^2}{\partial w \partial v}\Big|_{w=v=0} L(v,w) = \int_a^b (\langle \nabla_T V, \nabla_T W \rangle - \langle R(W,T)T, V \rangle - T\langle V,T \rangle T \langle W,T \rangle)\, dt + \langle \nabla_W V, T \rangle \Big|_a^b$$

where $T(t) := \dot{\gamma}(t)$ *is the velocity field along* γ
(3) Consider the orthogonal component $\widetilde{V}, \widetilde{W}$ *of* V, W *with respect to* T *, that is*

$$V^{\perp} := V - \langle V, T \rangle T$$

$$W^{\perp} := W - \langle W, T \rangle T$$

Show that

$$\frac{\partial^2}{\partial w \partial v}\Big|_{(0,0)} L(v,w) = \int_a^b \left(\left\langle \nabla_T V^{\perp}, \nabla_T W^{\perp} \right\rangle - \left\langle R(W^{\perp}, T)T, V^{\perp} \right\rangle \right) dt + \langle \nabla_W V, T \rangle \Big|_a^b$$

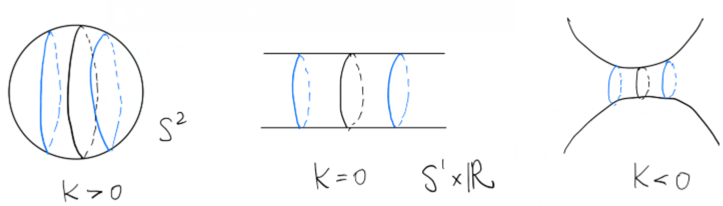
Remark 4.17. *Observe in the above proof, the "properness " of the variation F is only used to conclude* $E'(0) = 0$. *When we consider variation of closed geodesics, i.e. a geodesic*

$$\gamma : S^1 \longrightarrow M$$

When S^1 *is the unit circle parametrized by* $[0, 2\pi)$.
(in fact , $\gamma : S^1 \longrightarrow M$, $\gamma(0) = \gamma(2\pi)$, $\dot{\gamma}(0) = \dot{\gamma}(2\pi)$ *), the argument in the proof still works.*

Corollary 4.5. *On a Riemannian manifold with nonpositive (negative,resp) sectional curvature , closed geodesics are locally minimiing .(strict local minima,resp)*

Notice that on a manifold with vanishing curvature , closed geodesics are still locally minimizing , but not necessarily strictly so any more .On a manifold with positive curvature , closed geodesics in general do not minimize anymore.($\star$) The following picture is very imspiring .



We will derive a global consequence of this fact (★).

We give a general remark about how (SVF) implies minimizing property of geodesics.

Let $\gamma : [a,b] \longrightarrow M$ be a normal geodesic , F be a variation of γ , we have

$$\frac{d^2}{dv^2}\Big|_{v=0} E(v) = \langle \nabla_V V, T \rangle \Big|_a^b + \int_a^b (\langle \nabla_T V, \nabla_T V \rangle - \langle R(V,T)T, V \rangle)dt$$

where V is the variational field and T is the velocity field along γ.

Geometrically speaking , when F is a proper variation , or $\gamma : [a,b] \longrightarrow M$ is a closed geodesic ,($\Longrightarrow \gamma(a) = \gamma(b),\ T(a) = T(b)$),

we have $\frac{d^2}{dv^2}\Big|_{v=0} E(v) = \int_a^b (\langle \nabla_T V, \nabla_T V \rangle - \langle R(V,T)T, V \rangle)dt$

(1) If M has nonpositive (negative ,resp) curvature , E"(0) ≥ 0 (E"(0) > 0,resp) $\Longrightarrow \gamma$ is (strictly) locally minimizing .

(2) If M has positive curvature , -$\langle R(V,T)T, V \rangle < 0$.

If

$$\langle R(V,T)T, V \rangle > \langle \nabla_T V, \nabla_T V \rangle \qquad (\star)$$

then E"(0) < 0, and hence γ cannot be (locally) minimizing .

The philosophy if (2) leads to the applications of (SVF) we will discuss soon.

Synge Theorem ($\Longleftarrow$) When M is compact , orientable , even-dimension, of positive curvature , for any nontrivial closed geodesic γ , we can find V ,s.t. (★) holds.

That is , under the assumptions, any nontrivial (not homotopic to constant curve) geodesic can not be locally minimizing.

Bonnet-Myers Theorem ($\Longleftarrow$) When M is of sectional curvature $\geq k > 0$, geodesics of length $> \frac{\pi}{\sqrt{K}}$ can not be (locally) minimizing .

Next, let us discuss this two applications in more detail.

Synge Theorem [WSY, Chap 6][JJ, Chap 4,4.1][dC,Chap 9,3]

Lemma 4.3. *Let (M,g) be an , orientable, even-dimensional Riemannian manifold with positive sectional curvature. Then any closed geodesic which are not homotopic to a constant curve can not be minimizing in its (free) homotopy class.*

Lemma 4.4. *Let (M,g) be a compact Riemannian manifold . Then every (free) homotopy class of closed curve in M contains a shortest one (which is , therefore, a closed geodesic) [JJ,Theorem 1.5.1]*

Remark 4.18. *(1) A closed curve c can be considered as a continuous map $c : S^1 \longrightarrow M$, where S^1 is the unit circle.*

Recall that two contimuous maps

$$c_0, c_1 : S^1 \longrightarrow M$$

are called homotopic if there exists a continuous map

$$F : S^1 \times [0,1] \longrightarrow M$$

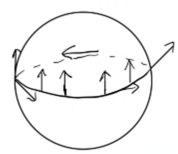
with $F(t,0) = c_0(t).F(t,1) = c_1(t), \forall t \in S^1$

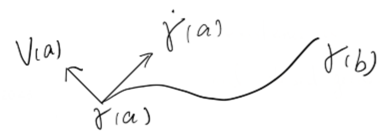
And the concept of homotopy defines an equivalence relation of all closed curves in M.

(2) Suppose (M, g) satisfy both the assumptions of Lemma 4.3 and that of Lemma 4.4 , then every homotopy class of closed curves in M contains the constant curve . That is , M us simply connected , i.e.,$\pi_1(M) = \{1\}$. This is exactly what Synge Theorem says.

Theorem 4.5. *[Synge,1936,On the connectivity of spaces of positive curvature ,Quartely Journal of Mathematics (Oxford Series),7,316-320] Any compact, orientable, even-dimensional Riemannian manifold,positive curvature is simply connected.*

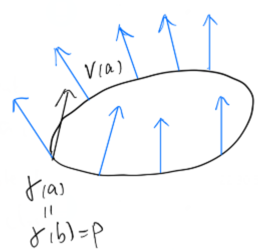
Now we start to prove Lemma 4.3 . In fact the restricitions "orientable , even-dimensional" guarantee the existence a parallel normal vector field along a nontrivial closed geodesic.





If $\gamma : [a, b] \longrightarrow M$ is not a closed one , it is not hard to find a parallel normal vector field along it . Just pick a vector $V(a) \in T_{\gamma(a)}M$, $\langle V(a), \dot{\gamma}(a)\rangle = 0$ and $V(t)$ is given by the parallel transport along γ.

But for a closed one , $V(b) = P_{\gamma.a.b}V(a)$ is not necessarily coincide with $V(a)$.



Note for the velocity field along a closed geodesic γ , we have

$$P_{\gamma,a,b}\dot{\gamma}(a) = \dot{\gamma}(b) = \dot{\gamma}(a)$$

That is , the orthogonal linear transformation

$$P_{\gamma,a,b} : T_pM \longrightarrow T_pM \ (p = \gamma(a) = \gamma(b))$$

has an eigenvalue +1 with eigenvector $\dot{\gamma}(a)$

If the multiplicity of eigenvalue $+1 \geq 2$, then we have a vector $V(p) \in T_pM$ lying in the orthogonal complement of $\dot{\gamma}(a)$ s.t. $P_{\gamma,a,b}V(p) = V(p)$.

Hence $V(t) := P_{\gamma,a,t}V(a)$ gives a parallel normal vector field along γ.

Next , we explain "orientable , even-dimensional " guarantee that the multiplicity of eigenvalue +1 of $P_{\gamma,a,b} \geq 2$.

Since $P_{\gamma,a,b}$ is orthogonal, we have det(V)=+1 or -1

Lemma 4.5. *If det($P_{\gamma,a,b}$ =)=+1,and M is even-dimensional then the multiplicity of the eigenvalue +1* ≥ 2

Proof. Since $P_{\gamma,a,b} : T_pM \longrightarrow T_pM$ is orthogonal , its eigenvalues can be listed as

$$\lambda_1, \bar{\lambda}_1, ..., \lambda_j, \bar{\lambda}_j, \underbrace{-1, -1, \cdots, -1}_{k}, \underbrace{1, 1, \cdots, 1}_{l}$$

where $\lambda_i, i = 1, 2, ..., j$ are complex numbers with $|\lambda_i| = 1$,

$$\begin{aligned} M\ even - dimensional &\Longrightarrow T_pM\ even - dimensional \\ &\Longrightarrow k + l\ is\ even \end{aligned}$$

Since $P_{\gamma,a,b} : \dot{\gamma}(a) \mapsto \dot{\gamma}(b) = \dot{\gamma}(a), l \geq 1 (i.e.\ l \neq 0)$
Hence l is even and $l \neq 0$. That is $l \geq 2$ □

In fact , det($P_{\gamma,a,b}$)=+1 is guaranteed by " orientability " of M . Let us recall briefly the concept of orientability.

Given a vector space V, let $\{e_i\}_{i=1}^n$ and $\{f_j\}_{j=1}^n$ be two basis , and $f_j = a^i_j e_i$. Then det(a^i_j) is either positive or negative. If det(a^i_j)¿0, we say the two basis have the same orientation . This defines an equivalence relation for all basis of V. There exactly 2 equivalant classes. We call each of them an orientation of V .

Alternatively , the orientation of V can be described as below : Consider the dual space V^* of V . Then we have

$$dim\ \Lambda^n(V^*) = 1$$

and let $\Omega(f_1, ..., f_n) = det(a^i_j)\Omega(e_1, e..., e_n)$.

That is , given a non-zero $\Omega \in \Lambda^n(V^*)$, two basis $\{e_i\}$,$\{f_j\}$ have the same orientation iff $\Omega(f_1, ..., f_n)$ and $\Omega(e_1, ..., e_n)$ have the same sign . In this sense , a nonzero $\Omega \in \Lambda^n(V^*)$ determines an orientation of V.

The second way of description can be generalized to the setting of a manifold. M is orientable if there exists a C^∞ nowhere vanishing n-form ω. At each $p \in M$, the basis of T_pM are divided into two classes, those with $\omega(e_1, ..., e_n) > 0$ and those with $\omega(e_1, ..., e_n) < 0$. The first class is called the basis coherent with the orientation .

Lemma 4.6. *Let* (M, g) *be an orientable Riemannian manifold and* $\gamma : [a, b] \longrightarrow M$ *be a closed curve . Then the parallel transport* $P_{\gamma,a,b} : T_pM \longrightarrow T_pM$ *has determinant 1.*

Proof. Since $P_{\gamma,a,b}$ is orthogonal , we only need to show

$$det(P_{\gamma,a,b}) > 0$$

Let ω be a C^∞ nowhere vanishing n-form ω on M , whose existence is guaranteed by orientability . Let $\{e_i\}$ be a basis of T_pM with $\omega(e_1, ..., e_n) > 0$.

Let $\{e_i(t)\} := \{P_{\gamma,a,b}(e_i)\}$ be the parallel transport of $\{e_i\}$ along γ . Then $t \mapsto \omega(e_1(t), ..., e_n(t))$ is a nowhere vanishing c^∞ function on [a,b] . In particular , $\omega(e_1(b), ..., e_n(b)) > 0$.

Note $\{e_i(b)_{i=1}^n\}$ is also a basis of T_pM , and

$$\omega(e_1(b),...,e_n(b)) = det(P_{\gamma.a.b})\omega(e_1(b),...,e_n).$$

Therefore, we have

$$det(P_{\gamma.a.b}) > 0$$

□

Proof of Lemma 1

Let $\gamma : [a,b] \longrightarrow M$ be a nontrivial closed geodesic in M (let $p = \gamma(a) = \gamma(b)$).

By lemma 4.5 and 4.6 , there exists $V(p) \in T_pM$,$\langle V(p), \dot{\gamma}(a)\rangle\, 0$ and $P_{\gamma,a,b}V(p) = V(p)$

Therefore $V(t) := P_{\gamma,a,b}V(p)$ is a parallel normal vector field along γ .

Since $\gamma([a,b])$ is compact , there exists $\delta > 0$,s.t.

$$\begin{aligned} F :&[a,b] \times (-\delta,\delta) \longrightarrow M \\ &(t,v) \mapsto exp_{\gamma(t)}vV(t) \end{aligned}$$

is a (geodesic) variation of γ. (existence of δ is shown by the argument we used in the proof of Corollary 4.4)

Since γ is a geodesic, we have $E'(0) = 0$. Moreover,

$$\begin{aligned} E\text{''}(0) &= \int_a^b (\langle\nabla_T V, \nabla_T V\rangle - \langle R(V,T)T, V\rangle)dt \\ &(\nabla_T V = 0 \textit{ since } V \textit{ is parallel}) \\ &= -\int_a^b \langle R(V,T)T, V\rangle\, dt < 0 \textit{ since sectional curvature } > 0 \end{aligned}$$

Therefore, for $v \neq 0$ small enough, $\gamma_v : [a,b] \longrightarrow M$ is a closed curve homotopic to γ but with $E(\gamma_v) < E(\gamma)$.

That is γ is not minimizing in its homotopy class. (In fact , for length we also have $l(\gamma_v)^2 \leq 2(b-a)E(\gamma_v) < 2(b-a)E(\gamma) = l(\gamma)^2$(since $\|\dot{\gamma}\| \equiv constant$))

Lemma 4.4 is a general result for compact Riemannian manifold (no curvature restriction is needed).

Proof of Lemma 4.4 Recall from Corollary (2.3) of Chapter 2 , that for a compact Riemannian manifold M , there exists a $\rho_0 > 0$, s.t. any $p,q \in M$ with $d(p,q) \leq \rho_0$ can be conneted by precisely one geodesic of shortest path.(Recall this is proved by using the concept of totally normal neighborhood).

Moreover,the geodesic depends continuously on (p,q). This implies immediately.

Claim. Let (M,g) be a compact Riemannian manifold, and $\rho_0 > 0$ be chosen as above. Let $\gamma_0, \gamma_1 : S^1 \longrightarrow M$ be closed curves with

$$d(\gamma_0(t), \gamma_1(t)) \leq \rho_0,\ \forall t \in S^1$$

Then γ_0, γ_1 are homotopic.

For any $t \in S^1$, let $c_t(s) : [0,1]$ be the unique shortest geodesic from $\gamma_0(t)$ to $\gamma_1(t)$. (paramatrized proportionally arclength). Since c_t depends continuously on its end points. The map

$$F(t,s) := c_t(s)$$

is contimuous and yields the desired homotopy.

Next we find the shortest curve in a given homotopy class by method of minimizing sequence .

Let $\{\gamma_n\}_{n\in\mathbb{N}}$ be a minimizing sequence for length in the given homotopy class. Here and in the sequal , all curves are parametrized proportionally to arc length. $\gamma_n : [0,2\pi] \longrightarrow M$.

We may assume each γ_n us piecewise geodesic. This is because: there exists m, and

$$0 = t_0 < t_1 < ... < t_m < t_{m+1} = 2\pi$$

s.t.,$L(\gamma_n|_[t_{j-1}, t_j]) \leq \rho_0/2$

(This is realizable since one can equally divide [0,2π],s.t.

$$|t_j - t_{j-1}| = \frac{\rho_0}{2|\dot{\gamma}|}, j = 1, ..., m, \; |t_{m+1} - t_m| < \frac{\rho_0}{2|\dot{\gamma}|}$$

and $m = \left\lceil \frac{2\pi}{\frac{\rho_0}{2|\dot{\gamma}|}} \right\rceil$)

Then replacing $\gamma_n\Big|_{[t_{j-1},t_j]}$ by the shortest geodesic arc from $\gamma_n(t_{j-1})$ to $\gamma_n(t_j)$. By the claim , this will not change the homotopy class of the curve.

Equivalently to say, we have a minimizing sequence $\{\gamma_n\}_n$ such that for each γ_n , there exists $p_{0,n}, ..., p_{m,n}$ for which $d(p_{j-1}, p_j) \leq \rho_0/2, \; j = 1, ..., m+1$ with $p_{m+1,n} = p_{0,n}$ and for which γ_n contains the shortest geodesic from p_{j-1} to p_j .

Since $\{\gamma_n\}_n$ is minimizing , the length of γ_n are bounded, say $L(\gamma_n) \leq C$. Then we can assume that m is independent of n. (This is because $L(\gamma_n) \leq C \Longrightarrow |\dot{\gamma}_n| \leq \frac{C}{2\pi} \Longrightarrow m \leq \frac{4\pi|\gamma|}{\rho_0} + 1 \leq \frac{2C}{\rho_0} + 1$)

Since M id compact , after selection of a subsequence , the points $p_{0,n}, ..., p_{m,n}$ converge to points $p_0, ..., p_m$ as $n \longrightarrow \infty$. The segment of γ_n from $p_{j-1,n}$ to $p_{j,n}$ then converges to the shortest geodesic from p_{j-1} to p_j (Recall such geodesic depends continuously on its end points).

The union of these geodesic segments yields a closed curve γ . By the claim ,γ is still in the given homotopy class and $L(\gamma) = \lim_{n\longrightarrow\infty} L(\gamma_n)$,i.e. ,$\gamma$ is the shortest one in its homotopy class. Therefore ,γ has to be geodesic.(Otherwise, there exists p and q on γ on which one of the two segments of γ) from p to q has length $\leq \rho_0/2$, but is not geodesic. Then replace this segment by the unique shortest geodesic from p to q. The resulting curve lies still in the same homotopy class but with a shorter length , which is a contradiction.)

Proof of Thoerem 4.5 Suppose M is not simply connected. Then there is a homotopy class of closed curves which are not homotopic to a constant curve. By Lemma 4.4 , there is a shortest closed geodesic γ in this given homotopy class . By Lemma 4.3 ,γ cannot be minimizing , this is a contradiction.

Remark 4.19. *(1) Synge theorem tells that any compact , orientable, even-dimensional manifold which is not simply connected does not admit a metric of positive curvature.*

(2) Examples: the real projective spave $P^2(\mathbb{R})$ of dimension two, is compact, non-orientable. Recall in Excercise (3),2(Homework 3 for 2017) ,we checked there is a Riemannian metric on $P^2(\mathbb{R})$,s.t. the covering map $\pi : S^2 \longrightarrow P^2(\mathbb{R})$ is a local isometry. Hence $P^2(\mathbb{R})$ has sectional curvature 1¿0.

But we know $\pi_1(P^2(\mathbb{R})) = \mathbb{Z}_2$. Hence "orientability" in the assumption of Synge Theorem is necessary.

Similarly, "evem-dimensional" assumption is also necessary. $P^3(\mathbb{R})$ is orientable,compact, odd-dimensional , of positive curvature , but $\pi_1(P^3(\mathbb{R})) = \mathbb{Z}_2$.

The above examples are inspiring and it is natural to ask what we can say when (M, g) is not orientable or not even-dimensional.

Corollary 4.6. *Let (M, g) be a compact,non-orientable,even-dimensional Riemannian manifold of positive sectional curvature, then $\pi_1(M) = \mathbb{Z}_2$.*

Theorem 4.6. *(Synge 1936) Let (M, g) be a compact , odd-dimensional Riemannian manifold of positive sectional curvature, then M is orientable.*

Remark 4.20. *In particular, Corollary 4.6 gives a geometric proof of the fact $\pi_1(P^n(\mathbb{R})) = \mathbb{Z}_2$, when n even (knowing $P^n(\mathbb{R})$ is non-orientable for n is even . Although $\pi_1(P^n(\mathbb{R})) = \mathbb{Z}_2$, $\forall n$).Theorem 4.6 gives a geometric proof of the fact $P^n(\mathbb{R})$ is orientable for n odd. But we can not say too much about the fundamental group $\pi_1(M)$ for a compact,odd-dimensional manifold admitting a metric of positive curvature.*

The proofs use property of the orientable double cover of a non-orientable manifold. In order not to interupt our current topic too much , we postpone the proofs.

Bonnet-Myers Theorem :[PP,Chap 6,4.1]

The following lemma was first priven by Bonnet for surfaces and later by Synge for general Riemannian manifolds as an application of his (SVF).

Lemma 4.7. *(Bonnet 1855 and Synge 1926) Let(M, g) be a Riemannian manifold with sectional curvature $\leq k > 0$. Then geodesics of length ¿ $\frac{\pi}{\sqrt{k}}$ cannot be (locally) minimizing*

Proof. Let $\gamma : [0, l] \longrightarrow M$ be a normal (i.e. $|\dot{\gamma}| = 1$) geodesic of length $l > \frac{\pi}{\sqrt{k}}$.

Let $E(0)$ be a unit vector in $T_{\gamma(0)}M$ with $\langle E(0), \dot{\gamma}(0)\rangle = 0$.

Then we obtain $E(t) := P_{\gamma,0,t}E(0)$ a parallel (vector field along γ).

Consider the following vector field along γ

$$V(t) := \sin(\pi t/l)E(t)$$

It corresponds to a proper variation since $V(0) = V(l) = 0$

By (SVF):

$$\frac{d^2}{dv^2}\Big|_{v=0} E(v) = E''(0) = \int_a^b (\langle \nabla_T V, \nabla_T V\rangle - \langle R(V, T)T, V\rangle)dt$$

Observe $\nabla_T V = \sin'(\pi t/l)E(t) = \frac{\pi}{l}\cos(\pi t/l)E(t)$ and hence $\langle \nabla_T V, \nabla_T V\rangle = (\frac{\pi}{l})^2 \cos^2(\pi t/l)$ and

$$\langle R(V,T)T,V\rangle = \sin^2(\pi t/l)\,\langle R(E,T)T,E\rangle = \sin^2(\pi t/l)K(E,T)$$

$$\begin{aligned}
\Longrightarrow E\text{"}(0) &= \int_0^l (\frac{\pi}{l})^2 \cos^2(\pi t/l)dt - \int_0^l \sin^2(\pi t/l)K(E,T)dt \\
&\leq (\frac{\pi}{l})^2 \int_0^l \cos^2(\pi t/l)dt - k\int_0^l \sin^2(\pi t/l)dt \\
&< k\int_0^l \left[\cos^2(\pi t/l) - \sin^2(\pi t/l)\right] dt \; (Since\ l > \frac{\pi}{\sqrt{k}} \Longrightarrow (\frac{\pi}{l})^2 < k) \\
&= k\int_0^l \cos(2\pi t/l)dt = 0
\end{aligned}$$

Hence all nearby curves in the variation are shorter than γ . (by the same argument as in the end of the proof for Lemma 4.3) □

In the above , we see that we actually has (n-1) choices of the parallel normal vector fields along γ. When sectional curvature $\geq k > 0$, our above argument works for each of these $(n-1)$ parallel vector field along γ. On the other hand , for our purpose here, it's enough to know that our above argument works for at least one of those $(n-1)$ vector fields along γ. This leads to the following extension due to Myers.

Lemma 4.8. *(Myers 1941). Let (M,g) be a Riemannian manifold with Ricci curvature Ric$\geq (n-1)k > 0$. Then geodesics of length $> \frac{\pi}{\sqrt{k}}$ cannot be minimizing.*

Proof. Similarly as in the proof of Lemma 4.7. Let $\gamma : [0,l] \longrightarrow M$ be a normal geodesic with $l > \frac{\pi}{\sqrt{k}}$.

Choose $E_2,...,E_n \in T_{\gamma(0)}M$ s.t. $\dot\gamma(0)$, $E_2,...,E_n$ form an orthonormal basis for $T_{\gamma(0)}M$. Then $E_i(t) := P_{\gamma,0,t}E_i$ and $\dot\gamma(t)$ form an orthonormal basis for $T_{\gamma(t)}M$.

Consider $n-1$ variational fields along γ

$$V_i(t) = \sin(\pi t/l)E_i(t), i = 2,3,...,n$$

We have for each i,

$$\frac{d^2}{dv_i^2}\Big|_{v_i=0} E(v_i) = (\frac{\pi}{l})^2 \int_0^l \cos^2(\pi t/l)dt - \int_0^l \sin^2(\pi t/l)K(E_i,T)dt$$

$$< k\int_0^l \cos^2(\pi t/l)dt - \int_0^l \sin^2(\pi t/l)K(E,T)dt$$

Taking the summation,

$$\sum_{i=2}^{n} \frac{d^2}{dv_i^2}\Big|_{v_i=0} E(v_i) < (n-1)k \int_0^l \cos^2(\pi t/l)dt - \int_0^l \sin^2(\pi t/l) \underbrace{\sum_i K(E_i, T)}_{Ric(T)} dt$$

$$\leq (n-1)k \int_0^l \cos^2(\pi t/l)dt - (n-1)k \int_0^l \sin^2(\pi t/l)dt$$

$$= 0$$

Hence there exists an $i_0 \in \{2, \cdots, n\}$, s.t.

$$\sum_{i=2}^{n} \frac{d^2}{dv_{i_0}^2}\Big|_{v_{i_0}=0} E(v_{i_0}) < 0$$

And hence γ is not (locally) minimizing . □

If we assume further that (M, g) is complete, the above lemma implies and upper bound of the diameter of (M, g). This seems to have first been pointed out by Hopf-Rinow(1931) for surfaces in their famous paper on completeness and then by Myers for general Riemannian manifolds.

(1935. Duke J. Math for sectional curvature restriction

1941. Duke J. Math for Ricci curvature restriction)

Corollary 4.7. *Suppose (M, g) is a complete Riemannian manifold with Ricci curvature $Ric \geq (n-1)k > 0$. Then*

$$diam(M, g) \leq \frac{\pi}{\sqrt{k}}$$

Further more, (M, g) has finite fundamental group.

Remark 4.21. *Corollary 4.7 is aftenly referred to as Bonnet-Myers Theorem.*

Proof. Lemma 4.8 tells no geodesic can realize distance between any $p, q \in M$ with $d(p, q) > \frac{\pi}{\sqrt{k}}$. Hopf-Rinow Theorem tells that completeness implies any $p, q \in M$ can be connected by a shortest geodesic. Hence $d(p, q) \leq \frac{\pi}{\sqrt{k}}, \forall p, q \in M$ □

Covering spaces and Fundaental groups

A continuous map $\pi : X \longrightarrow M$ is called a covering map if each $p \in M$ has a neighborhood U with the property that each connected component of $\pi^{-1}(U)$ is mapped homeomorphically onto U.

FACT 1: Let M be a differential manifold .X has a natural differentiable structure s.t. $\pi : X \longrightarrow M$ is a C^∞ and locally diffeomorphism.

Let $\{(U_\alpha, x_\alpha)\}$ be a differentiable structure of M . U_α small s.t. $\pi^{-1}(U_\alpha)$ are disjoint open sets of X,each connected component $U_\alpha^i \subset X$, we assign coordinate map $x_\alpha \circ \pi$ (note $\pi : U_\alpha^i \longrightarrow U_\alpha$ is homeomorphism)

This leads to a differentiable structure for X, under which π is C^∞ and locally diffeomorphism.

$$\begin{array}{ccc} \pi^{-1}(U_\alpha) & & \\ \downarrow{\scriptstyle \pi} & \searrow^{x_\alpha \circ \pi} & \\ U_\alpha & \xrightarrow[x_\alpha]{} & \mathbb{R}^n \end{array}$$

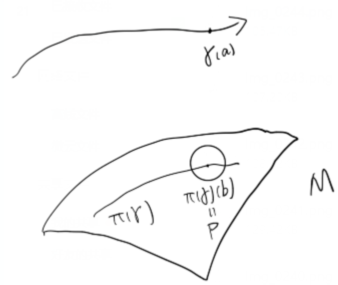
FACT 2 Let (M, g) be a Riemannian manifold . Note π is surjective . We can assign by $\widetilde{g} = \pi^* g$ a Riemannian metric for X . Then $\pi : (X, \widetilde{g} \longrightarrow (M, g))$ is a locally isometry .

FACT 3. If (M, g) is complete , then $(X, \widetilde{g})$ is also complete,

Proof. Suppose γ be a normal geodesic on $(X, \widetilde{g})$ with the maximal interval $[0, b)$, $b < \infty$

Since π is locally isometry, we have $\pi(\gamma) : [0, b) \longrightarrow M$ is a geodesic of (M, g). Since (M, g) is complete , we have the geodesic $\pi(\gamma)$ in (M, g) can be extended to

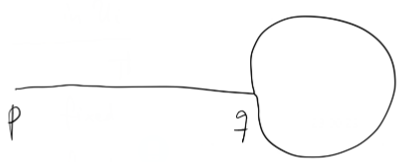
$$\pi(\gamma)(b) := o \in M$$



Pick a small normal neighborhood U of p in M. Then $\exists a < b$ s.t. $\pi(\gamma)(a) \in U$ and $\pi(\gamma)\Big|_{[a,b]} \in U$. Let U_i be the connected component of $\pi^{-1}(U)$ containing $\gamma(a)$.

Then the isometry $\pi^{-1} : U \longrightarrow U_i$ maps a geodesic to a geodesic. Hence the geodesic γ van be extended over b in U_i. This contradicts to the maximality of b $\square$

The equivalene or homotopy classes of closed curves with fixed base point $p \in M$ form a group $\pi_1(M, p)$, the fundamental group of M with base point p.



$\pi_1(M, p)$ and $\pi_1(M, q)$ are isomorphic for any $p, q \in M$. Hence , it make sense to speak of the fundamental group $\pi_1(M)$ of M without reference to a base point.

Let $\pi : X \longrightarrow M$ be a covering map . A deck transformation is a homeomorphism $\varphi : X \longrightarrow X$ with $\pi = \pi \circ \varphi$

FACT 4 A deck transformation φ of $(X, \widetilde{g})$ is an isometry .

Proof. Since π is locally isometry, and $\pi = \pi \circ \varphi$, we know φ is locally isometry. Since φ us homeomorphic , we have φ is an isometry. □

A deck transformation which has a fixed point is the identity.
If $\pi : \widetilde{M} \longrightarrow M$ be the universal covering of M . $\pi(x_0) = p_0 \in M$
(1)$\pi_1(M, p_0)$ is in 1-1 correspondance with $\pi^{-1}(p_0)$.
$x_1 \in \pi^{-1}(p_0)$ corresponds to the homotopy class of $\pi(\gamma_{x_1})$ where $\gamma_{x_1}(0) = x_0, \gamma_{x_1}(1) = x_1$.
(2) The set $\mathcal{D}$ of all deck transformation is in 1-1 correspondance with $\pi_1(M, p_0)$.
Associate each deck transformation φ with $\varphi(p_0) \in \pi^{-1}(p_0)$ So muc for the general facts. Let's come back to our discussion about Bonnet-Myers Theorem and Synge Theorem .

We have shown if (M, g) is a complete Riemannian manifold with Ricci curvature $Ric \geq (n-1)k > 0$.Then

$$diam(M, g) \leq \frac{\pi}{\sqrt{k}}$$

(Corolarry 4.7) and hence , in particular (M, g) is compact .

(The last assertion follows from Hopf-Rinow (The whole manifold is a bounded closed set))

Moreover . (M, g) has finite fundamental group .

Proof for the last statement:

Let $\pi : \widetilde{M} \longrightarrow M$ be the universal covering . From our previous discussion, $(\widetilde{M}, \widetilde{g})$ is a C^∞ Riemannian manifold , and $\pi : (\widetilde{M}, \widetilde{g}) \longrightarrow (M, g)$ is a locally isometry. Hence the Ricci curvature of $(\widetilde{M}, \widetilde{g})$ is also bounded from below by $(n-1)k$

Moreoveer , (M, g) is complete $\Longrightarrow$ $(\widetilde{M}, \widetilde{g})$ is complete. Hence $diam(\widetilde{M}, \widetilde{g}) \leq \frac{\pi}{\sqrt{k}}$ and $(\frac{\pi}{\sqrt{k}})$ is compact.

Then $\forall p \in M,\ \pi^{-1}(p)$ is finite . Since otherwise , $\pi^{-1}(p)$ has an accumulated point $\widetilde{p} \in \widetilde{M}$, and π is not a locally diffeomorphism . Therefore , the fundamental group is finite.

Extension: Cheeger-Gromoll [JDG,1971] $Ric \geq 0$,positive at one point ,thn $\pi_1(M)$ is finite. Next , we discuss Synge Thoerem further.

Firstly , recall the remarks on 4.3 that "orientable","even-dimensional" are all necessary.

By Bonnet-Myers, the "compactness" can be replaced by the assumption that M is complete and has sectional curatures bounded away from 0 .

In fact , the "compactness" alone , due to a theorem of Gromoll-Meyer.

Theorem 4.7. *(Gromoll-Meyer On complete open manifolds of positive curvature , Ann of Math,go(1639),75-90)*

If M is a connected , complete , non-compact n -dimensional manifold with all sectional curvatures positive, then M is diffeomorphic to $\mathbb{R}^n$.

What happens if M is not orientable?
We give a proof of Corollary 4.6

Proof. Every non-orientable differential manifold M has an orientable double cover $\bar{M}$:

A brief description : At each point $p \in M$, the T_pM can be separated into two disjoint sets : Recall the orientation of a vector space, two bases are equivalent if their transformation matrix has determinant > 0 . This is an equivalente relation . Let $\mathcal{O}_p$ be the quotient space of T_pM w.r.t. the equivalence relation. $O_p \in \mathcal{O}_p$ will be called an orientation of T_pM

$$\bar{M} = \{(p, O_p) : p \in M, O_p \in \mathcal{O}_p\}$$

$\bar{M}$ has a natural differentiable structure s.t. $\pi : \bar{M} \longrightarrow M$ is C^∞ and surjective . $\forall p \in M,\ \exists U \in M,\ p \in U$ s.t.

$$\pi^{-1}(U) = V_1 \sqcup V_2$$

$\pi : V_i \longrightarrow U$ is a diffeomorphism [dC. Chap 0, Ex12]

Example:S^2 is the orientable double cover if $P^2(\mathbb{R})$.

Then by our previous discussion , $\bar{M}$ is orientable , and compact , even-dimensional nd positive sectional curvature. Hence $\bar{M}$ is simply connected. Therefore $\pi_1(M) = \mathbb{Z}_2$

□

What happens if not "even-dimensional"

Theorem 4.6

For that purpose, we prove a more general result

Theorem 4.8. *(Weinstein 1968) Let f be an isometry of a compact orientable Riemannian manifold M^n . Suppose that M has positive sectional curvature , and f reverses the orientatiion of M and n is odd. Then f has a fixed point .*

Proof. Suppose to the contrary , $f(q) \neq q, \forall q \in M$

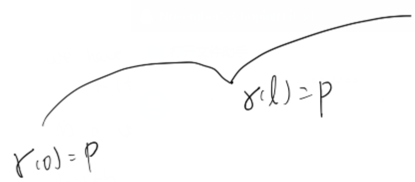
Let $p \in M$ such that $d(p, f(p))$ attains the minimum

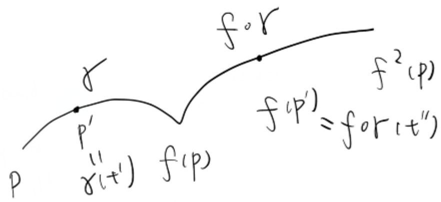
$$d(p, f(p)) = \inf_{q \in M} d(q, f(q))(We\ use\ M\ is\ compact)$$

Since M is compact $\Longrightarrow$ complete, $\exists$ a normal minimizing geodesic

$$\gamma : [0, l] \longrightarrow M,$$

$\gamma(0) = p,\ \gamma(l) = f(p)$ and $l = d(p, f(p))$





Claim: $(f \dot{\circ} \gamma)(0) = \dot{\gamma}(l)$
proof of claim:
Let $p' = \gamma(t'),\ t' \neq 0,\ t' \neq l,\ f(p') = f \circ \gamma(t')$
We have

$$\begin{aligned} d(p', f(p')) &\leq d(p', f(p')) + d(f(p), f(p')) \\ &= d(p', f(p')) + d(p, p') \ \ (Since\ f\ is\ isometry) \\ &= d(p, f(p))\ (Since\ \gamma\ is\ minimal) \end{aligned}$$

Then by $d(p, f(p)) = \inf_{q\in M} d(q, f(q))$, we know the "$\leq$ " is an "=" i.e.,$d(p', f(p')) = d(p', f(p)) + d(f(p), f(p'))$

That is the curve $\gamma\Big|_{[t',l]} \cup f \circ \gamma\Big|_{[0,t']}$ is a shortest curve and hence a geodesic.
In particular , this implies $(f \dot{\circ} \gamma)(0) = \dot{\gamma}(l)$.
Next consider $P^{-1}_{\gamma,0,l} \circ df_p : T_pM \longrightarrow T_pM$
Then it is an isometry and hence , an orthogonal transformation.
Note $df_p(\dot{\gamma}(0)) = (f \dot{\circ} \gamma)(0),\ (since\ f(p) = f \circ \gamma(0))$
We have

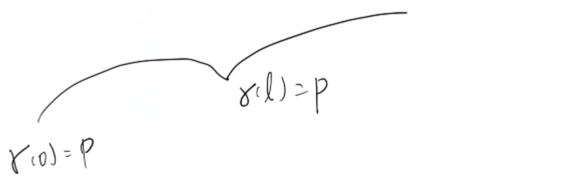
$$\begin{aligned} (P^{-1}_{\gamma,0,l} \circ df_p)(\dot{\gamma}(0)) &= P^{-1}_{\gamma,0,l}((f \dot{\circ} \gamma)(0)) \\ &= P^{-1}_{\gamma,0,l}(\dot{\gamma}(l)) = \dot{\gamma}(0) \end{aligned}$$

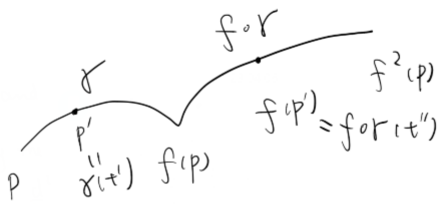
That is $P^{-1}_{\gamma,0,l} \circ df_p$ has eigenvalue $+1$ with multiplicity $ge1$.
SInce $P_{\gamma,0,t}$ preserves orientation and f reverse the orientation , we have $det(P^{-1}_{\gamma,0,l} \circ df_p) = -1$
List all its eigenvalues as

$$\lambda_1, \bar{\lambda}_1, ..., \lambda_j, \bar{\lambda}_j, \underbrace{-1, -1, \cdots, -1}_{l}, \underbrace{1, 1, \cdots, 1}_{k}$$

We have $\left.\begin{array}{l} n \text{ odd } \Rightarrow k + l \text{ odd} \\ \det = -1 \Rightarrow l \text{ odd} \end{array}\right\} \Rightarrow \left.\begin{array}{l} k \text{ even} \\ k \geqslant 1 \end{array}\right\} \Rightarrow k > 2$

$\Rightarrow \exists V \in T_pM, < V, \dot{\gamma}(0) >= 0,$ and $\left(P^{-1}_{\gamma,0,l} \circ df_p\right)(V) = V$
i.e. $P_{\gamma,0,l}V = df_p(V)$





Define $V(t) = P_{\gamma,0,t}V$,
We have $F(t, s) = exp_{\gamma(t)}(sV(t)),\ s \in (-\epsilon, \epsilon),\ t \in [0, l]$ is a variation of γ , with

$$F(t, 0) = \gamma(t)$$
$$F(0, s) = \beta(s)$$
$$F(l, s) = f \circ \beta(s)$$
$$and\ \frac{\partial F}{\partial s}\Big|_{s=0} = V(t)$$

By the (SVF), we have

$$\frac{d^2}{ds^2}\Big|_{s=0} E(s) = \int_a^b (\langle \nabla_T V, \nabla_T V\rangle - \langle R(V, T)T, V\rangle)dt < 0$$

This shows that $\exists$ small enough s,s.t. the curve γ_s has the property $L(\gamma_s)^2 \leq 2lE(\gamma_s) < 2lE(\gamma) = L(\gamma)^2$

Hence let $p_s = \gamma_s(0)$, we have

$$d(p_s, f(p_s)) \leq L(\gamma_s) < L(\gamma) = d(p, f(p))$$

which contradicts to the minimality of $d(p, f(p))$

□

Proof of Theorem 4.6

Suppose M is not orientable , let $\bar{M}$ be the orientable double cover of M. Then $(\bar{M}, \pi^* g)$ is a compact orientable manifold with positve sectional curvature . Let φ be a deck transformation of $\bar{M}$ with $\varphi \neq id$.

Because M is not orientable ,φ is an isometry which reverse the orientation of $\bar{M}$. Since n is odd, we can apply Weinstein's theorem to conclude φ has a fixed point . Therefore $\varphi = id$, which is a contradicition.

Exercise 4.2. *(1)Prove Weinstein Theorem for even-dimensional case: Let f be an isometry of a compact orientable Riemannian manifold M^n. Suppose M has positive sectional curvature , and f preserve the orientation of M and n is even. Then f has a fixed point.*

(2) Prove Synge Theorem (even-dimensional) as a corollary.

Chapter 5

Space forms and Jacobi fields

We start our further investigation on geometry and topology of Riemannian manifolds by studying the simplest cases: complete Riemannian manifolds with constant sectional curvature, which are called space forms. We again will study the behavior of geodesics in order to reveal the underlying geometry and topology.

The first problem we're concerned about space forms is the existence. Recall if a Riemannian manifold (M, g) has constant sectional curvature k, then $(M, \lambda g)$ has constant sectional curvature $\frac{k}{\lambda}$ for $\lambda > 0$. Therefore, we only need to consider space forms with sectional curvature 0,+1,-1.

Obviously, R^n with the Euclidean metric has 0 sectional curvature.(For example, by the formula in local coordinate:

$$\langle R(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})\frac{\partial}{\partial x^l}, \frac{\partial}{\partial x^k}\rangle = \frac{1}{2}(\frac{\partial^2 g_{jk}}{\partial x^i \partial x^l} - \frac{\partial^2 g_{ik}}{\partial x^j \partial x^l} - \frac{\partial^2 g_{jl}}{\partial x^i \partial x^k} + \frac{\partial^2 g_{il}}{\partial x^j \partial x^k}) + g_{mp}(\Gamma^m_{il}\Gamma^p_{jk} - \Gamma^m_{jl}\Gamma^p_{ik})$$

recall from the previous discussion. For $\mathbb{R}^n$, $R_{klij} = 0$, $\forall i, j, k, l$.) Hence $\mathbb{R}^n$ is a space form with sectional curvature 0. In fact, we have the following result:

Theorem 5.1. *For any $c \in \mathbb{R}$ and any $n \in \mathbb{Z}^+$, there exists a unique(upto isometries) simply connected n-dimensional space form with constant sectional curvature c.*

In order to discuss the existence for the other two cases $c = +1$ or -1, we first introduce same useful ideas.

5.1 Isometries and totally geodesic submanifold

Let (M, g), $(\overline{M}, \overline{g})$ be two Riemannian manifolds, and $f : M \to \overline{M}$ be an immersion. If $f^*\overline{g} = g$, then we say f is a isometric immersion and M is called the Riemannian embedded submanifold, or regular submanifold.

Let $dim\, M = n$, $dim\, M = n + k$, we say M has codimension k in $\overline{M}$. In particular, if $k = 1$, M is called a hypersurface in $\overline{M}$.

Definition 5.1. *(totally geodesic submanifold). Let M be a submanifold of $\overline{M}$. We identify $p \in M$ with $f(p) \in \overline{M}$. Then $T_p\overline{M} = T_pM \bigoplus T_p^{\perp}M$, where $T_p^{\perp}M$ is the orthonormal complement of T_pM in $T_p\overline{M}$.*

M is called a totally geodesic submanifold if $\forall$ geodesic γ in $\overline{M}$ with $\gamma(0) \in M$, $\dot{\gamma}(0) \in T_pM$, we have $\gamma \subset M$.

Remark 5.1. *Recall from the Final Remark of our discussions about Levi-Civita connection, we know for the Levi-Civita connection $\overline{\nabla}$ and ∇ for $\overline{M}$ and M respectively, we heve*

$$\overline{\nabla}_{\dot{\gamma}}\dot{\gamma} = 0 \Rightarrow \nabla_{\dot{\gamma}}\dot{\gamma} = 0.$$

That is, γ is also a geodesic in M.

There is a characterization of totally geodesic submanifold by the second fundamental form.

In fact, the decomposition $T_p\overline{M} = T_pM \bigoplus T_p^{\perp}M$ is differentiable, and, hence, the tangent bundle $T\overline{M} = TM \bigoplus NM$ where NM is the normal bundle.

For any $X, Y \in \Gamma(TM)$, define

$$B(X, Y) = \overline{\nabla}_X Y - \nabla_X Y \in \Gamma(NM).$$

First, we observe that $\forall$function f on M, we have

$$\left.\begin{array}{l} B(fX, Y) = fB(X, Y) \ \ (easy) \\ B(X, fY) = fB(X, Y) \end{array}\right\} \tag{5.1.1}$$

We also have $B(X, Y) = B(Y, X).(using torsion-free property.)$ B is called the second fundamental form of the submanifold M in $\overline{M}$.

Theorem 5.2. *M is a totally geodesic submanifold of $\overline{M}$ if and only if $B \equiv 0$.*

Proof. Due to the property (5.1.1), we can speak of the map for all p

$$B : T_pM \times T_pM \to N_pM$$

which is bilinear and symmetric. Let M be a totally geodesic submanifold of $\overline{M}$, then $\forall V \in T_pM$, let γ be the geodesic in $\overline{M}$ with $\gamma(0) = p, \dot{\gamma}(0) = V$. $\overline{\nabla}_{\dot{\gamma}}\dot{\gamma} = 0$, then we have $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$.

That is

$$\overline{\nabla}_{\dot{\gamma}}\dot{\gamma} - \nabla_{\dot{\gamma}}\dot{\gamma} = \overline{\nabla}_v v - \nabla_v v = B(v, v) = 0.$$

Since B is bilinear and symmetric, we have

$$B(v, v) = 0, \ \forall v \in T_pM,$$

Conversely, suppose $B \equiv 0$. Then $\forall p \in M, V \in T_pM$, let γ be a geodesic in $\overline{M}$ with $\gamma(0) = p, \dot{\gamma}(0) = V$. Let ξ be the geodesic in M. Due to the uniqueness of the geodesic with initial data $\gamma(0), \dot{\gamma}(0)$, we conclude $\xi = \gamma$. Hence $\gamma \subset M$. ■ □

Remark 5.2. *Notice that by $\overline{\nabla}_{\dot\gamma}\dot\gamma$, $\nabla_{\dot\gamma}\dot\gamma$ we are actually using the induced connection, One can check $B \equiv 0 \Rightarrow \overline{\nabla}_{\dot\xi(t)}\dot\xi$* ■

Totally geodesic submanifold can be considered as a generalization of geodesics. $\mathbb{R}^k \hookrightarrow \mathbb{R}^{k+1}$ is a totally geodesic submanifold but $S^2 \subset R^3$ is not.

we have

Proposition 5.1. *Let M be a totally geodesic submanifold of $\overline{M}$, denote by K and $\overline{K}$ for their sectional curvature respectively. Then every 2-dim sections $\pi_p T_p M$ and any $p \in M$, we have $K(\pi_p) = \overline{\pi_p}$.*

Proof. By definition and Theorem 6.2.9. ■ □

Next, we give a relation between isometry and totally geodesic submanifold.

Theorem 5.3. *[WSY, p59, Lemma 3] Let $f : (\overline{M}, g) \to (M, g)$ be an isometry. Then every connected component of the fixed point set $M = Fix(f) = \{p \in \overline{M} | f(p) = p\}$ is totally geodesic submanifold.*

Proof. Observe that $Fix(f)$ is a closed subset: It is the preimage of the diagonal in $\overline{M} \times \overline{M}$ under the differentiable mapping $p \mapsto (p, f(p)) \in M \times M$. Let $p \in Fix(f)$. If p is not isolated, consider $H = \{v \in T_p\overline{M}$.

Let δ be small enough s.t.

$$\underset{p}{exp} : B(0,\delta) \subset T_p\overline{M} \to B_p(\delta) \subset \overline{M}$$

is a diffeomorphism.

Claim: $exp_p(H \bigcap B(0,\delta)) = M \bigcap B_p(\delta)$. This claims implies immediately that M is submanifold of $\overline{M}$.

Proof of the claim:

(1) $\forall q \in M \bigcap B_p(\delta)$, choose $V \in B(0,\delta) \subset T_pM$, s.t. $exp_p V = q$ and $\gamma : [0,1] \to \overline{M}$, $\gamma(t) = exp_p tV$ is the unique shortest geodesic. By our previous discussion, $V \in H$. Hence $M \bigcap B_p(\delta) \subset exp_p(H \bigcap B(0,\delta))$.

(2) Let $V \in H \bigcap B(0,\delta)$, let $q = exp_p V$. Let $\gamma : [0,1] \to \overline{M}$ be the geodesic $\gamma(t) = exp_p tV$, then $\gamma(0) = p$, $\gamma(1) = q$. Then $f \circ \gamma$ is also a geodesic with $(f \circ \gamma)(0) = f(p) = p$. Moreover $(f \dot\circ \gamma)(0) = df_p(\dot\gamma(0)) = df_p(V) = V = \dot\gamma(0)$, then by uniqueness, $f \circ \gamma = \gamma$, and in particular

$$f(q) = f \circ \gamma(1) = \gamma(1) = q.$$

Hence $exp\, PV \subset B_p(\delta) \bigcap M$ i.e. $exp(H \bigcap B(0,\delta)) = B_p(\delta) \bigcap M$. This complete the proof of the claim. ■

The above arguement (2) also tells that any geodesic γ in $\overline{M}$ with $\gamma(0) \in M$, $\dot\gamma(0) \in T_{\gamma(0)}M$ satisfies $f(\gamma) = \gamma$. Hence M is a totally geodesic submanifold M. ■ □

5.2 Space forms

We continue the discussion about the existence of sapce forms with sectional curvature +1 or -1.

Example 5.1. *$S^2 \subset \mathbb{R}^3$ with the induced metric of the Euclidean metric of $\mathbb{R}^3$. Since*

$$\begin{cases} x = r\cos\varphi\cos\theta \\ y = r\cos\varphi\sin\theta \\ z = r\sin\varphi \end{cases} \tag{5.2.1}$$

$g|_{S^2} = (dx^2 + dy^2 + dz^2)|_{S^2} = d\varphi^2 + \cos^2\varphi d\theta^2.(r = 1)$ *Then*

$$\langle\frac{\partial}{\partial\theta},\frac{\partial}{\partial\theta}\rangle\langle\frac{\partial}{\partial\varphi},\frac{\partial}{\partial\varphi}\rangle - \langle\frac{\partial}{\partial\theta}\rangle,\frac{\partial}{\partial\varphi}^2 = \cos^2\varphi.$$

$$\langle R(\frac{\partial}{\partial\varphi},\frac{\partial}{\partial\theta})\frac{\partial}{\partial\theta},\frac{\partial}{\partial\varphi}\rangle$$

$$=\frac{1}{2}(g_{\theta\varphi,\varphi\theta} - g_{\varphi\varphi,\theta\theta} - g_{\theta\theta,\varphi\varphi} + g_{\varphi\theta,\theta\varphi}) + g_{mp}(\Gamma^m_{\varphi\theta}\Gamma^p_{\theta\varphi} - \Gamma^m_{\theta\theta}\Gamma^p_{\varphi\varphi})$$

$$(check\ \Gamma^{\varphi}_{\varphi\theta} = \Gamma^{\theta}_{\theta\theta} = \Gamma^{\varphi}_{\varphi\varphi} = 0,\ \Gamma^{\theta}_{\varphi\theta} = -\frac{\cos\varphi\sin\varphi}{\cos^2\varphi}.) = \quad -\frac{1}{2}g_{\theta\theta,\varphi\varphi} + g_{\theta\theta}\Gamma_{(\varphi\theta)^{\theta}}\Gamma^{\theta}_{\theta\varphi} = \cos^2\varphi$$

$\Rightarrow$ *sectionalcurvature* $K \equiv 1$.

Proposition 5.2. *The unit sphere $S^n \subset \mathbb{R}^{n+1}(n \geq 2)$ has constant sectional curvature +1.*

Proof. $n = 2$ has been checked in Example 1.

When $n \geq 3$, define an isometry $f : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ as below

$$f : (x^1, x^2, x^3, -x^4, \cdots, x^{n+1}) = (x^1, x^2, x^3 - x^4, \cdots, -x^{n+1}).$$

It induce an isometry $f : S^n \to S^n$.

Observe the set of fixed point of $f : S^n \to S^n = \{(x^1, x^2, x^3, 0, \cdots, c)| \sum_{i=1}^{3} x^{i^2} = 1\} = S^2$. Therefore, S^2 is a totally geodesic submanifold of S^n. Since sectional curvature of S^2 is 1, we have S^n has sectional curvature $K(\pi_p) = 1$ for some $\pi_p \subset T_pS^2$. For any $\pi'_q \subset T_qS^n$. Suppose $\pi_p = \{e_1, e_2\}$, the positive vector of p be e_{n+1}, $\pi'_q = span\{e'_1, e'_2\}$, the positive vector of q be e'_{n+1}. First let rotate ϕ in $span\{e_{n+1}, e'_{n+1}\}$ be s.t. $\phi(e_{n+1}) = e'_{n+1}$ and $d\phi(\pi_p) = \pi_q$. Then let ϕ' be the ratation which fix q and send π_q to π'_q. Then the isometry $\phi' \circ \phi$ send p to q, and $\pi_p \subset T_pS^n$ to $\pi'_q \subset T_qS^n$. Hence $K(\pi'_q) = 1$. ■ □

Proposition 5.3. *The unit ball $B^n = \{x \in \mathbb{R}^n|||x||\langle 1\} \subset R^n$ with the hyperbolic metric*

$$g = \frac{4}{(1 - \sum_i (x^i)^2)^2} \sum_i dx^i \otimes dx^i$$

is a space form with constant sectional curvature -1.

Proof. First, we show $(B^n, g) := H^n$ is complete. Consider the curve $\gamma(s) := (\frac{e^s-1}{e^s+1}, 0, \cdots, 0)$. We compute

$$\begin{aligned}\langle\dot{\gamma}(s), \dot{\gamma}(s)\rangle_g &= \frac{4}{(1-(\frac{e^s-1}{e^s+1})^2)^2}(\frac{\partial}{\partial s}(\frac{e^s-1}{e^s+1}))^2 \\ &= \frac{4}{(\frac{4e^s}{(e^s+1)^2})^2}(\frac{e^s(e^s+1)-(e^s-1)e^s}{(e^s+1)^2})^2 \\ &= \frac{(e^s+1)^4}{4e^{2s}}\frac{(2e^s)^2}{(e^s+1)^4} = 1\end{aligned}$$

That is γ is paramertrized by arc length.

Observation: Any orthonormal transformation of $\mathbb{R}^n$ induces an isometry $(B^n, g) \to (B^n, g)$.

Let $f : B^n \to B^n$ be the isometry induced by $(x^1, x^2, \cdots, x^n) \mapsto (x^1, -x^2, \cdots, -x^n)$. Note $Fix(f) = \gamma((-\infty, \infty))$. By Theorem 6.2.10, γ is a geodesic. Use the Observation againm $A(\gamma)$ is a geodesic for any isometry A induced by a orthonormal transformation $\mathbb{R}^n \to \mathbb{R}^n$. That is all geodesic starting from 0 can be defined on $[0.\infty)$. We conclude the completeness by Hopf-Rinow Theorem.

Next,we show $\mathbb{H}^n$ has constant sectional curvature -1: i.e. $\forall p \in B_n$, $\forall$ 2-dim section $\pi_p \subset T_pB_n$, we have to show $K(\pi_p) = -1$.

Let $\vec{p}$ be the position vector of p. Identifying T_pB^n with $\mathbb{R}^n$. Let E be a 3-dimensional linear subspace of $\mathbb{R}^n$ containing $\vec{p}$ and π_p. If $\vec{p} \in \pi_p$, or $\vec{p} = 0$, the choice of E is not unique. The reason we have to consider such a $3-dim$ subspace: There is no obvious way to say $\mathbb{R}^n$ is homogeneous, i.e. $\forall p, q \in B^n$, $\exists$ isometry $f : B^n \to B^n$ s.t. $f(p) = f(q)$. However, we will show this later.

Let $\mathbb{R}^n = E \bigoplus E^\perp$, let $f : B^n \to B^n$ be the isometry induced by the orthonormal transformation

$$(e, e^1) \mapsto (e, -e^1),\ e \in E,\ e^\perp \in E^\perp.$$

Observe $\mathbb{F}\beth\curvearrowleft(f) = E \bigcap B^n$. Use the Observation again, choose orthonormal transformation A s.t. $A(E) = \{(x_1, x_2, x_3, 0, \cdots, 0)\} \subset \mathbb{R}^n$. A induce an isometry $B^n \to B^n$. Hence, it remains to show B^3 with the hyperbolic metric has constant sectional curvature -1.

Use the spherical coordinate $\{\rho, \varphi, \theta\}$ on $B^3 \setminus \{0\}$, the hyperbolic metric can be written as

$$\frac{4}{(1-\rho^2)^2}(d\rho^2 + \rho^2 d\theta^2 + \rho^2\cos^2\theta d\varphi^2)$$

where $d\rho^2 := d\rho \otimes d\rho$ and, similarly, $d\theta^2$, $d\varphi^2$.

Consider vector fields

$$X_1 = \frac{1-\rho^2}{2}\frac{\partial}{\partial\rho},\ X_2 = \frac{1-\rho^2}{2\rho}\frac{\partial}{\partial\theta},\ X_3 = \frac{1-\rho^2}{2\rho\cos\theta}\frac{\partial}{\partial\varphi}.$$

Then we have $\langle X_i, X_j\rangle = \delta_{ij}$. We calculate

$$\begin{aligned}
[X_1, X_2](f) &= \frac{1-\rho^2}{2}\frac{\partial}{\partial\rho}(\frac{1-\rho^2}{2\rho}\frac{\partial f}{\partial\theta}) - \frac{1-\rho^2}{2\rho}\frac{\partial}{\partial\theta}(\frac{1-\rho^2}{2}\frac{\partial f}{\partial\rho}) \\
&= \frac{1-\rho^2}{2}\frac{\partial}{\partial\rho}(\frac{1-\rho^2}{2\rho})\frac{\partial f}{\partial\theta} + \frac{1-\rho^2}{2}\frac{1-\rho^2}{2\rho}\frac{\partial^2 f}{\partial\rho\partial\theta} - \frac{1-\rho^2}{2\rho}\frac{1-\rho^2}{2}\frac{\partial^2 f}{\partial\theta\partial\rho} \\
&= \frac{1-\rho^2}{2}\frac{\partial}{\partial\rho}(\frac{1-\rho^2}{2\rho})\frac{\partial}{\partial\theta}f \\
\Rightarrow [X_1, X_2] &= \frac{1-\rho^2}{2}\frac{\partial}{\partial\rho}(\frac{1-\rho^2}{2\rho})\frac{\partial}{\partial\theta} = \frac{1-\rho^2}{2}\frac{-(\rho^2+1)}{2\rho^2}\frac{\partial}{\partial\theta}. \\
&(\frac{\partial}{\partial\rho}(\frac{1-\rho^2}{2\rho}) = \frac{-2\rho(2\rho) - 2(1-\rho^2)}{4\rho^2} = \frac{-(\rho^2+1)}{2\rho^2})
\end{aligned}$$

$$\Rightarrow [X_1, X_2] = -\frac{1+\rho^2}{2\rho}X_2. \tag{1}$$

Similarly,

$$[X_2, X_3] = +\frac{1-\rho^2}{2\rho}\tan\theta X^3 \tag{2}$$

$$[X_1, X_3] = -\frac{-(\rho^2+1)}{2\rho^2}X_3 \tag{3}$$

Recall for orthonormal vector fields X, Y, Z, we have by koszul formula

$$2\langle\nabla_X Y, Z\rangle = -\langle X, [Y, Z]\rangle + \langle Y, [Z, X]\rangle + \langle Z, [X, Y]\rangle.$$

Employing (1),(2), and (3), we have

$$\begin{aligned}
\nabla_{X_1}X_1 &= \nabla_{X_1}X_2 = \nabla_{X_1}X_3 = 0 \\
\nabla_{X_2}X_2 &= -\frac{\rho^2+1}{2\rho}X_1,\ \nabla_{X_2}X_3 = 0 \\
\nabla_{X_3}X_3 &= -\frac{\rho^2+1}{2\rho}X_1 + \frac{1-\rho^2}{2\rho}\tan\theta X_2.
\end{aligned}$$

By torsion-free property $(\nabla_X Y - \nabla_Y X) = [X, Y]$, the above information is enough to calculate the sectional curvature.

$$\begin{cases} K(X_1, X_2) = \langle R(X_1, X_2)X_2, X_1\rangle \\ K(X_2, X_3) = \langle R(X_2, X_3)X_3, X_2\rangle \\ K(X_1, X_3) = \langle R(X_1, X_3)X_3, X_1\rangle \end{cases} \tag{5.2.2}$$

$$\begin{aligned}R(X_1,X_2)X_2 =&\nabla_{X_1}\nabla_{X_2}X_2-\nabla_{X_2}\nabla_{X_1}X_2-\nabla_{[X_1,X_2]}X_2\\ =&\nabla_{X_1}(-\frac{\rho^2+1}{2\rho}X_1)-\nabla_{-\frac{1+\rho^2}{2\rho}X_2}X_2\\ =&\frac{1-rho^2}{2}\frac{\partial}{\partial\rho}(-\frac{\rho^2+1}{2\rho})X_1+\frac{1+\rho^2}{2\rho}(-\frac{\rho^2+1}{2\rho})X_1\\ \frac{\partial}{\partial\rho}(\frac{\rho^2+1}{2\rho}) =&\frac{2\rho\,.\,2\rho-2(\rho^2+1)}{4\rho^2}=\frac{2\rho^2-2}{4\rho^2}=\frac{\rho^2-1}{2\rho^2}\\ \Rightarrow R(X_1,X_2)X_2 =&-\frac{1-\rho^2}{2}\frac{\rho^2-1}{2\rho^2}X_1-\frac{(\rho^2+1)^2}{4\rho^2}X_1\\ =&\frac{(\rho^2-1)^2-(\rho^2+1)^2}{4\rho^2}X_1=-X_1.\end{aligned}$$

Hence $K(X_1,X_2)=-1$.

$$\begin{aligned}&R(X_1,X_3)X_3=\nabla_{X_1}\nabla_{X_3}X_3-\nabla_{X_3}\nabla_{X_1}X_3-\nabla_{[X_1,X_3]}X_3\\ =&\nabla_{X_1}(-\frac{\rho^2+1}{2\rho}X_1+\frac{1-\rho^2}{2\rho}\tan\theta X_2)-\nabla_{\frac{1+\rho^2}{2\rho}X_3}X_3\\ =&X_1(-\frac{\rho^2+1}{2\rho})X_1+X_1(\frac{1-\rho^2}{2\rho}\tan\theta)X_2+\frac{\rho^2+1}{2\rho}\nabla_{X_3}X_3\\ =&-\frac{1-\rho^2}{2}\frac{\rho^2-1}{2\rho^2}X_1-\frac{1-\rho^2}{2}\frac{\rho^2+1}{2\rho^2}\tan\theta X_2-\frac{1+\rho^2}{2\rho}\frac{\rho^2+1}{2\rho}X_1+\frac{1+rho^2}{2\rho}\frac{1-\rho^2}{2\rho}\tan\theta X_2\\ =&\frac{(\rho^2-1)^2-(\rho^2+1)^2}{4\rho^2}X_1=-X_1\end{aligned}$$

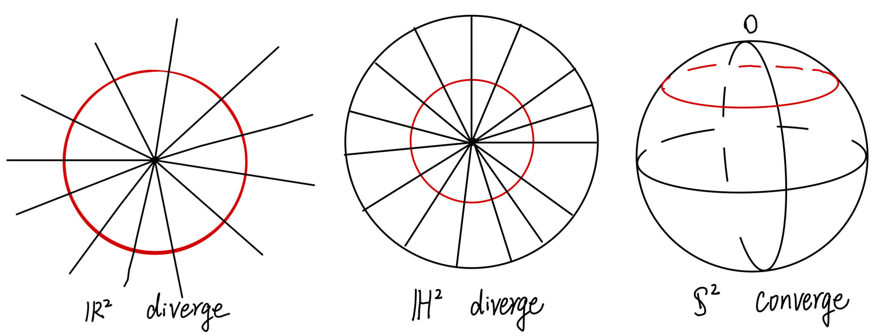
Hence $K(X_1, X_3) = -1$.

$$
\begin{aligned}
R(X_2, X_3)X_3 &= \nabla_{X_2}\nabla_{X_3}X_3 - \nabla_{X_3}\nabla_{X_2}X_3 - \nabla_{[X_2,X_3]}X_3 \\
&= \nabla_{X_2}(-\frac{\rho^2+1}{2\rho}X_1 + \frac{1-\rho^2}{2\rho}\tan\theta X_2) - \nabla_{\frac{1-\rho^2}{2\rho}\tan\theta X_3}X_3 \\
&= \frac{1-\rho^2}{2\rho}\frac{\partial}{\partial\theta}(-\frac{\rho^2+1}{2\rho})\nabla_{X_2}X_1 + \frac{1-\rho^2}{2\rho}\frac{\partial}{\partial\theta}(\frac{1-\rho^2}{2\rho}\tan\theta)X_2 \\
&\quad + \frac{1-\rho^2}{2\rho}\tan\theta\nabla_{X_2}X_2 - \frac{1-\rho^2}{2\rho}\tan\theta\nabla_{X_3}X_3 \\
&= -\frac{\rho^2+1}{2\rho}\cdot\frac{\rho^2+1}{2\rho}X_2 + \frac{(1-\rho)^2}{4\rho^2}\frac{1}{\cos^2\theta}X_2 \\
&\quad - \frac{1-\rho^2}{2\rho}\tan\theta\frac{\rho^2+1}{2\rho}X_1 + \frac{1-\rho^2}{2\rho}\tan\theta\frac{\rho^2+1}{2\rho}X_1 \\
&\quad - \frac{\rho^2+1}{2\rho}\tan\theta\frac{\rho^2+1}{2\rho}\tan\theta X_2 \\
&= -\frac{(\rho^2+1)^2}{4\rho^2}X_2 + \frac{(1-\rho^2)^2}{4\rho^2}[\frac{1}{\cos^2\theta} - \tan^2\theta]X_2 \\
&= \frac{(\rho^2-1)^2 - (\rho^2+1)^2}{4\rho^2}X_2 = -X_2.
\end{aligned}
$$

Hence $K(X_2, X_3) = -1$ ■ □

5.3 Geodesics in $\mathbb{R}^n$, S^n, $\mathbb{H}^n$

Since $\mathbb{R}^2$, S^2, $\mathbb{H}^2$ are totally geodesic submanifold of $\mathbb{R}^n$, S^n, $\mathbb{H}^n$, respectively, we only need to consider geodesics in $\mathbb{R}^2, S^2.\mathbb{H}^2$.



Let us measure the convergence/divergence properties of geodesic emanating from a reference point 0 by the length of the circle

$$c(r) := \{x \in M : d(0, x) = r.$$

When r is small eougn, $c(r)$ is the image of $S(0, r) \subset T_0M$ under the diffeomorphism exp_0, i.e. $c(r) = exp_0 S(0, r)$.

Let $c_0(r),\ c_+(r),\ c_-(r)$ be the length of $c(r)$ in $R^2,\ S^2,\ H^2$, respectively.

(1) $c_0(r) = 2\pi r$ is linear in r.

(2) $c_+(r)$, i.e. $M = S^2$.

(3) $c_-(r)$, i.e. $M = H^2$.(Here, we choose 0 to be the centre of the disc. Later, we will see $c_-(r)$ does not depend on the choice of 0.) Recall the normal geodesic emanating from 0 is given by

$$\gamma : [0,\infty) \to \mathbb{B}^2,\ s \mapsto \frac{e^s - 1}{e^s + 1} \cdot \overline{p},\ \forall \overline{p} \in \partial\mathbb{B}^2.$$

Then

$$\begin{aligned} c_-(r) = &\int_0^{2\pi} \frac{2}{(1-\rho^2)} \cdot \rho d\theta|_{\rho = \frac{e^r-1}{e^r+1}} \\ =&\frac{2 \cdot \frac{e^r-1}{e^r+1}}{1-(\frac{e^r-1}{e^r+1})^2} 2\pi = 2\pi \frac{e^{2r}-1}{2e^r} = 2\pi \frac{e^r - e^{-r}}{2} \end{aligned}$$

$\Rightarrow c_-(r) = 2\pi \sinh r$.

We see that $c_-(r)$ grows much faster than $c_0(r)$.

In the above 3 particular cases, we see the sign of the curvature is closely related to the behavior of geodesic. What happens in genernal?

In order to answer this question, we consider the quantity $c(r)$ for a Riemannian manifold (M, g). Let $0 \in M$, and $\delta\rangle 0$ be a small number such that exp_0 is a diffeomorphism on $B(0,\delta) \subset T_0M$.

Consider the polar coordinate (ρ, θ) in T_0M. Then for any fixed r, $\widetilde{r}(\theta) = (r, \theta)$ is a curve in T_0M. $\frac{d}{d\theta}(r,\theta)$ is the velocity field along $\widetilde{r}(\theta)$.

Let $r < \delta$, we have

$$c(r) = \int_0^{2\pi} \langle d \underset{0}{exp}(\frac{d}{d\theta}(r,\theta)), d \underset{0}{exp}(\frac{d}{d\theta}(r,\theta)) > d\theta$$

So, for our purpose, we have to explore the interaction between the norm of $d\, exp_0(\frac{d}{d\theta}(r,\theta))$ and the curvature of (M, g). Note that $R^n \cong T_0M \ni \vec{p} = (r, \theta_p)$, if we write

$$d \underset{0}{exp}(\frac{d}{d\theta}(r,\theta)) = d \underset{0}{exp}(\vec{p})(\frac{d}{d\theta}).$$

In order to calculate its norm, we first observe it can be extended to be a variational field of a geodesic variation of

$$\gamma(t) = \underset{0}{exp} \frac{t}{r}\vec{p},\ t \in [0, r].$$

In fact, we pick

$$F:\ [0,r] \times (-\epsilon, \epsilon) \to M,\ (t,s) \mapsto \underset{0}{exp} \frac{t}{r}(\vec{p} + s\frac{d}{d\theta}).$$

We observe that $F(t,0) = \gamma(t)$, and $\frac{\partial F}{\partial s}(t,0) = \frac{\partial}{\partial s}|_{s=0}\, exp_0\, \frac{t}{r}(\vec{p} + s\frac{d}{d\theta})$ is the variational field along γ. In particular

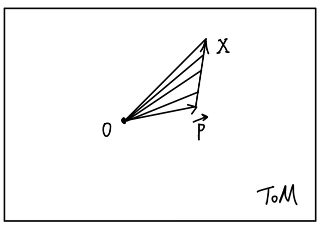
$$\begin{aligned}\frac{\partial F}{\partial s}(r,0) =& \frac{\partial}{\partial s}|_{s=0}\, \underset{0}{exp}(\vec{p} + s\frac{d}{d\theta})\\ =& d\, \underset{0}{exp}(\vec{p})(\frac{d}{d\theta})\\ =& d\, \underset{0}{exp}(\frac{d}{d\theta}(r,\theta)).\end{aligned}$$

In order to calculate $\frac{\partial F}{\partial s}(r,0)$, we calculate the whole variational field. $V(t) = \frac{\partial F}{\partial s}(t,0),\ t \in [0,r]$.

Here, we can be slightly more general: consider a general vector $X \in T_{\vec{p}}(T_0M)$ and the variation

$$F(t,s) = \underset{0}{exp}\, \frac{t}{r}(\vec{p} + sX),\ t \in [0,r],\ s \in (-\epsilon,\epsilon).$$

Let $V(t) := \frac{\partial F}{\partial s}(t,0)$ be the geodesic variational field along γ.



To calculate $V(t),\ t \in [0,r]$, we derive the equations it satisfies:

$$\begin{aligned}\widetilde{\nabla}_{\frac{\partial}{\partial t}}\widetilde{\nabla}_{\frac{\partial}{\partial t}}\frac{\partial F}{\partial s} =& \widetilde{\nabla}_{\frac{\partial}{\partial t}}\widetilde{\nabla}_{\frac{\partial}{\partial s}}\frac{\partial F}{\partial t}\\ =& \widetilde{\nabla}_{\frac{\partial}{\partial s}}\widetilde{\nabla}_{\frac{\partial}{\partial t}}\frac{\partial F}{\partial t} + \widetilde{\nabla}_{\frac{\partial}{\partial t}}\widetilde{\nabla}_{\frac{\partial}{\partial s}}\frac{\partial F}{\partial t} - \widetilde{\nabla}_{\frac{\partial}{\partial s}}\widetilde{\nabla}_{\frac{\partial}{\partial t}}\frac{\partial F}{\partial t}\\ =& R(\frac{\partial F}{\partial t}, \frac{\partial F}{\partial s})\frac{\partial F}{\partial t}.\end{aligned}$$

Restricting to the normal geodesic γ, we have

$$\nabla_T\nabla_T V = R(T,V)T,\ or\ \nabla_T\nabla_T V + R(V,T)T = 0 \qquad (*)$$

Definition 5.2. *(Jacobi field), Let $\gamma : [a,b] \to M$ be a geodesic, and T be the velocity field along γ. If a vector field V along γ satisfies*

$$\nabla_T\nabla_T V + R(V,T)T = 0, \qquad (5.3.1)$$

we call V a Jacobi field(along γ). The equation (5.3.1) is called the Jacobiequation.

Choose parallel vector fields $Y_1, \cdots, Y_n$ along γ which are orthonormal at $\gamma(a)$, and hence orthonormal everywhere along γ, then $\exists f^i(t)$ s.t. $V(t) = f^i(t)Y_i(t)$, and

$$\begin{aligned}
&\nabla_T\nabla_T V + R(V,T)T = \frac{d^2 f^i(t)}{dt^2}Y_i + f^i R(Y_i,T)T = 0\\
\Leftrightarrow&\langle\frac{d^2 f^i(t)}{dt^2}Y_i, Y_j\rangle + \langle f^i R(Y_i,T)T, Y_j\rangle = 0,\ \forall\ j = 1,\cdots,n.\\
\Leftrightarrow&\frac{d^2 f^j}{dt^2} + f^i(R(Y)i,T)T, Y_j) = 0,\ \forall\ j = 1,\cdots,n.
\end{aligned}$$

Hence, $V(t)$ the solution of the above system of second order linear ODE. It will be determined by its initial conditions $V(0)$ and $\dot{V}(0) := \widetilde{\nabla}_T V \in T_0 T_{\gamma(0)}M$. Recall

$$\begin{aligned}
V(0) =&\frac{\partial}{\partial s}|_{s=0}F(t,s)|_{t=0}\\
=&\frac{\partial}{\partial s}|_{s=0}F(0,s) = \frac{\partial}{\partial s}|_{s=0}\underset{0}{exp}\,0 = 0\\
\dot{V}(0) =&\nabla_T V(0) = \widetilde{\nabla}_{\frac{\partial}{\partial t}}\frac{\partial F}{\partial s}|_{s=0}(0,s) = \widetilde{\nabla}_{\frac{\partial}{\partial s}}\frac{\partial F}{\partial t}(0,0)\\
=&\widetilde{\nabla}_{\frac{\partial}{\partial t}}\frac{1}{r}(\vec{p} + sX).
\end{aligned}$$

Note $\frac{1}{r}(\vec{p} + sX)$ is a vector field along the constant curve. By definition of induced connection, we have

$$\begin{aligned}
\dot{V}(0) =&\widetilde{\nabla}_{\frac{\partial}{\partial s}}(\frac{1}{r}(\vec{p} + sX)) = \frac{X}{r} \in T_0(T_0M)\\
&i.e.\ \dot{f}^i(0)Y_i(0) = \frac{X}{r} = \langle\frac{X}{r}, Y_i(0)\rangle Y_i(0)\\
&i.e.\ \dot{f}^i(0) = \langle\frac{X}{r}, Y_i(0)\rangle.
\end{aligned}$$

So, in order to solve $V(t)$, where $V(r) = d\,exp_0(\vec{p})(X)$ is what we need, we have solve

$$\begin{cases}
\ddot{f}^j + \langle R(Y_i,T)T, Y_j\rangle f^i = 0,\ j = 1,\cdots,n\\
f^j(0) = 0\\
\dot{f}^j(0) = \frac{1}{r}\langle X, Y_j(0)\rangle
\end{cases}\tag{5.3.2}$$

Next, we come back to the calculation of $c(r)$:

$$c(r) = \int_0^{2\pi}\langle d\,\underset{0}{exp}(\frac{d}{d\theta}(r,\theta)), d\,\underset{0}{exp}(\frac{d}{d\theta}(r,\theta))\rangle^{\frac{1}{2}}d\theta$$

where

$$\begin{cases}
\ddot{f}^j + \langle R(Y_i,T)T, Y_j\rangle f^i = 0,\ j = 1,\cdots,n\\
f^j(0) = 0\\
\dot{f}^j(0) = \frac{1}{r}\langle\frac{d}{d\theta}(r,\theta), Y_j(0)\rangle
\end{cases}\tag{5.3.3}$$

In particular, for the cases of $\mathbb{R}^2,\ S^2,\ \mathbb{H}^2$. Let $Y(t)$ be a unit parallel vector field along γ s.t. $\langle Y(t), T(t)\rangle = 0,\ \forall t$.

By Gauss's lemma, $\frac{d}{d\theta}(t,\theta),\ t \in [0, r]$ is vertical to the radial geodesic at everywhere t. In fact, the variational field $V(t)$ along γ is perpendicular to γ everywhere.(by First variational formula.) So, we can write

$$V(t) = f(t)Y(t).$$

Then the equation (5.3.2) become

$$\begin{cases} \ddot{f} + \langle R(Y,T)T, Y\rangle f = 0 \\ f(0) = 0 \\ \dot{f}(0) = \frac{1}{r}\langle \frac{d}{d\theta}(r,\theta), Y(0)\rangle = \frac{1}{r}|\frac{d}{d\theta}(r,\theta)| = 1 \end{cases} \tag{5.3.4}$$

Recall we have constant sectional curvature in $\mathbb{R}^2,\ S^2,\ \mathbb{H}^2$. Therefore, we need solve

$$\begin{cases} \ddot{f} + Kf(t) = 0 \\ f(0) = 0 \\ \dot{f}(0) = 1 \end{cases} \tag{5.3.5}$$

The solution is given by

$$f(t) = \begin{cases} t,\ K = 0 \\ \sin t,\ K = +1 \\ \sinh t,\ K = -1 \end{cases} \tag{5.3.6}$$

Therefore, we recover the results:

$$\begin{cases} c_0(r) = 2\pi r \\ c_+(r) = 2\pi \sin r \\ c_-(r) = 2\pi \sinh r \end{cases} \tag{5.3.7}$$

Therefore, (**) establish the relations between $c(r)$ and the curvature.

5.4 What is a Jacobi field?

We have already seen the definition of Jacobi fields . Now, we want to understand this concept further.

As we have explained, it is a solution of a system of second order ODE:

$$\frac{d^2 f^j}{dt^2} + f^i\langle R(Y_i, T)T, Y_j\rangle = 0,\ j = 1, \cdots, n.$$

and $Y_1, \cdots, Y_n$ are parallel orthonormal vector fields along γ. And the Jacobi field is given by $V(t) = f^i(t)Y_i(t)$.

Proposition 5.4. *Let $\gamma : [a,b] \to M$ be any geodesic.*

(1) Given $V, W \in T_{\gamma(a)}M$, there exists a unique Jacobi field $U(t),\ t \in [a,b]$ such that $U(0) = V,\ \widetilde{\nabla}_{\frac{\partial}{\partial t}} U(t) := \dot{U}(0) = W$.

(2) The linear space of all Jacobi fields along γ is of $2n$ dim'l.

(3) The zero points of a Jacobi field U along γ are discrete, if U is not identically 0 along γ.

Proof. (1),(2) follows directly from the theory of 2^{nd} linear ODEs. Given $U(0), \dot{U}(0)$, the 2^{nd} order linear ODE has a unique solution.

For (3), assume the zero points are not discrete. Then there is an accermulated point $\gamma(t_0)$. Then $U(t_0) = 0$, and

$$\dot{U}(0) = \widetilde{\nabla}_{\frac{\partial}{\partial t}}(f^i(t)\frac{\partial}{\partial x^i}) = \frac{df^i}{dt}(t_0)\frac{\partial}{\partial x^i} + f^i(t_0)\widetilde{\nabla}_{\frac{\partial}{\partial t}}(t_0)$$

pick (x^i) to be the normal coordinate around t_0, then

$$\dot{U}(t_0) = \frac{df^i}{dt}(t_0)\frac{\partial}{\partial x^i} = 0.$$

Then U is identically zero along γ ■ □

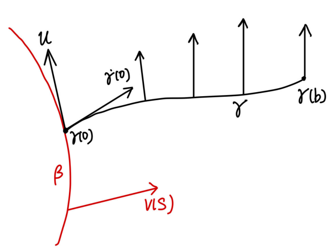
From our discussion in section 5.3, we have seen that the variational dield odf a geodesic variation of a geodesic γ is a Jacobi field along γ. In fact, the converse also hold.

Proposition 5.5. *Let $\gamma : [a,b] \to M$ be a geodesic and U be a vector field along γ. Then U is a Jacobi field if and only if U is the variational field of a geodesic variation of X.*

Proof. ($\Leftarrow$) The calculations in the end of !!! proves this directions.

($\Rightarrow$) Let U be a Jacobi field along γ. Let $\beta : (-\epsilon, \epsilon) \to M$ be the geodesic with

$$\begin{cases} \beta(0) = \gamma(0) \\ \dot{\beta}(0) = U(0) \end{cases} \tag{5.4.1}$$



We put

$$F : [0,b] \times (-\epsilon, \epsilon) \to M,\ (t,s) \mapsto \underset{\beta(s)}{exp}\, t(V(s) + sW(s)).$$

where V,W are parallel vector fields along β with

$$V(0) = \dot{\gamma}(0),\ W(0) = \dot{U}(0) = \widetilde{\nabla}_{\frac{\partial}{\partial t}} U(0).$$

Then

$$F(t, 0) = \underset{\beta(0)}{exp}\, tV(0) = \underset{\gamma(0)}{exp}\, t\dot{\gamma}(0) = \gamma$$

and $F(t, s) = exp_{\beta(s)}\, t(V(s) + sW(s))$ are all geodesics for $s \in (-\epsilon, \epsilon)$. That is F is a geodesic variation of γ. Therefore its variational field

$$Y(t) =: \frac{\partial F}{\partial s}(t, 0) = \frac{\partial F}{\partial s}(t, s)|_{s=0}$$

is a Jacobi field. Meanwhile, we have

$$Y(0) = \frac{\partial}{\partial s}|_{s=0} F(0, s) = \frac{\partial}{\partial s}|_{s=0} \underset{\beta(s)}{exp}\, 0 = \dot{\beta}(0) = U(0)$$

and

$$\begin{aligned}
\dot{Y}(0) =& \widetilde{\nabla}_{\frac{\partial}{\partial t}} \frac{\partial}{\partial s} F(t, s)|_{s=0} \\
=& \widetilde{\nabla}_{\frac{\partial}{\partial s}} \frac{\partial}{\partial t} F(t, s)|_{s=0} \\
=& \widetilde{\nabla}_{\frac{\partial}{\partial s}} (V(s) + sW(s))|_{s=0} \\
=& W(0) = \dot{U}(0).
\end{aligned}$$

Then Proposition 5.4(1) implies that $U = Y$. thqat is U is the variational field of F. ■ □

Remark 5.3. *We summarize what we learned about Jacobi fields up to now:*

(1) Let $\beta : (-\epsilon, \epsilon) \to M$ be a curve, $V(s)$, $W(s)$ are parellel vector fields along β. Then the family of geodesics

$$\gamma_s(t) := \underset{\beta(s)}{exp}\, t(V(s) + sW(s))$$

leads to a geodesic variation $F(t, s) := \gamma_s(t)$ whose variational field along $\gamma_0(t)$ is a Jacobi field $U(t)$ with

$$U(0) = \dot{\beta}(0),\ \dot{U}(0) = W(0).$$

In particular, when $\beta(s) = p \in M$ is a constant curve, we have.

(2) The 1-parameter family of geodesics

$$\gamma_s(t) = \underset{0}{exp}\, t(V + sW),\ V,\ W \in T_pM$$

gives the Jacobi field $U(t)$ along γ_0 with

$$U(0) = \dot{\beta}(0) = 0,\ \dot{U}(0) = W.$$

Since T_pM is an inner product space, we can restrict $\langle V, W\rangle = 0$. Then, we have

(3) The 1-parameter family of geodesics

$$\gamma_s(t) = \exp_p t(V + sW),\ \langle V, W\rangle = 0$$

givea a "normal Jacobi field" $U(t)$ with $U(0) = 0$, $\dot{U}(0) = W$, and $\langle U(t), \gamma t_0(t)\rangle = 0$, $\forall t$.

Observe $\langle tW, tV\rangle = 0$. Recall the Gauss lemma we derived from the First variation formula, we have, since $U(t)$ is the variational fields, $\langle U(t), \dot{\gamma}_0(t)\rangle = 0$. ■

We will se later that normal Jacobi fields with $U(0) = 0$ are very important for any further investigation.

Relations with the SVF:

Recall for proper variations of a geodesic γ, we have

$$\frac{\partial^2}{\partial v \partial w}|_{(0,0)} E(v, w) = I(V, W) = \int_a^b \langle \nabla_T W, \nabla_T W\rangle - \langle R(W, T)T, V\rangle dt.$$

Observe that $T(V, W)$ is bilinear. $I(V, W)$ is also symmetric since E is C^∞ in (u, w).

Resturning to the original problem: to determine whether a geodesic is (locally) minimizing. For that purpose, we hope to decide whether $det(\frac{\partial^2}{\partial v \partial w})|_{(0,0)} E(v, w)$ is positive or not. We will see the existance of Jacobi field(vanishing at the two ends $\gamma(a), \gamma(b)$) will be an obstruction.

Proposition 5.6. *Let $\gamma : [a, b] \to M$ be a geodesic and U be a vector field along γ. Then U is a Jacobi field if and only if*

$$I(U, Y) = 0$$

for all vector fields Y along γ with $Y(a) = Y(b) = 0$.

Proof.

$$\begin{aligned} I(U, Y) &= \int_a^b \langle \nabla_T U, \nabla_T Y\rangle - \langle R(U, T)T, Y\rangle dt \\ &= \int_a^b \langle -\nabla_T \nabla_T U, Y\rangle - \langle R(U, T)T, Y\rangle dt \\ &\quad \textit{since } \nabla \textit{ is compatible with } g \textit{ and } Y(a) = Y(b) = 0. \\ &= \int_a^b \langle -\nabla_T \nabla_T U - R(U, T)T, Y\rangle dt \end{aligned}$$

for all Y with $Y(a) = Y(b) = 0$.

Therefore $\nabla_T \nabla_T U + R(U, T)T = 0$ holds by the fundamental lemma of the calculus of variatiuons. ■ □

Proposition 5.7. *Let $\gamma : [a, b] \to M$ be a geodesic and U be a vector field along γ. Then U is a Jacobi field if and only if it is a critical point of $I(X, X)$ w.r.t. all variations with fixed endpoints, i.e.*

$$\frac{d}{ds}|_{s=0} I(X + sY, X + sY) = 0$$

for all vector fields Y along γ with $Y(a) = Y(b) = 0$

Proof. We compute

$$\frac{d}{ds}|_{s=0} I(X + sY, X + sY) == 2I(X, Y).$$

Then Proposition 5.6 follows directly from Proposition 5.5. □

Remark 5.4. *The Jacobi equation is the Euler-Lagrange equation for $I(X) = I(X, X)$. In fact, one can consider the second variation for each critibal point os a variational problem. The second variation then is a quadratic integral in the variation fields, and the second variation may be considered as a new variational problem. This new variational problem is called accessary variational problem of the original one.*

5.5 Conjugate Points and ,Minimizing Geodesics

From Proposition 5.5, we see that if there exists nonzero Jacobi field U along the geodesic $\gamma : [a, b] \to M$ with $U(a) = U(b) = 0$, then $I(U, U) = 0$, i.e. I is not positive definite, and hence $\gamma|_{[a,b]}$ may not be strictly local minimizing. This phenomena can be observed explicitly. For any semicircle from the north pole p to the south pole q, $\exists$ nonzero Jacobi field U along it with $U(a) = U(b) = 0$, each semicircle has the same length π.

Definition 5.3. *(Conjugate points) Let $\gamma : [a, b] \to M$ be a geodesic. For $t_0, t_1 \in [a, b]$, if there exists a Jacobi field $U(t)$ along γ that does not vanish identically, but satisfies*

$$U(t_0) = U(t_1) = 0$$

then t_0, t_1 are called conjudate values along γ. The multiplicity of t_0 and t_1 as conjugate values is defined as the dimensions of the vector space consisting of all such Jacobi fields. We also say $\gamma(t_0)$, $\gamma(t_1)$ are conjugate points of γ.(This terminology is ambiguous when γ has self-intersetions).

Recall a Jacobi field U is determined by its initial values $U(t_0)$, $\dot{U}(t_0)$ at any point to. Hence, the multiplicity of two conjugate values t_0, t_1 is clearly $\leq n$. Actually, it is $\leq n - 1$. This is because a Jacobi field which is tangent to γ and vanish at to will not vanish at t_1.

Proposition 5.8. *Let $\gamma : [a, b] \to M$ be a geodesic with velocity field $T(t) = \dot{\gamma}(t)$.*

(1) The vector field fT along γ is a Jacobi field if and only if f is linear.

(2) Every Jacobi field U along γ can be written uniquely as

$$fT + U^{\perp},$$

where f is linear and $U^{\perp}$ is a Jacobi field prependicular to γ.

(3) If a Jacobi field U along γ is prependicular to γ at two points t_0 and t_1, then U is prependicular to γ everywhere. In particular, if $U(t_0) = U(t_1) = 0$, then U is perpendicular to γ everywhere.

Proof. (1) fT is a Jacobi field$\Rightarrow$

$$0 = \nabla_T\nabla_T(fT) = R(fT,T)T = f''(t)T.$$

Hence f is linear.

(2) Let U be a Jacobi field along γ, we can write $U = fT + U^{\perp}$ for some f and some $U^{\perp}$ with $\langle U^{\perp}, T\rangle = 0$.

U is Jacobi$\Rightarrow$ $0 = \nabla_T\nabla_T(fT + U^{\perp}) + R(fT + U^{\perp}, T)T = f''T + \nabla_T\nabla_T U^{\perp} + R(U^{\perp}, T)T$.

In particular, we have

$$0 = f''T + \langle\nabla_T\nabla_T U^{\perp}, T\rangle + \langle R(U^{\perp}, T)T, T\rangle.$$

By symmetry, $\langle R(U^{\perp}, T)T, T\rangle = 0$.

$$\begin{aligned} 0 = \langle U^{\perp}, T\rangle \Rightarrow & 0 = \frac{d}{dt}\langle U^{\perp}, T\rangle = \langle\nabla_T U^{\perp}, T\rangle \\ \Rightarrow & 0 = \frac{d}{dt}\langle\nabla_T U^{\perp}, T\rangle = \langle\nabla_T\nabla_T U^{\perp}, T\rangle \end{aligned}$$

Hence $0 = f''$ and

$$\nabla_T\nabla_T U^{\perp} + R(U^{\perp}, T)T = 0,$$

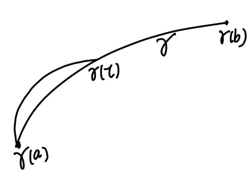
i.e. $U^{\perp}$ is a Jacobi field. Uniqueness is obvious.

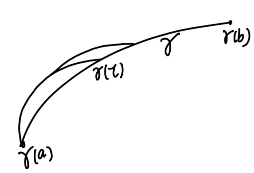
(3) Write $U = fT + U^{\perp}$. When $\langle U(t_0), T\rangle = \langle U(t_1), T\rangle = 0$ implies $f(t_0) = f(t_1) = 0$. Recall f is linear, we have $f \equiv 0$. Therefore $U = U^{\perp}$. ■ □

Proposition 5.9(3) shows that for the purpose of investigating conjugate values, we need consider only normal Jacobi fields.

Conjugate points play an important role in the study of local minima for length. A geodesic $\gamma : [a,b] \to M$ can not locally minimizi length if $\exists\tau \in (a,b)$ conjugate to a.

Intuitive arguement:





$\gamma(\tau)$ conjugate to $\gamma(a)$, $\exists$ a geodesic η from $\gamma(a)$ to $\gamma(\tau)$ with nearly the same length as $\gamma|_{[0,\tau]}$.

Then η followed by $\gamma|_{[\tau,b]}$ has nearly the same length as γ. By the first curve has a corner, and can be shorten by replacing the corner with a minimal geodesic. Therefore γ is not a minimizing curve.

In fact we have the following theorem of Jacobi.

Theorem 5.4. *(Jacobi) Let $\gamma : [a,b] \to M$ be a geodesic from $p = \gamma(a)$ to $q = \gamma(b)$.*

(1) If there is no conjugate points of p along γ, then there exists $\epsilon > 0$ so that for any piecewise smooth curve $c : [a,b] \to M$ from p to q satisfying $d(\gamma(t), c(t))\langle\epsilon$, we have

$$L(c) \geq L(\gamma)$$

with equality holds if and only if c is a reparametrization of γ.

(2) If there exists $\bar{t} \in (a,b)$ so that $\bar{q} = \gamma(\bar{t})$ is a conjugate point of p, then there is a proper variation of γ so that

$$L(\gamma_s) < L(\gamma)$$

for any $0 < |s| < \epsilon$.

The above results are direct consequences of the corresponding properties of index forms, which will be discussed in the next subsection,

Next, we derive a characterization of the conjugate points in terms of critical point of the exponential map.

Theorem 5.5. *Let $\gamma : [0,1] \to M$ be a geodesic with $\gamma(0) = p \in M$ and $\dot{\gamma} = V \in T_pM$, so that γ can be described as*

$$t \mapsto \exp_p tV.$$

Then 0 and 1 are conjugate values for γ if and only if V is a critical points of $\exp_p$. Moreoverm the multiplicity of the conjugate values 0 and 1 is the dimension of the kernel of $d\exp_p : T_V(T_pM) \to T_{\gamma(1)}M$.

Proof. "$\Leftarrow$" Suppos that $V \in T_pM$ is a critical point for $\exp_p$. That is $0 = d\exp_p(\dot{V})(X)$ for some nonzero $X \in T_V(T_pM)$. Let c be a path in T_pM with $c(0) = V$, $\dot{c}(0) = X$.

We put

$$F(t,s) = \exp_p t(c(s)),\ t \in [0,1].$$

Then $F(t,0) = \exp_p tV = \gamma$, and $\gamma_s(t) = \exp_p tc(s)$ is a geodesic. That is, F is a geodesic variation of γ. So the variational field

$$U(t) := \frac{\partial}{\partial s}|_{s=0} \exp_p tc(s)$$

is a Jacobi field along γ. We compute $U(0) = \frac{\partial}{\partial s}|_{s=0} \exp_p 0 = 0$, and $U(1) = \frac{\partial}{\partial s}|_{s=0} \exp_p c(s) = d\exp_p(c(s))(\dot{c}(0)) = d\exp_p(\dot{V})(X) = 0$. Next, we hope to show U is not identically zero. This is because

$$\begin{aligned}\dot{U}(0) =&\widetilde{\nabla}_{\frac{\partial}{\partial t}} U(t)|_{t=0} = \widetilde{\nabla}_{\frac{\partial}{\partial t}}|_{t=0} \frac{\partial}{\partial s}|_{s=0} \underset{p}{exp}\, t(c(s)) \\ =&\widetilde{\nabla}_{\frac{\partial}{\partial s}}|_{s=0} \frac{\partial}{\partial t}|_{t=0} \underset{p}{exp}\, tc(s) = \widetilde{\nabla}_{\frac{\partial}{\partial s}}|_{s=0} c(s) \\ &(the\ covariant\ derivative\ of\ the\ vector\ field\ s \mapsto c(s)\ along\ the\ constant\ curve\ s \mapsto p) \\ =&\dot{c}(0) = X \neq 0.\end{aligned}$$

Therefore, we show 0 and 1 are conjugate values for γ.

("$\Rightarrow$") We argue by contridiction. Suppose V is not a critical point for exp_p. If $X_1, \cdots, X_n \in T_V(T_pM)$ are m linearly independent vectors, then $d\,exp_p(V)(X_1), \cdots, d\,exp_p(V)(X_n) \in T_{\gamma(1)}M$ are linearly independent. Choose paths $c_1, \cdots, c_n$ in T_pM with

$$\begin{cases} c_i(0) = V \\ \dot{c}_i(0) = X_i,\ i = 1, \cdots, n, \end{cases} \tag{5.5.1}$$

And $F(t,s) := d\,exp_p\, tc_1(s)$ are geodesic variation of γ with variational fields $V_i(t)$. The V_i are Jacobi fields along γ which vanish at 0. Moreover, the $V_i(1) := d\,exp_p(V)(X_i)$ are independent, so no nontrivial linear combination of the V_i can vanish at 1. Since the vector space of Jacobi fields along γ which vanish at 0 has dimension exactly n, it follows that no nonzero Jacobi field along γ vanishes at 0 and also at 1. ■ □

5.6 Index forms

In this section, we discuss the minimizing property of a geodesic via Index forms: For that purpose, we need consider a piecewise C^∞ proper variation of a geodesic γ. That is, we compare the length of a geodesic $\gamma : [a,b] \to M$ with the length of any piecewise C^∞ curve from $\gamma(a)$ to $\gamma(b)$. The corresponding variational field of γ is then a piecewise C^∞ vector field along γ. Recall our calculations for the second variation formula(SVF), the result is the same as the case of smooth variation:

$$\frac{\partial^2}{\partial v \partial w}|_{(v,w)=(0,0)} E(v,w) = \langle \nabla_W V, T\rangle|_a^b + \int_a^b \langle \nabla_T V, \nabla_T W\rangle - \langle R(W,T)T, V\rangle dt,$$

where $\langle \nabla_W V, T\rangle \int_a^b = 0$ when the variation is proper.

Definition 5.4. *(Index form) The index form of a geodesic γ is*

$$I(V,W) = \int_a^b (\langle \nabla_T V, \nabla_T W\rangle - \langle R(W,T)T, V\rangle) dt$$

where V, W are two piecewise smooth vector fields along γ.

Remark 5.5. *(1) If V, W are C^∞ on each $[t_i, t_{i+1}]$ where*

$$a = t_0 < t_1 < \cdots < t_n < t_{n+1} = b$$

is a subdivision of $[a,b]$. *Then by integration by parts,*

$$\begin{aligned} I(V,W) &= \int_a^b \langle \nabla_T V, \nabla_T W\rangle - \langle R(W,T)T, V\rangle dt \\ &= \sum_{i=0}^{k} \langle V, \nabla_T W\rangle|_{t_i}^{t_{i+1}} + \int_a^b \langle -\nabla_T\nabla_T W, V\rangle - \langle R(W,T)T, V\rangle dt \\ \Rightarrow I(V,W) &= -\int_a^b \langle \nabla_T\nabla_T W + R(W,T)T, V\rangle dt + \langle \nabla_T W, V\rangle|_a^b \\ &\quad - \sum_{j=1}^{k} \langle \nabla_{T(t_j^+)} W - \nabla_{T(t_j^-)} W, V\rangle. \end{aligned} \tag{5.6.1}$$

(2) Note for a proper variation

$$\frac{\partial^2}{\partial v \partial w}|_{(v,w)=(0,0)} E(v,w) = I(V,W).$$

Let $\mathcal{V}$:= *the set of all piecewise smooth vector fields along* $\gamma : [a,b] \to M$, *and*

$$\mathcal{V}_0 = \{x \in \mathcal{V} | X(a) = 0, X(b) = 0\}.$$

We need extend Proposition 5.6 to piecewise smooth vector fields.

Proposition 5.9. *Let* $\gamma : [a,b] \to M$ *be a geodesic and* $U \in \mathcal{V}$. *Then* U *is a Jacobi field if and only if* $I(U,Y) = 0,\ \forall Y \in \mathcal{V}_0$.

Proof. Note that, comparing with Proposition 5.6, we here have $U \in \mathcal{V}$ may be piecewise smooth, and so does Y. However, a Jacobi field is smooth. (The result and proof here is very much similar in sprit to the characterization of geodesic.(see previous exercise); A piecewise smooth curve c is a geodesic if and only if, for every proper variation $F$of c, we have $E'(0) = 0$.)

($\Rightarrow$) If U is a Jacobi field, then $I(U,Y) = 0,\ \forall Y \in \mathcal{V}_0$

$$\begin{aligned} I(U,Y) &= \int_a^b (\langle \nabla_T U, \nabla_T Y\rangle - \langle R(Y,T)T, U\rangle) dt \\ &\overset{Remark(1)}{=} -\int_a^b \langle \nabla_T\nabla_T U + R(U,T)T, Y\rangle dt + \langle \nabla_T U, Y\rangle|_a^b \\ &\quad - \sum_{j=1}^{k} \langle \nabla_{T(t_j^+)} U - \nabla_{T(t_j^-)} U, Y\rangle. \\ &\quad Y \in \mathcal{V} \Rightarrow \langle \nabla_T U, Y\rangle|_a^b = 0 \\ &\quad U \text{ is smooth} \Rightarrow \sum_{j=1}^{k} \langle \nabla_{T(t_j^+)} U - \nabla_{T(t_j^-)} U, Y\rangle = 0 \\ &\overset{U\ is\ Jacobi}{=} 0 \end{aligned}$$

($\Leftarrow$) Assume $I(U,Y) = 0,\ \forall Y \in \mathcal{V}_0$. Let $f : [a,b] \to \mathbb{R}$ be a smooth function s.t.

$$f(t_i) = 0,\ i = 0, \cdots, k+1$$

and $f\rangle 0$ otherwise. Set $W = U,\ V = Y = f(\nabla_T\nabla_T U + R(U,T)T)$. Note that Y is well-defined and $Y \in \mathcal{V}_0$.

Therefore

$$0 = T(U,Y) = -\sum_i \int_{t_i}^{t_{i+1}} f(t)|\nabla_T\nabla_T U + R(U,T)T|^2 dt.$$

Hence, we have $\nabla_T\nabla_T U + R(U,T)T = 0$ on each $[t_i, t_{i+1}]$.

That is, "piecewisely", U is a Jacobi field. $(*)$

Next, for any $j = 1, \cdots, k$, let $Y \in \mathcal{V}_0$ s.t.

$$\begin{cases} Y(t_j) = 0,\ \forall i \neq j \\ Y(t_j) = \nabla_{T(t_j^+)}U - \nabla_{T(t_j^-)}U \end{cases} \tag{5.6.2}$$

Then $0 = I(U,Y) = |\nabla_{T(t_j^+)}U - \nabla_{T(t_j^-)}U|^2$.

Hence $\nabla_{T(t_j^+)}U = \nabla_{T(t_j^-)}U$.

Therefore U is a C^1 vector field along γ. Combining with the fact (*) and using the uniqueness of Jacobi fields with given initial data, we conclude U is the Jacobi field on $[a,b]$. ■ □

Recall our previous discussions about SVF, we say the property "γ is loccaly minimizing" is equivalent to "$I(V,V) = 0,\ \forall 0 \neq V \in \mathcal{V}_0$". Since $I(V,W)$ is a biliear, symmetric from on the vector space $\mathcal{V}_0$, the later condition is equivalent to say "I is positive definite on $\mathcal{V}_0$".

To illustrate the idea, we can compare the index form with the Hessian of a function $f : \mathbb{R}^n \to \mathbb{R}$. Consider a curve ξ in $\mathbb{R}^n$, with $\xi(0) \in \mathbb{R}^n$. Then the second order derivative of f along ξ is $\frac{d^2}{ds^2}f(\xi(s))$. Hessian of f valued at the vector $\dot{\xi}(0)$ is

$$\frac{d^2}{dt^2}f(\xi(s))|_{s=0} = Hess\, f(\dot{\xi}(0), \dot{\xi}(0)).$$

In particular $\frac{d^2}{ds^2}f(\xi(s))|_{s=0}$ only depends on $\dot{\xi}(0)$. Once we know $\frac{d^2}{ds^2}f(\xi(s))|_{s=0},\ \forall \xi$, then we have $Hess\, f(v,v)$ for any v, and hence

$$Hess\, f(v,w) = \frac{1}{2}(Hess\, f(v+w, v+w) - Hess\, f(v,v) - Hess\, f(w,w)).$$

Analogously, we replace $\mathbb{R}^n$ by the space $\wp$ of <u>all curves $c : [a,b] \to M$.</u> Given a "point" of $\wp$, i.e. a curve $\gamma \in \wp$, consider a "curve" through it, i.e. a 1-parameter family of curves $\{\gamma_s\}$. Let E be a function $E : \wp \to \mathbb{R}$. The restriction $R \circ \gamma_s := E(s)$, and

$$\frac{d^2}{ds^2}E(\gamma_s)|_{s=0} = \frac{d^2}{ds^2}E(s)|_{s=0} = \text{``}\, Hess\, E\langle V(t), V(t)\rangle''.$$

By polaritation, one have "$Hess\,E(V, W)$", the Hessian of E on the "Hilbert space of curves". All formal discussion here can ve made rigorous,

In particular, when considering $\wp_0$ of all curves $c : [a, b] \to M$ s.t. $c(a) = \gamma(a)$, $c(b) = \gamma(b)$, the "Hessian of E" is given by the index form.

Next, our aim is discuss the relation between Algeraic properties of the index form of the geodesic γ and Minimizing properties of the geodesic γ. Given a normal geodesic $\gamma : [a, b] \to M$, we can imagine the end point $\gamma(b)$ move from $\gamma(a)$ slowly to $\gamma(b)$. When $|b - a|$ is small enough, $\gamma|_{[a.b]}$ is minimizing, hence we can expect I is positive definite on $\mathcal{V}_0$.

By the rough idea we explained before Theorem 5.4, when $|b - a|$ is large, s.t. there is a conjugate value of a in (a, b), $\gamma|_{[a,b]}$ is not(locally) minimizing, then we can expect $\exists\, X$ s.t. $I(X, X) < 0$.

In the case of a and b are conjugate values of γ, we have from Proposition 5.6, for any Jacobi field U along γ with $U(a) = U(b) = 0$, we have $I(U, U) = 0$.

Theorem 5.6. *Let $\gamma : [a, b] \to M$ be a geodesic from $p = \gamma(a)$ to $q = \gamma(b)$.*

(1) $p = \gamma(a)$ has no conjugate point along $\gamma \Leftrightarrow$ the index form I is positive definite on $\mathcal{V}_0$.

(2) $q = \gamma(b)$ is a conjugate point of p along γ, and $\forall t \in (a, b)$, $\gamma(a)$ and $\gamma(t)$ are not conjugate point.(i.e. q is the first conjugate point of p)$\Leftrightarrow I$ is positive semidefinite but not positive definite on $\mathcal{V}_0$.

(3) $\exists\, \bar{t} \in (a, b)$, s.t. $p = \gamma(a)$ and $\overline{q} = \gamma(\bar{t})$ are conjugate points$\Leftrightarrow I(X, X) < 0$ for some $X \in \mathcal{V}_0$

Remark 5.6. *Theorem refz.23 tells if $\gamma(a)$ has no conjugate point along $\gamma|_{[a,b]}$, then for any $[a, \beta] \subset [a, b]$, $(\alpha < \beta)$, $\gamma(\alpha)$ also has no conjugate point along $\gamma|_{[\alpha,\beta]}$. Since otherwise, let $\widetilde{J}$ be a nonzero Jacobi field along $\gamma|_{[\alpha,\beta]}$ with $\widetilde{J}(\alpha) = 0 = \widetilde{J}(\beta)$. Let I_r^s be the index form of $\gamma|_{[r,s]}$. Then let $J|_{[a,\alpha]} \equiv 0 \equiv J|_{[\beta,b]}$, $J|_{[\alpha,\beta]} = \widetilde{J}$.*

$$I_a^b(J, J) = I_a^\alpha(0, 0) + I_\alpha^\beta(\widetilde{J}, \widetilde{J}) + I_\beta^b(0, 0) = I_\alpha^\beta(\widetilde{J}, \widetilde{J}) = 0.$$

Hence (1) tells, $p = \gamma(a)$ does have a conjugate along $\gamma|_{[a,b]}$. ∎

To show Theorem 5.6(1), we first prove the following useful Lemma.

Lemma 5.1. *Let $\gamma : [a, b] \to M^n$ be a geodesic, and $\gamma(1)$ has no conjugate point along γ. Then for any $V_a \in T_{\gamma(a)}M$ and $V_b \in T_{\gamma(b)}M$, there exists a unique Jacobi field U such that*

$$U(a) = V_a,\ U(b) = V_b.$$

Proof. By proposition 5.4(2), the vector space of all Jacobi fields along γ is of dimension $2n$. Let ℓ' be the subspace of Jacobi fields U with $U(a) = V_a$. Then $dim\,\ell' = n$. Note that $T_{\gamma(b)}M$ is also a vector space with $dim\,T_{\gamma(b)}M = n$. In fact, the linear transformation

$$A : \ell' \to T_{\gamma(b)}M,\ U \mapsto U(b)$$

is injective. This is because if we have $U_1,\ U_2 \in \ell'$ s.t. $U_1(b) = U_2(b)$.

Then $U_1 - U_2$ is again a Jacobi field along γ, we check

$$U_1 - U_2(a) - 0,\ U_1 - U_2(b) = 0.$$

Since $\gamma(a)$ and $\gamma(b)$ are not conjugate points, we have $U_1 - U_2 \equiv 0$. Therefore A is injective, and hence, an isomorphism,

Proof of (1):

($\Rightarrow$) Let $\{\dot{\gamma}(b), E_2, \cdots, E_n\}$ be an orthonormal basis of $T_{\gamma(b)}M$. From Lemma 1, $\exists!$ Jacobi field J_i along γ s.t.

$$J_i(0) = 0,\ J_i(b) = E_i,\ i = 2, \cdots, n.$$

Moreover, Proposition 5.9(3) tells $\langle J_i(t), \dot{\gamma}(t)\rangle = 0,\ \forall t \in [a, b]$. By the arguement in the proof of Theorem **??**, $\{J_i(t)\}$ are linearly independent in any $T_{\gamma(t)}M$.

For any $U \in \mathcal{V}_0$, note

$$I(f\dot{\gamma}(t), f\dot{\gamma}(t)) = \int_a^b \langle \dot{f}(t)\dot{\gamma}(t), \dot{f}(t)\dot{\gamma}(t)\rangle dt \geq 0,$$

and "$=$"$\Rightarrow f \equiv 0$. Only consider $U = f^i J_i$ for some functions f^i s.t. $f^i(a) = f^i(b) = 0$. Next, we compute

$$\begin{aligned}
I(U, U) &= \int_a^b \langle \nabla_T(f^i J_i), \nabla_T(f^j J_j)\rangle - \langle R(f^i J_i, T)T, f^j J_j\rangle dt \\
&= \int_a^b \langle \dot{f}^i J_i, \dot{f}^j J_j\rangle dt + \int_a^b \langle \dot{f}^i J_i, f^j \nabla_T J_j\rangle + \int_a^b \langle f^i \nabla_T J_i, \dot{f}^j J_j\rangle \\
&+ \int_a^b f^i f^j \langle \nabla_T J_i, \nabla_T J_j\rangle dt - \int_a^b f^i f^j \langle R(J_i, T)T, J_j\rangle dt.
\end{aligned}$$

$$\begin{aligned}
Suppose\ C &= \int_a^b \langle f^i \nabla_T J_i, \dot{f}^j J_j\rangle,\ D = \int_a^b f^i f^j \langle \nabla_T J_i, \nabla_T J_j\rangle dt \\
E &= \int_a^b f^i f^j \langle R(J_i, T)T, J_j\rangle dt
\end{aligned}$$

Observe that

$$\begin{aligned}
D &= \int_a^b f^i f^j \langle \nabla_T J_i, \nabla_T J_j\rangle dt \\
&= \int_a^b \{\frac{d}{dt}(f^i f^j \langle \nabla_T J_i, J_j\rangle) - \dot{f}^i f^j \langle \nabla_T J_i, J_j\rangle - f^i \dot{f}^j \langle \nabla_T J_i, J_j\rangle - f^i f^j \langle \nabla_T \nabla_T J_i, J_j\rangle\} dt \\
&= f^i f^j \langle \nabla_T J_i, J_j\rangle|_a^b - \int_a^b \dot{f}^i f^j \langle \nabla_T J_i, J_j\rangle - C + E \\
&= -\int_a^b \dot{f}^i f^j \langle \nabla_T J_i, J_j\rangle - C + E
\end{aligned}$$

In fact, $\langle \nabla_T J_i, J_j\rangle = \langle J_i, \nabla_T J_j\rangle$. This is because

$$\langle \nabla_T J_i, J_j\rangle - \langle J_i, \nabla_T J_j\rangle|_{t=0} = 0\ (since\ J_i(0) = J_j(0) = 0)$$

and

$$\frac{d}{dt}(\langle\nabla_T J_i, J_j\rangle - \langle J_i, \nabla_T J_j\rangle) = \langle\nabla_T\nabla_T J_i, J_j\rangle - \langle J_i, \nabla_T\nabla J_j\rangle$$
$$= -\langle R(J_i, T)T, J_j\rangle + \langle R(J_i, T)T, J_i\rangle = 0, \ \forall t.$$

Therefore $D = -B - C + E$, and hence

$$I(U, U) = \int_a^b \langle \dot{f}^i J_i, \dot{f}^j J_j\rangle dt \geq 0.$$

Moreover

$$\text{“}='' \ holds \Leftrightarrow \left.\begin{matrix} \dot{f}^i = 0 \\ f^i(0) = 0 = f^i(b) \end{matrix}\right\} \Leftrightarrow f^i = 0 \Leftrightarrow U = 0. \tag{5.6.3}$$

This proves the positive definiteness of I on $\mathcal{V}_0$. ∎

Proof of (2):

(⇒) Choose any $c \in (a, b)$. Pick a parallel orthonormal vector fields $\{\dot{\gamma}(t), E_2(t), \cdots, E_n(t)\}$. Then any $U \in \mathcal{V}_0 = \mathcal{V}_0(a, b)$,

$$U(t) = \sum_{i=2}^{n} f^i(t)E_i(t)$$

for some functions f^i with $f^i(a) = f^i(b) = 0$.

Define $\tau : \mathcal{V}_0(a, b) \to \mathcal{V}_0(a, c)$ by

$$\tau(V)(t) = \sum_{i=2}^{n} f^i(a + \frac{b-a}{c-a}(t-a))E_i(a + \frac{b-a}{c-a}(t-a)) = .$$

By Theorem **??**(1), we know $I^c(\tau(V), \tau(V)) > 0$. We can check by definition that

$$\lim_{c\to b} I^c(\tau(V), \tau(V)) = I(V, V) = \int_a^b (\dot{f}^i)^2 - f^i f^j \langle R(E_i, T)T, E_j\rangle dt \geq 0.$$

Hence I is positive sedefinite, since for any nonzero Jacobi field U with $U(a) = U(b) = 0$, we have $I(U, U) = 0$. ∎

Proof of (3):

(“⇒”) Let $\bar{t}$ is conjugate to a along γ, and there is a nonzero Jacobi field J along γ s.t. $J(a) = J(\bar{t}) = 0$. Let $\widetilde{J}$ be the vector field along γ with

$$\widetilde{J}(t) = J(t), \ a \leq t \leq \bar{t}$$
$$\widetilde{J}(t) = 0, \ \bar{t} \leq t \leq b$$

Notice that the discontinuity of $\widetilde{\nabla}_{\frac{\partial}{\partial t}}\widetilde{J} - \nabla_T\widetilde{J}$ since

$$\nabla_{T(\bar{t}^+)}\widetilde{J} - \nabla_{T(\bar{t}^-)}\widetilde{J} = -\nabla_{T(\bar{t}^-)}\widetilde{J} \neq 0.$$

(Otherwise, together with $\widetilde{J}(\bar{t}) = 0$, this implies $\widetilde{J} \equiv 0$.)

Choose a vector field along γ which satisfies

$$U(a) = 0 = U(b),\ \langle U(\bar{t}), \nabla_{T(\bar{t}^+)}\widetilde{J} - \nabla_{T(\bar{t}^-)}\widetilde{J}\rangle = -1.$$

Define the vector field along γ

$$X := \frac{1}{c}\widetilde{J} - cU,$$

where c is a small number.

Then $I(X, X) = \frac{1}{c^2}I(\widetilde{J}, \widetilde{J}) - 2I(\widetilde{J}, U) + c^2 I(U, U)$, where

$$I(\widetilde{J}, \widetilde{J}) = 0 \ \ \textit{since}\ \widetilde{J} \in \mathcal{V}_0(a, b).$$
$$I(\widetilde{J}, U) = -\langle U(\bar{t}, \nabla_{T(\bar{t}^+)}\widetilde{J} - \nabla_{T(\bar{t}^-)}\widetilde{J})\rangle = 1.$$

Hence $I(X, X) = -2 + c^2 I(U, U)$.

For sufficiently small c, this is < 0. ■

(1)⇐)follows from (2)⇒(3)⇒.

Similarly, (2)(⇐), (3)⇐ are proved. ■ □

Let us mention a very useful lemma. Recall in Proposition 5.7, we have shown a Jacobi field U is the critical point of $I(X, X)$.

Lemma 5.2. *(Minimizing property of Jacobi field) Let $\gamma : [a, b] \to M$ be a geodesic without conjugate points, U be a Jacobi field along γ, and X a piecewise C^∞ vector field along γ with*

$$X(a) = U(a),\ X(b) = U(b).$$

Then $I(U, U) \le I(X, X)$ where "=" holds iff $X = U$.

Proof. From (5.6.1), we see for any piecewise C^∞ vector field W along γ, we have

$$I(U, W) = \langle \nabla_T U, W\rangle|_a^b.$$

Since $X - U \in \mathcal{V}_0(a, b)$, Theorem **??**(1)(⇒) tells

$$\begin{aligned} 0 \le & I(X - U, X - U) = I(X, X) + I(U, U) - 2I(X, U) \\ = & I(X, X) + \langle \nabla_T U, U\rangle|_a^b - 2\langle \nabla_T U, X\rangle|_a^b \\ = & I(X, X) - \langle \nabla_T U, U\rangle|_a^b \\ = & I(X, X) - I(U, U) \end{aligned}$$

If $I(X, X) = I(U, U)$, we have $I(X - U, X - U) = 0$.

Therefore Theorem **??**(1) tells $X - U = 0$. ■ □

Remark 5.7. *In the case, we derive Lemma 5.6.2 from Theorem* **??***(1)(⇒). In face, the converse is also true.*

*In fact, the results in <u>Theorem**??**</u> can be pushed forward much further to the celebrated <u>Morse index Theorem.</u>*

We particularly observe that for a geodesic $\gamma : [a, b] \to M$ $\gamma(a)$ has a conjugate point in (a, b)⇔ $\exists X \in \mathcal{V}_0(a, b),\ I(X, X) < 0$.

Definition 5.5. *(index and nullity of γ). We call for a geodesic $\gamma : [a, b] \to M$*

$$ind(\gamma) = \max dim\{\mathcal{A} \subset \mathcal{V}_0 | I \text{ is negatively definite on the subspace } \mathcal{A}\}$$

the index of γ.

We call

$$N(\gamma) = dim\{X \in \mathcal{V}_0 | I(X, Y) = 0, \ \forall Y \in \mathcal{V}_0\}$$

the nullity of γ.

Remark 5.8. *In fact, $N(\gamma)$ is equal to the multiplicity of a and b as conjugate values. If $\gamma(b)$ is not conjugate to $\gamma(a)$, then $N(\gamma) = 0$.*

*In this language, Theorem **??**(3) can be restated as*

$$\exists\, \bar{t} \text{ s.t. } N(\gamma|_{[a,\bar{t}]}) \geq 1 \Leftrightarrow index(\gamma) \geq 1.$$

A far-reaching generalization is the following celebrated Theorem.

Theorem 5.7. *(**Morse index Theorem:**) The index of $\gamma : [a, b] \to M$ is the number of $\bar{t} \in (a, b)$ which are conjugate to a, each conjugate value being counted with its multiplicity. The index is always finite. That is*

$$ind(\gamma) = \sum_{a<t<b} N(\gamma|_{[a,t]}) < \infty.$$

In particular, $\gamma(a)$ has only finite many conjugate points along γ.

For the proof, one need to show the index of γ increases by at least v as t passes a conjugate value $\bar{t}$ with multiplicity v. This can be handled by essentially the same trick which was used in the proof of Theorem **??**(3). We refer to [WSY, Chapter 9] for details of proof.(see [JJ, section 4.3] for an analytic proof!!)

It is a good point to reflect our proof of Bonet-Myers Theorem. We show that if *sectionalcurvature* $\geq k\rangle 0$, for a geodesic γ of length $l > \frac{\pi}{\sqrt{k}}$, we have for

$$V(t) = \sin(\frac{\pi}{l} t) E(t)$$

where $E(t)$ is a parallel vector field along γ,

$$I(V, V) = 0.$$

Note when $l = \frac{\pi}{\sqrt{k}}$, sectional curvature=$k > 0$

$$V(t) = \sin(\sqrt{k} t) E(t)$$

is a Jacobi field along γ.(Exercise)

In particular, when *sectional curvature* $= k > 0$, a geodesic γ of length $l > \frac{\pi}{\sqrt{k}}$ contains at least a conjugate point of $\gamma(0)$. Hence $ind(\gamma) \geq 1$, and γ is not (locally) minimizing.

The proof of Bonnet-Myers tells when *sectional curvature* $\geq k$, a geodesic of length $l > \frac{\pi}{\sqrt{k}}$ also contains at least a conjugate point, and $ind(\gamma) \geq 1$.

On the other hand, if *sectional curvature* $= 0$ or *sectional curvature* $= -k$, $k > 0$, the Jacobi field along γ with $J(0) = 0$ are linearly combinations of $dtE(t)$ and $d\sinh(kt)E(t)$ which will never vanish anywhere other than 0. Hence γ does not contain conjugate points. In fact, this is true for the case *sectional curvature* ≤ 0. This is the our next topic.

5.7 The proof of Morse index theorem

Recall we have $I(fT, fT) = \int_a^b (\dot{f})^2 dt \geq 0$, " $=" \Leftrightarrow f \equiv 0$. And

$$I(fT, U) = 0,\ \forall U \in \mathcal{V}_0(a,b),\ \langle U, T\rangle = 0.$$

So we can restrict ourself to the subspace

$$\mathcal{V}_0^{\perp}(a,b) := \{X \in \mathcal{V}_0(a,b) | \langle X, T\rangle = 0\}$$

when studying index and nullity of γ.

For simplicity, let's take $(a,b) = (0,1)$. Firstly,we show $\overline{ind(\gamma) < \infty}$:

We first explain that we can find a finite-dim subspace $\overline{T-1 \text{ of } \mathcal{V}}_0^{\perp}(0,1)$ s.t. the index, nullity of I do not change when restricting to T_1.

By considering the open covers of $\gamma|_{[0,1]}$ by the totally normal neighborhood of each $\gamma(t)$, $t \in [0,1]$, we can find a finite subdivision, $0 = t_0 < t_1 < \cdots < t_k < t_{k+1} = 1$ such that $\gamma|_{[t_i, t_{i+1}]}$ lies in a totally normal neighborhood U_i. In particular, $\gamma|_{[t_i,t_{i+1}]}$ contains no conjugate point for each i.

Define:

$$T_1 := T_1(1) := \{X \in \mathcal{V}_0 : X \text{ is Jacobian along each } \gamma|_{[t_i,t_{i+1}]},\ \forall\ i = 0, \cdots, k\}.$$
$$T_2 := T_2(1) := \{X \in \mathcal{V}_0 : X(t_i) = 0,\ \forall\ i = 0, \cdots, k+1\}$$

Lemma 5.3. *we have*

$$(i)\ \mathcal{V}_0^{\perp}(0,1) = T_1 \bigoplus T_2$$
$$(ii)\ I(T_1, T_2) = 0$$
$$(iii)\ I|_{T_2} \text{ is positive definite.}$$

Proof. Consider the map

$$\varphi : T_1 \to T_{\gamma(t_1)}M \bigoplus \cdots \bigoplus T_{\gamma(t_k)}M,\ J \mapsto (J(t_1), \cdots, J(t_k)).$$

Clearly, this is a linear map, and 1-1.(Since on $\gamma|_{[t_i,t_{i+1}]}$, J is uniquely determined by $J(t_i)$ and $J(t_{i+1})$.)

Therefore φ is a linear isometry, In particular, $dim\, T_1 = nk < \infty$.

Given any $X \in \mathcal{V}_0^{\perp}(0,1)$. Let $J_X := \varphi^{-1}(X(t_1), \cdots, X(t_k))$. Then we have $J_X \in T_1$, $X - J_X \in T_2$.

Moreover, $T_1 \bigcap T_2 = \{0\}$ since $J(t_i) = 0 = J(t_{i+1}) \Rightarrow J \equiv 0$ on $\gamma|_{[t_i,t_{i+1}]}$.
This shows (i) $\mathcal{V}_0^{\perp}(0,1) = T_1 \bigoplus T_2$.
For (ii), we have $\forall\ X_1 \in T_1,\ X_2 \in T_2$

$$I(X_1, X_2) = \langle \nabla_T X_1, X_2 \rangle|_0^1 - \sum_{j=1}^{k} \langle \nabla_{T(t_j^+)} X_1 - \nabla_{T(t_j^-)} X_1, X_2 \rangle = 0.$$

For (iii), any $X \in T_2$

$$I(X, X) = \sum_{i=0}^{k} I_{t_i}^{t_{i+1}}(X_i, X_i) \rangle 0$$

■ □

By Lemma 5.7.1, we obtain immediately

$$ind(\gamma) \leq dim(T_1) \leq \infty$$

and the index, nullity of $I|_{T_1}$ equal to $ind(\gamma)$, $N(\gamma)$ respectively.

Lemma 5.4. *(i)* $\forall\ \tau \in (0,1],\ \exists\ \delta > 0,\ s.t.\ \forall\ \epsilon \in [0,\delta],$ *we have* $ind(\tau - \epsilon) = ind(\tau)$.
(ii) $\forall\ \tau \in [0,1),\ \exists\ \delta > 0,\ s.t.\ \forall\ \epsilon \in [0,\delta],$ *we have* $ind(\tau + \epsilon) = ind(\tau) + N(\tau)$.

Proof. For given τ, we can assume the division we choose previously has the property that $\tau \in (t_j, t_{j+1})$.
Define $T_1(\tau) := \{X \in \mathcal{V}_0^{\perp}(0,1), X|_{[t_i,t_{i+1}]}\ is\ Jacobian,\ i = 0, \cdots, j{-}1.X|_{[t_j,\tau]}\ is\ also\ Jacobian.\}$.
Similarly, consider

$$\varphi^{\tau} : T_1(\tau) \to T_{\gamma(t_1)}M \bigoplus \cdots \bigoplus T_{\gamma(t_j)}M,\ X \mapsto (X(t_1), \cdots, X(t_j))$$

is a linear isometry.
$I^{\tau}|_{T_1(\tau)}$ can be considered as a quadratic form over $T_{\gamma(t_1)}M \bigoplus \cdots \bigoplus T_{\gamma(t_j)}M$ in the sense

$$I_0^{\tau}(x, y) := I_0^{\tau}((\varphi\tau)^{-1}(x), (\varphi\tau)^{-1}(y)).$$

Note $ind(\tau)$ is the index of $I_0^{\tau}|_{T_1(\tau)}$.
Hence $\forall\ X,\ Y \in T_1(\tau)$, denote $X_i = X|_{[t_i,t_{i+1}]},\ X_j = X|_{[t_j,\tau]}$.

$$I_0^{\tau}(x, y) = \sum_{i=0}^{j-1} \langle \nabla_T X_i, Y \rangle|_{t_i}^{t_{i+1}} - \langle \nabla_T X_j(t_j), Y(t_j) \rangle.$$

For given division $0 = t_0 < t_1 < \cdots < t_j < t_{j+1} < \cdots < t_{k+1} = 1$,

$$T_1(\tau) \cong T_{\gamma(t_1)}M \bigoplus \cdots \bigoplus T_{\gamma(t_j)}M$$

is a fixed vector space. So when τ changes, only $\langle \nabla_T \dot{X}_j(t_j), Y(t_j) \rangle$ change in $I_0^{\tau}(x, y)$, where $x = (x^1, \cdots, x^j),\ y = (y^1, \cdots, y^j) \in \bigoplus_{i=1}^{j} T_{\gamma(t_i)}M$.

X_j and Y are Jacobi field on $[t_j, \tau)$ with

$$X_j(t_j) = x^j, \ X_j(\tau) = 0$$
$$Y(t_j) = y^j, \ Y(\tau) = 0$$

By construction, $\gamma|_{[t_j, t_{j+1}]} \subset U_j$ a totally normal neighborhood.

Hence geodesics lying inside U_j depends smoothly on their endpoints.

Since Jacobi fields are variational vector fields geodesic variations, we have $X_j, \ Y|_{[t_j, \tau]}$ also depends continuouly on endpoint $\gamma(t_j)$ and $\gamma(\tau)$.

Therefore $-\langle \nabla_T X_j(t_j), Y(t_j) \rangle$ are smooth w.r.t. τ. That is $I_0^\tau(x, y)$ is a continuous w.r.t τ for given x, y. So for $x \in \bigoplus_{i=1}^{j} T_{\gamma(t_i)}M, \ I_0^\tau(x, y) < 0 (> 0)$ implies

$$\exists\, \delta > 0 \ s.t. \ I_0^{\tau \pm \epsilon}(x, x) < 0, \ \forall \epsilon \in [0, \delta].$$

This tells

$$ind(\tau \pm \epsilon) \geq ind(\tau) \tag{5.7.1}$$
$$ind_+(\tau \pm \epsilon) \geq ind_+(\tau) \tag{5.7.2}$$

By linear algebraic theory, the linear space $T_1(\tau)$ can be decomposed into:

. maximal positive definite subspace

. maximal negative definite subspace

. null space $\{X \in T_1(\tau) | I(X, Y) = 0, \ \forall\ Y \in T_1(\tau)\}$

$n_j = dim(\bigoplus_{i=1}^{j} T_{\gamma(t_i)}M) = ind_+(\tau) + ind(\tau) + N(\tau)$.

Using $ind_+(\tau) = n_j - ind(\tau) - N(\tau)$, we derive from (5.7.2) that

$$n_j - ind(\tau \pm \epsilon) - N(\tau \pm \epsilon) \geq n_j - ind(\tau) - N(\tau)$$

i.e.

$$ind(\tau \pm \epsilon) \leq ind(\tau) + N(\tau) - N(\tau + \epsilon) \leq ind(\tau) + N(\tau) \tag{5.7.3}$$

Combining (5.7.1) and (5.7.3) gives

$$ind(\tau) \leq ind(\tau \pm \epsilon) \leq ind(\tau) + N(\tau) \tag{5.7.4}$$

For any $\xi, \ t_j \langle \xi < \tau < t_{j+1}, \ \forall x \in \bigoplus_{i=1}^{j} T_{\gamma(t_i)}M$, we have

$$I_0^\xi(x, x) - I_0^\tau(x, x) = I_{t_j}^\xi(X_{j,\xi}, X_{j,\xi}) - I_{t_j}^\tau(X_{j,\tau}, X_{j,\tau})$$

where $X_{j,\xi}$ is the Jacobi field with $X_{j,\xi}(t_j) = x_j, X_{j,\xi}(\xi) = 0$, $X_{j,\tau}$ is the Jacobi field with $X_{j,\tau}(t_j) = x_j, X_{j,\tau}(\tau) = 0$.

By minimizing property of Jacobian field, we have

$$I(X_{j,\tau}, X_{j,\tau}) \leq I(X_{j,\xi}, X_{j,\xi})$$

and "=" holds iff $X_{j,\tau} = X_{j,\xi} \Leftrightarrow X_{j,\tau} = 0$.

That is,

$$I_0^\tau(x,x) \le I_0^\xi(x,x),$$

and " =″ holds iff $X_{j,\tau} = 0$

Hence

(i) $I_0^\xi(x,x) < 0 \Rightarrow I_0^\tau(x,x) < 0$

(ii)*Let x be in the null space of* I_0^ξ. *Then* $(\varphi^\xi)^{-1}(x) \in \mathcal{V}_0^\perp(0,\xi)$ *is a Jacobi field vanishing at* 0 *and* ξ.

Observe that $x_j = (\varphi^\xi)^{-1}(x)(t_j) \neq 0$. Since otherwise, we have $(\varphi^\xi)^{-1}(x)|_{[t_j,\xi]} \equiv 0$. $(\gamma|_{[t_j,\xi]}$ contains no conjugate point) and therefore $(\varphi^\xi)^{-1}(x)(x) \equiv 0 \Rightarrow x \equiv 0$, contradicting to $x \neq 0$.

Therefore, we have

$$I_0^\tau(x,x) < I_0^\xi(x,x) = 0,\ \forall\ x \in \textit{the null space of } I_0^\xi.$$

In conclusion, (i)+(iiimplies

$$ind(\tau) \ge ind(\xi) + N(\xi).$$

We have

$$ind(\tau) \le ind(\tau - \epsilon) \le ind(\tau) - N(\tau - \epsilon) \le ind(\tau).$$

$\Rightarrow ind(\tau) = ind(\tau - \epsilon)$.

$$ind(\tau + \epsilon) \le ind(\tau) + N(\tau) \le ind(\tau + \epsilon).$$

$\Rightarrow ind(\tau + \epsilon) = ind(\tau) + N(\tau)$. □

Proof pf Morse:

Lemma 4.7.2⇒if $\gamma(\tau)$ is not a conjugate point of $\gamma(0)$, $\exists\ \delta > 0$ s.t. $ind(t)|_{(\tau-\delta,\tau+\delta)}$ is constant.

. If $\gamma(\tau)$ is conjugate to $\gamma(0)$, $\exists\ \delta > 0$ s.t. $ind(t)|_{(\tau-\delta,\tau)}$ is constant and $ind(t)|_{(\tau,\tau+\delta)}$ is also constant.

And the jump size of $ind(t)$ at $t = \tau$ is $N(\tau)$.

So when t changes from 0 to 1, $ind(t)$ changes from 0, and jump where τ is conjugate value. Since $ind(1) < \infty$, we know this jump can only happen finitely many times.

$\Rightarrow ind(1) = \sum_{0<\tau<1} N(\tau)$. ■

5.8 Cartan-Hadamard Theorem

Recall in (IV) §5 we have shown that when sec ≤ 0, every geodesic is locally minimizing. This indicates that no conjugate points exist in this setting.

Proposition 5.10. *If all sectional curvature of* (M, g) *are* ≤ 0*, the no two points of* M *are conjugate along any geodesic.*

Proof. Let γ be a geodesic with velocity field along velocity field $T(t) = \dot{\gamma}(t)$ it. Let $U(t)$ be a Jacobi field along γ. Then

$$\nabla_T \nabla_T U + R(U, T)T = 0.$$

So $\langle \nabla_T \nabla_T U, U \rangle = -\langle R(U, T)T, U \rangle \geq 0$.

Therefore, $\frac{\mathrm{d}}{\mathrm{d}t} \langle \nabla_T U, U \rangle = \langle \nabla_T \nabla_T U, U \rangle + \langle \nabla_T U, \nabla_T U \rangle \geq 0$, that is, $\langle \nabla_T U, U \rangle$ is non-decreasing.

Note that $\frac{\mathrm{d}}{\mathrm{d}t} \langle \nabla_T U, U \rangle = \frac{1}{2} \frac{\mathrm{d}^2}{\mathrm{d}t^2} \langle U, U \rangle$.

That is, we have shown $\frac{\mathrm{d}^2}{\mathrm{d}t^2} |U(t)|^2 \geq 0$, i.e. $|U(t)|^2$ is convex. So $U(t_0) = 0 = U(t_1)$ $\Rightarrow U \equiv 0$.

□

Remark: In fact, for a normal Jacobi field $U(t)$ with $U(0) = 0$, define $f : (0, \infty) \to \mathbb{R}$ by $f(t) = |U(t)| = \langle U(t), U(t) \rangle^{\frac{1}{2}}$. At the values t with $U(t) \neq 0$, we compute

$$\begin{aligned}
\dot{f} =& \frac{\mathrm{d}}{\mathrm{d}t} f(t) = \frac{\frac{\mathrm{d}}{\mathrm{d}t} \langle U(t), U(t) \rangle}{2 \langle U(t), U(t) \rangle^{\frac{1}{2}}} = \frac{\langle \dot{U}(t), U(t) \rangle}{|U(t)|} \\
\ddot{f}(t) =& \frac{\Big(\langle \ddot{U}(t), U(t) \rangle + \langle \dot{U}(t), \dot{U}(t) \rangle \Big) |U(t)| - \langle \dot{U}(t), U(t) \rangle \frac{\langle \dot{U}(t), U(t) \rangle}{|U(t)|}}{|U(t)|^2} \\
=& - \frac{\langle \dot{U}(t), U(t) \rangle^2}{|U(t)|^3} + \frac{|\dot{U}(t)|^2}{|U(t)|} - \frac{1}{|U(t)|} \langle R(U, T)T, U \rangle \text{ (By Cauchy} - \text{Schwarz)} \\
\geq& - \frac{|\dot{U}(t)|^2}{|U(t)|} + \frac{|\dot{U}(t)|^2}{|U(t)|} - \frac{1}{|U(t)|} K(U, T) \Big(\langle U, U \rangle \langle T, T \rangle - \langle U, T \rangle^2 \Big) \\
=& - K(U, T) |U(t)|.
\end{aligned}$$

That is, $\frac{\mathrm{d}^2}{\mathrm{d}t^2} f(t) \geq -K(U, T) f(t)$, where $f(t) = |U(t)|$, and $f(0) = 0$.

A comparison result:

$$\begin{cases} f''(t) \geq & -\beta f(t) \\ f(0) = & 0 \\ \dot{f}(0) = & 1 \end{cases}$$

and

$$\begin{cases} g''(t) = & -\beta g(t) \\ g(0) = & 0 \\ \dot{g}(0) = & 1 \end{cases}$$

Then $f(t) \geq g(t)$. (Use $(f - g)'' \geq 0$ and $(f - g)(0) = 0 = \frac{\mathrm{d}}{\mathrm{d}t}(f - g)(0)$.)

This is a very useful principle to investigate the geometry of a Riemannian manifold via its Jacobi field and that of the space form. (Lecture 20. 2017.05.02)

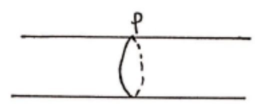
Theorem 5.8 (Cartan-Hadamard)**.** *A complete, simply-connected, n-dimensional Riemannian manifold* (M, g) *with all sectional curvature* ≤ 0 *is diffeomorphic to* $\mathbb{R}^n$*, more precisely,*

$$\exp_p : T_pM \to M$$

is a diffeomorphism.

Remark: In 1898, Hadamard proved such properties for a complete, simply-connected surface with non-positive Gauss curvature. In 1928, E.Cartan extended it to n-dimensional Riemannian manifolds. In fact, Hadamard's result has been proved by von Mongoldt in 1881.

The assumption of 'simply-connectivity' is necesary. For example, the cylinder $C \equiv \{(x, y, z) \in \mathbb{R}^3 : x^2 = z^2 = 1, y \in \mathbb{R}\}$



is complete and with secitonal curvature zero. Its exponential map $\exp_p : T_pM \to M$ is a non-trival covering map.

An important feature of theorem 5.8 is that it not only asserts that M and $\mathbb{R}^n$ are diffeomorphic, but also gives the diffeomorphism map explicitly. Recall we have mentioned Gromall – Meyer(1969) Theorem: any non-compact complete Riemannian manifold (M^n, g) with positive sectional curvature is diffeomorphic to $\mathbb{R}^n$. But in this case, the diffeomorphism map is not necessarily given explicitly by $\exp_p$. This is a big difference between our understanding about nonpositively curved complete simply-connnected Riemannian manifold and positively curved non-compact complete Riemannian manifold, although their topology are both trival.

Theorem 5.8 is a direct consequence of Proposition 5.10 and the following general result.

Theorem 5.9. *Let* (M, g) *be a complete, connected, n-dimensional Riemannian manifold, and let p be a point of M such that no point of M is conjugate to p along any geodesic. Then*

$$\exp_p : T_pM \to M$$

is a covering map. In particular, if M is simply-connected, then M is diffeomorphic to $\mathbb{R}^n$*.*

Proof. We first make it clear what the assuption "p has no conjugate point" tell us: For $\exp_p : T_pM \to M$, we have the tensor $\left(\exp_p\right)^* g$ on T_pM which is defined as $\forall V, W \in T_X(T_pM), \forall X \in T_pM, \left(\exp_p\right)^* g(V, W) = g\left((d\exp_p)_X V, (d\exp_p)_X W\right)$.

"p has no conjugate point" $\Rightarrow \left(d\exp_p\right)_X : T_X(T_pM) \to T_{\exp_p}M$ is 1-1. Therefore $(\exp_p)^* g(V, V) = 0 \Leftrightarrow \left(d\exp_p\right)_X (V) = 0 \Leftrightarrow V = 0$. And, hence, $\left(\exp_p\right)^* g$ is a Riemannian metric on T_pM. That is,

$$\exp_p : (T_pM, \left(\exp_p\right)^* g) \to (m, g)$$

is a local isometry.

Moreover, we claim $(T_pM, \left(\exp_p\right)^* g)$ is complete.

That is beacuase, all straight lines throgh $0 \in T_pM$ are geodesics of $(T_pM, \left(\exp_p\right)^* g)$ since their images under the local isometry $\exp_p : T_pM \to M$ are geodeisc in M. That is, all geodesics through $0 \in T_pM$ can be defined for all t. It follows that T_pM is geodesic complete and, hence, complete, by Hopf-Rinow.

In concluision, "p has no conjugate point" $\Rightarrow$ "$\exp_p$ is a local isometry and T_pM is complete". Then theorem 5.9 is a consequence of the following lemma:

Lemma 5.5. *Let M and N be connected Riemannian manifolds with M complete and let $\phi : M \to N$ be a local isometry. Then N is complete$_1$ and ϕ a covering map$_2$ onto$_3$ N.*

Remark: Lemma 5.5 has 3 conclusions. Note that the completeness of M is needed. For example the inclusion map $i : B(0,1) \subset \mathbb{R}^2 \to \mathbb{R}^2$ is a local isometry but the open disk $B(0,1)$ is not complete. i is not a covering map.

Proof of Lemma 5.5.

1, N is complete: We will show any geodesic on N is depend for all t. For any geodesic $\gamma : (-\epsilon, \epsilon) \to N$, there exists a $p_0 \in M$ s.t. $\gamma(0) = \phi(p_0)$.

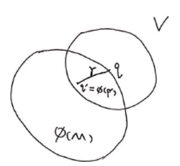
Then we find the geodesic c in M with $c(0) = p_0$, $\dot{c}(0) = \left(\mathrm{d}\phi_{p_0}\right)^{-1} \dot{\gamma}(0)$. This is possible since ϕ is a local isometry.

One can check the curve $\phi \circ c$ is a geodesic (since ϕ is a local isometry which preserves geodesic), and $\phi \circ c(0) = \phi(p_0)$, $(\phi \dot{\circ} c)(0) = \mathrm{d}\phi_{p_0}\dot{c}(0) = \dot{\gamma}(0)$.

Hence $\gamma = \phi \circ c$.

M is complete $\Rightarrow$ c is defined for all $t \Rightarrow \gamma$ is defined for all t . Hopf-Rinow tells N is complete.

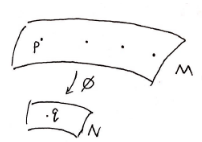
2, ϕ is onto N: That is, we have to show $\phi(M) = N$. Since ϕ is everywhere regular (i.e. $\forall p \in M$, $\mathrm{d}\phi_p$ is 1-1), by iverse function theorem, ϕ is open and, in particular, $\phi(M)$ is open.



In fact, $\phi(M)$ is also closed, and, hence, $\phi(M) = N$. Following is the reason.

Let $q \in \overline{\phi(M)}$, and let V be a totally normal neighborhood of q. There is a $q' \in V$ of the form $q' = \phi(p')$ for $p' \in M$. Let γ be the geodesic in V s.t. $\gamma(0) = q'$, $\gamma(1) = q$. Consider the geodesic c in M s.t. $c(0) = p'$, $\dot{c}(0) = \left(\mathrm{d}\phi_{p'}\right)^{-1} \dot{\gamma}(0)$. Then $\gamma = \phi \circ c$. Define $\underline{p = c}(1)$, then $\phi(p) = \phi(c(1)) = phi \circ c(1) = \gamma(1) = q$. Therefore $q \in \phi(M)$. That is $\overline{\phi(M)} \subset \phi(M)$, so $\phi(M)$ is closed.

3, ϕ is a covering map:



For fixed $q \in N$, let $B(0, 2\epsilon) = \{Y \in T_qN : \|Y\| < 2\epsilon\} \subset T_qN$, where $\epsilon > 0$ is small enough such that $\exp_p$ is a diffeomorphism. Suppose $p \in \phi^{-1}(q)$. We have

$$\begin{array}{ccc} B(0,2\epsilon) \subset T_pM & \xrightarrow{\mathrm{d}\phi_p} & B(0,2\epsilon) \subset T_qN \\ \Big\downarrow \exp_p & & \Big\downarrow \exp_q \\ B_p(2\epsilon) \subset M & \xrightarrow{\phi} & B_q(2\epsilon) \subset N \end{array} \qquad (\heartsuit)$$

where $\phi \circ \exp_p = \exp_q \circ \mathrm{d}\phi_p$.

This is because, $\forall V \in B(0, 1) \subset T_pM$, $\gamma(t) = \phi \circ \exp_p(tV)$, $t \in [0, 2\epsilon)$ is a geodesic in N s.t. $\gamma(0) = \phi(p) = q$, $\dot\gamma(0) = \mathrm{d}\phi_p(V)$. On the other hand, $c(t) = \exp_p \circ \mathrm{d}\phi_p(tV) = \exp_p\big(t\mathrm{d}\phi_p(V)\big)$ is a geodesic in N with $c(0) = \exp_q O = q$, $\dot c(0) = \mathrm{d}\phi_p(V)$. Hence $\exp_q \circ \mathrm{d}\phi_p(tV) = \phi \circ \exp_p(tV)$, $\forall V \in B(0, 1) \subset T_pM$, $\forall t \in [0, 2\epsilon) \Rightarrow$

$$\exp_q \circ \mathrm{d}\phi_p|_{B(0,2\epsilon)\subset T_pM} = \phi \circ \exp_p(tV)|_{B(0,2\epsilon)\subset T_pM} \tag{5.8.1}$$

That is, the diagram $(\heartsuit)$ commutes.

Therefore, we have (using the fact $\exp_q : B(0, 2\epsilon) \subset T_pM \to B_q(2\epsilon) \subset N$ is a diffeomorphisms) the LHS of (5.8.1) is a diffeomorphism. Since $\exp_p : B(0, 2\epsilon) \subset T_pM \to B_p(2\epsilon) \subset M$ is surjective, and $\phi \circ \exp_p$ is a diffeomorphism by (5.8.1), we have $B(0, 2\epsilon) \subset T_pM \to B_p(2\epsilon) \subset M$ is a diffeomorphism. Therefore, (5.8.1) $\Rightarrow$ $\phi = \exp_q \circ \mathrm{d}\phi_p \circ \exp_p^{-1}$ is also a diffeomorphism.

Now let, $W = \exp_q(B(0, \epsilon)) \subset N$ and $\forall p \in M$, $W_p = \exp(B(0, 2\epsilon)) \subset M$. We claim that

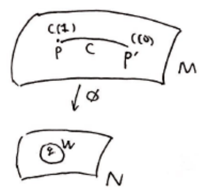
(1): $\phi^{-1}(W) = \cup_{p\in\phi^{-1}(q)} W_p$.

Now that $\phi : W_p \to W$ is a diffeomorphism by our previous argument. So in order to show $\phi : M \to N$ is a covering map, we only need to show the claim and

(2): $W_{p_i} \cap W_{p_j} \neq \phi$, $\forall p_i, p_j \in \phi^{-1}(q)$, $p_i \neq p_j$.

Proof of (1)

$\phi : W_p \to W$ is a diffeomorphism tells $\cup_{p\in\phi^{-1}(q)} W_p \subset \phi^{-1}(W)$. Now $\forall p' \in \phi^{-1}(W)$, let γ be the geodesic in W with $\gamma(0) = \phi(p')$, $\gamma(1) = q$ and of length $d(\phi(p'), q)$.



Let c be the geodesic in M with $c(0) = p'$, $\dot{c}(0) = \left(\mathrm{d}\phi_{p'}^{-1}\right)\dot{\gamma}(0)$. Then $\gamma = \phi \circ c$. M is complete $\Rightarrow q = \gamma(1) = \phi \circ c(1)$ is well defined. In particular, $c(1) \in \phi^{-1}(q)$ and $p' = c(0) \in W_{c(1)}$. This implies $\phi^{-1}(W) \subset \cup_{p\in\phi^{-1}(q)} W_p$. Hence, we prove the claim (1).

Proof of (2)

Suppose $\exists p_1, p_2 \in \phi^{-1}(q)$, $p_1 \neq p_2$ s.t. $W_{p_1} \cap W_{p_2} \neq \phi$, then we have $p_2 \in \exp_{p_1}(B(0, 2\epsilon))$.

But ϕ is a diffeomorphism on the $B(0, 2\epsilon) \subset T_{p_1}M \to B_{p_1}(2\epsilon) \subset M$. Hence $\phi(p_1) = \phi(p_2) \to p_1 = p_2$.

□

5.9 Uniqueness of simply-connected space forms

Now we can prove the 'uniqueness part' of Theorem (5.1) which we started in the very beginning of this Chapter. In fact, we can prove.

Theorem 5.10 (Uniqueness)**.** *Let (M, g) and $(\overline{M}, \overline{g})$ be two n-dimensional simply-connected space form with sectional curvature $c \in \mathbb{R}$. Let $p \in M$, $\overline{p} \in \overline{M}$, $\{e_1, \ldots, e_n\}$, $\{\overline{e}_1, \ldots, \overline{e}_n\}$ be orthonormal basis of T_pM, $T_{\overline{p}}\overline{M}$, respectively. Then there exists a unique isometry $\phi : M \to \overline{M}$ such that $\phi(p) = \overline{p}$, $\mathrm{d}\phi_p(e_i) = \overline{e}_i$, $\forall i$.*

Proof. Since $K(Ag) = \frac{1}{A}K(g)$, we only need consider the cases $c = 0, +1, -1$. We first show the existence of such an isometry.

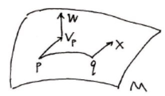
Case 1. $c = 0, -1$. By Cartan-Hadamard, the maps $\exp_p : T_pM \to M$ and $\exp_{\overline{p}} : T_{\overline{p}}\overline{M} \to \overline{M}$ are both diffeomorphisms. Let Φ be the unique isometry from T_pM to $T_{\overline{p}}\overline{M}$ (as inner product) such that $\Phi(e_i) = \overline{e}_i$, $i = 1, \ldots, n$.

$$\begin{array}{ccc} T_pM & \xrightarrow{\Phi} & T_{\overline{p}}\overline{M} \\ \downarrow{\scriptstyle \exp_p} & & \downarrow{\scriptstyle \exp_{\overline{p}}} \\ M & \xrightarrow{\phi} & \overline{M} \end{array}$$

This leads to $\phi : M \to \overline{M}$ where $\phi = \exp_{\overline{p}} \circ \Phi \circ (\exp_p)^{-1}$. Notice that ϕ is a diffeomorphism. So it remains to show $\phi^*\overline{g} = g$.

It is enough to show $\forall q \in M\ \forall X \in T_qM$, $\phi^*\overline{g}(X, X) = g(X, X)$ i.e. the lenghth of $\mathrm{d}\phi(X)$ equals the length of X, $\forall X \in T_pM$, $\forall q$.

Recall Lemma ... tells, there is a unique Jacobi field U along the geodesic from p to q s.t. $U(0) = 0$ and $U(1) = X$. So we can calculate $|X|^2 = g(X, X)$ by calculate the whole Jacobi field $U(t)$. In fact we can construct $U(t)$ explicitly.



Let $V_p \in T_pM$ be such that $\exp_p V_p = q$. Let $W \in T_{V_p}(T_pM)$ be such that $\left(\mathrm{d}\exp_p\right)_{V_p}(W) = X$. Consider the variation $F(t, s) = \exp_p t\left(V_p + sW\right)$. We know the variational field $U(t) = \frac{\partial}{\partial s}|_{s=0}F(t, s)$ is the Jacobi field with $U(0) = p$, $U(1) = \frac{\partial}{\partial s}\exp_p\left(V_p + sW\right) = \left(\mathrm{d}\exp_p\right)_{V_p}(W) = X$, $\dot{U}(0) = X$, $\dot{U}(0) = W$.

Next, we show $\overline{g}\left(\mathrm{d}\phi_p(X), \mathrm{d}\phi_p(X)\right)$ can also be calculated by computing a whole Jacobi field.

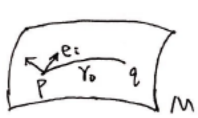
Consider $\overline{F}(t, s) = \exp_{\overline{p}} t\left(\Phi(V_p) + s\Phi(W)\right)$ (we identify $T_{V_p}(T_pM)$ with T_pM). Similarly, the variation field $\frac{\partial}{\partial s}|_{s=0}\overline{F}(t, s) = \overline{U}(t)$ is a Jacobi field with $\overline{U}(0) = 0$, $\dot{\overline{U}}(t)$. We claim $\overline{U}(1) = \mathrm{d}\phi_q(X)$. This is seen from

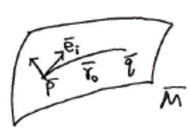
$$\begin{aligned}
\phi \circ F(t, s) &= \exp_{\overline{p}} \circ \Phi \circ (\exp_p)^{-1} \circ \exp_p t(V_p + sW) \\
&= \exp_{\overline{p}} \circ \Phi(t(V_p + sW)) \\
&= \exp_{\overline{p}} t(\Phi(V_p) + s\Phi(W)) \\
&= \overline{F}(t, s)
\end{aligned}$$

Hence

$$\begin{aligned}
\mathrm{d}\phi_{F(t,s)}((U(t)) &= \mathrm{d}\phi \circ \frac{\partial}{\partial s}|_{s=0}F(t, s) \\
&= \frac{\partial}{\partial s}|_{s=0}\left(\phi \circ F(t, s)\right) \\
&= \frac{\partial}{\partial s}|_{s=0}\overline{F}(t, s) \\
&= \overline{U}(t)
\end{aligned}$$

In particular, $\overline{U}(1) = \mathrm{d}\phi_{F(1,0)}((U(1)) = \mathrm{d}\phi_q(X)$. Hence, it remains to show $|\overline{U}(1)| = |U(1)|$.





Pick parallel orthonormal vector fields $\{e_1(t), \ldots, e_n(t)\}$, $\{\overline{e}_1(t), \ldots, \overline{e}_n(t)\}$ along γ_0, $\overline{\gamma}_0$ respectively such that $e_i(0) = e_i$, $\overline{e}_i(0) = \overline{e}_i$. Then $U(t) = f^i(t)e_i(t)$, $\overline{U}(t) = \overline{f}^i(t)\overline{e}_i(t)$ for some functions f^i, $\overline{f}^i$.

Solving Jacobi equation in M, $\overline{M}$ respectively:

in M: $\nabla_T\nabla_T U + R(U,T) = 0$, $T = \dot{\gamma}_0(t) \Leftrightarrow \ddot{f}^i(t)e_i(t) + R(f^j e_j, T)T = 0 \Leftrightarrow \ddot{f}^i(t) + f^j\langle R(e_j, T)T, e_i\rangle = 0$, $i = 1, \ldots, n$. (sectional curvature=c $\Rightarrow \langle R(e_j, T)T, e_i\rangle = c(\delta_{ij}\langle T, T\rangle - \langle T, e_i\rangle\langle T, e_j\rangle) \Leftrightarrow \langle V_p, V_p\rangle = \langle \dot{\gamma}_0(0), \dot{\gamma}_0(0)\rangle = 0$, $i = 1, \ldots, n$.). Recall we have further $U(0) = 0$, $\dot{U}(0) = W$. Hence f^i, $i = 1, \ldots, n$ satisfing the following equation.

$$\begin{cases} \ddot{f}^i(t) + cf^j(t)\left(\delta_{ij}\langle V_p, V_p\rangle - \langle V_p, e_i\rangle\langle V_p, e_j\rangle\right) = 0, i = 1, \ldots, n \\ f^i(0) = 0 \\ \dot{f}^i(0) = \langle W, e_i\rangle \end{cases}$$

in $\overline{M}$: $\overline{f}^i$, $i = 1, \ldots, n$ satisfies

$$\begin{cases} \ddot{\overline{f}}^i(t) + c\overline{f}^j(t)\left(\delta_{ij}\langle \Phi(V_p), \Phi(V_p)\rangle - \langle \Phi(V_p), \overline{e}_i\rangle\langle \Phi(V_p), \overline{e}_j\rangle\right) = 0, i = 1, \ldots, n \\ \overline{f}^i(0) = 0 \\ \dot{\overline{f}}^i(0) = \langle \Phi(W), \overline{e}_i\rangle \end{cases}$$

Since $\Phi : T_pM \to T_{\overline{p}}\overline{M}$ is an isometry, we have $\langle V_p, V_p\rangle = \langle \Phi(V_p), \Phi(V_p)\rangle$, $\langle V_p, e_i\rangle = \langle \Phi(V_p), \Phi(e_i)\rangle = \langle \Phi(V_p), \overline{e}_i\rangle$, $\langle W, e_i\rangle = \langle \Phi(W), \Phi(e_i)\rangle = \langle \Phi(W), \overline{e}_i\rangle$. By the uniqueness of the solution, we have $f^i(t) = \overline{f}^i(t)$, $\forall t$, $\forall i \in \{1, \ldots, n\}$. In particular, $|U(1)|^2 = \sum_i f^i(1)^2 = \sum_i \overline{f}^i(1)^2 = |\overline{U}(1)|^2$. This proves the existence of an isometry claimed in Theorem for the case $c = 0$ or t. The uniqueness of the isometry ϕ follows from the following lemma.

Lemma 5.6. *Let M be a connected Riemannian manifold and N be a Riemannian manifold. Let $\phi_1, \phi_2 : M \to N$ be two locally isometry such that $\exists x \in M$, $\phi_1(x) = \phi_2(x) = x' \in N$, $(\mathrm{d}\phi_1)_x = (\mathrm{d}\phi_2)_x : T_xM \to T_{x'}N$. Then $\phi_1 = \phi_2$.*

Proof. Define $A \subset M$ to be $A = \{z \in M : \phi_1(z) = \phi_2(z), (\mathrm{d}\phi_1)_z = (\mathrm{d}\phi_2)_z\}$. By assumption, x in A, i.e. $A \neq \phi$. From the definition, A is closed.

Next, we show A is open. Thus since M is connected, we have $A = M$. Suppose $z \in A$, then $z' = \phi_1(z) = \phi_2(z) \in N$. Choose $\delta > 0$ small enough, such that $\exp_z : B(0, \delta) \subset T_zM \to B_z(\delta) \subset M$ is a diffeomorphism and $\exp_{z'} : B(0, \delta) \subset T_{z'}N$ is defined.

$$\begin{array}{ccc} B(0,\delta) \subset T_zM & \xrightarrow{(\mathrm{d}\phi_i)_z} & B(0,\delta) \subset T_{z'}N \\ \Big\downarrow \exp_z & & \Big\downarrow \exp_{z'} \\ B_z(\delta) \subset M & \xrightarrow{\phi_i} & B_{z'}(\delta) \subset N \end{array}$$

By similar argument in the proof of Lemma 5.5, we have $\phi_i \circ \exp_z = \exp_{z'} \circ (\mathrm{d}\phi_i)_z$, $i = 1, 2$. Notice that $\exp_z |_{B_z(\delta) \subset M}$ is invertible, we have $\phi_i = \exp_{z'} \circ (\mathrm{d}\phi_i)_z \circ (\exp_z)^{-1}$. Now we check $\forall y \in B_z(\delta)$,

$$\phi_1(y) = \exp_{z'} \circ (\mathrm{d}\phi_1)_z \circ (\exp_z)^{-1}(y) = \exp_{z'} \circ (\mathrm{d}\phi_2)_z \circ (\exp_z)^{-1}(y) = \phi_2(y)$$

and $(\mathrm{d}\phi_1) = (\mathrm{d}\phi_2)$. Therefore, we have $B_z(\delta) \subset A \Rightarrow A$ is open. □

Case 2. $c = +1$. We can suppose $M = \mathbb{S}^n$. $\forall p \in \mathbb{S}^n$, any two geodesic from p will together at its antipodal point p'. Therefore, $\exp_p^{-1} : \mathbb{S}^n \backslash \{p'\} \to T_p\mathbb{S}^n$ is a well-defined smooth map.

$$\begin{array}{ccc} T_p\mathbb{S}^n & \xrightarrow{\ \Phi\ } & T_{\overline{p}}\overline{M} \\ \uparrow{\scriptstyle \exp_p^{-1}} & & \downarrow{\scriptstyle \exp_{\overline{p}}} \\ \mathbb{S}^n \backslash \{p'\} & & \overline{M} \end{array}$$

where Φ is the isometry (of inner product spaces) with $\Phi(e_i) = \overline{e}_i$.

Then $\phi : exp_{\overline{p}} \circ \Phi \circ \exp_p^{-1}$ is a local isometry by the same argument as in the first case.

Next, we extend ϕ to be defiend on the whole $\mathbb{S}^n$. Pick any $z \in \mathbb{S}^n \backslash \{p'\}$, $z \neq p$. Let $z^1 = -z$ is the antipodal point of z. Let $\phi(z) = \overline{z} \in \overline{M}$, then $(\mathrm{d}\phi)_z : T_z\mathbb{S}^n \to T_{\overline{z}}\overline{M}$. Define $\psi : \mathbb{S}^n \backslash \{z'\} \to \overline{M}$ as $\psi = \exp_{\overline{z}} \circ (\mathrm{d}\phi)_z \circ \exp_z^{-1}$.

Similar arguments tell that ψ is also a locally isometry. Consider the connected Riemannian manifold $W = \mathbb{S}^n \backslash \{p', z'\}$. We have two local isometries $\phi, \psi : W \to \overline{M}$.

Oberserve that $\psi(z) = \exp_{\overline{z}} \circ (\mathrm{d}\phi)_z \circ \exp_z^{-1}(z) = \overline{z} = \phi(z)$, $(\mathrm{d}\psi)_z = (\mathrm{d}\phi)_z$. By lemma 5.6, we have $\phi = \psi|_W$. Now define $\theta : \mathbb{S}^n \to \overline{M}$ by

$$\theta(y) = \begin{cases} \phi(y), \text{if } y \in \mathbb{S}^n \backslash \{p'\} \\ \psi(y), \text{if } y \in \mathbb{S}^n \backslash \{z'\} \end{cases}$$

This is a well-defined c^∞ map on $\mathbb{S}^n$, and θ is a local isometry. By lemma 5.5, we have θ is a covering map. Since $\overline{M}$ is simply-connected, θ is a diffeomorphism and hence an isometry. Moreover $\mathrm{d}\theta(e_i) = \overline{e}_i$. This proves the existence. The uniqueness follows again from lemma 5.6.

□

Theorem 5.10 has ery interesting consequnces. When $M = \overline{M}$, we have

Corollary 5.1. *Let M be a n-dimensional complete simply-connected Riemannian manifold. Then M is a space-form iff $\forall p, \overline{p} \in M$, and any orthonormal basis $\{e_1, \ldots, e_n\}$, $\{\overline{e}_1, \ldots, \overline{e}_n\}$ of T_pM, $T_{\overline{p}}M$, respectively, there exists an isometry $\phi : M \to M$ s.t. $\phi(p) = \overline{p}$, $\mathrm{d}\phi(e_i) = \overline{e}_i$, $\forall i$.*

Definition 5.6 (Homogenous Riemannian manifolds)**.** *A Riemannian manifold (M, g) is called homogenous is $\forall p, q \in M$, there exists an isometry*

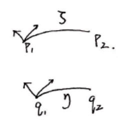
$$\phi : M \to M$$

such that $\phi(p) = q$

(M, g) is called two-point homogenous, if for any two pairs of points p_1, p_2 and $q_1, q_2 \in M$ with $d(p_1, p_2) = d(q_1, q_2)$, there exists an isometry $\phi : M \to M$ s.t. $\phi(p_i) = q_i$, $i = 1, 2$.

Corollary 5.2. *All simply-connected space forms are two-point homogenous.*

Proof.



Let $d(p_1, p_2) = d(q_1, q_2) = d$. Let $\eta, \xi : [0, \alpha] \to M$ be two normal geodesics with $\xi(0) = p_1$, $\xi(\alpha) = p_2$, $\eta(0) = q_1$, $\eta(\alpha) = q_2$. (The existence is guaranted by completeness via Hopf-Rinow).

Pick orhtonormal basis $\{e_1, \dots, e_n\}$ and $\{e'_1, \dots, e'_n\}$ of $T_{p_1}M$ and $T_{q_1}M$, respectively, where $e_1 = \dot{\xi}(0)$, $e'_1 = \dot{\eta}(0)$. Then Theorem 5.10 $\Rightarrow \exists$ an isometry $\phi : M \to M$ with $\phi(p_1) = q_1$, $\mathrm{d}\phi(e_i) = e'_i$ $\forall i$. So $\phi \circ \xi$ is a geodesic with $\phi \circ \xi(0) = \phi(p_1) = q_1$, $(\dot{\phi \circ \xi}) = \mathrm{d}\phi(\dot{\xi}(0)) = \mathrm{d}\phi(e_1) = e'_1 = \dot{\eta(0)}$.

Therefore $\phi \circ \xi = \eta$. In particular $\phi(p_2) = \phi(\xi(\alpha)) = \eta(\alpha) = q_2$.

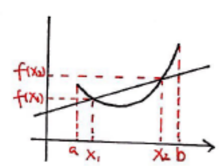
□

5.10 Convexity: Another application of Cartan-Hadamard Theorem

Convex functions and convex (sub)sets are important and useful concepts in analysis. We discuss these topics on Riemannian manifolds in this section.

What is a convex function?

Recall that we call a function $f : [a, b] \to \mathrm{R}$ to be convex if $f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda f(x_1) + (1 - \lambda) f(x_2)$, $\forall x_1, x_2 \in [a, b]$, $\lambda \in [0, 1]$. One can prove that a convex function must be Lipschitz continuous.



Recall for a C^∞ function $f : [a, b] \to \mathrm{R}$, it is convex iff $f'' \geq 0$ on $[a, b]$. This can be shown via its Taylor expansion. That is

$$\begin{aligned} f(x) =& f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + \dots \\ =& f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x^*)}{2}(x - x_0)^2 \end{aligned}$$

for some x^* lying between x_0 and x.

($\Rightarrow$) Apply to the case $x = x + h, x_0 = x$ and $x = x - h, x_0 = x$, we have $f''(x) = \lim_{h\to 0} \frac{f(x+h)+f(x-h)-2f(x)}{h^2}$. Also convexity implies $f(x) = f\left(\frac{x+h}{2} + \frac{x-h}{2}\right) \leq \frac{f(x+h)+f(x-h)}{2}$. Hence $f''(x) \geq 0$.

($\Leftarrow$) Apply to $x_0 = \lambda x_1 + (1-\lambda)x_2, x = x_1$ gives $f(x_1) \geq f(\lambda x_1 + (1-\lambda)x_2) + f'(x_0)(1-\lambda)(x_1 - x_2)$. Apply to $x_0 = \lambda x_1 + (1-\lambda)x_2, x = x_2$ gives $f(x_2) \geq f(\lambda x_1 + (1-\lambda)x_2) + f'(x_0)\lambda(x_2 - x_1)$. Multiply the first by λ, multiply the second by $(1-\lambda)$, and add them up providing $\lambda f(x_1) + (1-\lambda)f(x_2) \geq f(\lambda x_1 + (1-\lambda)x_2)$.

We say a C^∞ function f is strictly convex, if $f'' > 0$. Now consider a function $f : M \to \mathbb{R}$ where M is a Riemannian manifold. A suitable definition of convexing is:

Definition 5.7 (convex functions)**.** *We call a function $f : M \to \mathbb{R}$ a convex function if for any geodesic $\gamma : [a,b] \to M$, $f \circ \gamma$ is convex, i.e. if $\forall t_1, t_2 \in [a,b]$, $\forall \lambda \in [0,1]$, it holds that $f(\lambda(\lambda t_1 + (1-\lambda)t_2)) \leq \lambda f(\gamma(t_1)) + (1-\lambda)f(\gamma(t_2))$.*

Proposition 5.11. *A C^∞ fucntion $f : M \to \mathbb{R}$ is convex iff $(f\circ\gamma)'' \geq 0$ for all geodesic γ, which is further equivalent to* Hess $f \geq 0$*, i.e.* Hess f *is positive semidefinite.*

Proof. C^∞ $f : M \to \mathbb{R}$ is convex iff $(f\circ\gamma)'' \geq 0$, $\forall\gamma$ follows from our previous discussions.

Notice further that for any $p \in M$, any $V_p \in T_pM$, letting $\gamma(t)$ be the geodesic with $\gamma(0) = p$, $\dot\gamma(0) = V_p$, we have

$$\begin{aligned}&\mathrm{Hess} f(V_p, V_p) = \mathrm{Hess} f(\dot\gamma(t), \dot\gamma(t))|_{t=0}\\ =&\nabla^2 f(\dot\gamma, \dot\gamma)|_{t=0} = \nabla(\nabla f)(\dot\gamma(t), \dot\gamma(t))|_{t=0}\\ =&\nabla_{\dot\gamma(0)}(\nabla_{\dot\gamma(t)} f) - \nabla_{\nabla_{\dot\gamma}\dot\gamma} f = (f\circ\gamma)''.\end{aligned}$$

Hence $(f\circ\gamma)'' \geq 0, \forall\gamma \Leftrightarrow \mathrm{Hess} f(V_p, V_p) \geq 0, \forall p, \forall V_p \in T_pM$.

□

We say a C^∞ function $f : M \to \mathbb{R}$ to be a strictly convex funciton if Hess$f > 0$.

Next, let us consider a particular function on M. Given a fixed point $O \in M$, consider the funtion $\varrho(\cdot) = d(\cdot, O) : M \to \mathbb{R}$.

Theorem 5.11. *Let M be a complete, simply-connected Riemanian manifold with non-positive sectional curvature. Let $O \in M$. Then the function ϱ^2 is C^∞ and strictly convex.*

Example: In $\mathbb{R}^n$ with the canonical Euclidean metric, let $O = 0 \in \mathrm{R}^n$, we compute

$$\begin{aligned}\mathrm{Hess}\varrho^2(X,X) =& X^iX^j\frac{\partial^2}{\partial x^i\partial x^j}\varrho^2(x) = X^iX^j\frac{\partial^2}{\partial x^i\partial x^j}\sum_k (x^k)^2\\ =& X^iX^j 2\sum_k \delta_{kj}\delta_{ki} = 2\sum_k (x^k)^2 = 2|X|^2\end{aligned}$$

First, we observe, without any curvature restriction, ϱ^2 is always "locally" strictly convex.

Lemma 5.7. *Let M be a Riemannian manifold, and $O \in M$. Then there exists a neighbor U_0 of O s.t. ϱ^2 is smooth and strictly convex in U_0.*

Proof. Let $(U, x^1, \dots, x^n)$ be a normal coordinate neighborhood of $O \in M$, such that $x^i(O) = 0$. Then $\varrho^2(0) = \sum_{i=1}^n (x^i)^2$, $\forall x \in U$.

Recall any geodesic γ with $\gamma(0) = O$, $\dot\gamma(0) = V(= V^1, \dots, V^n) \in T_O M$, can be written as $\gamma(t) = (x^1(t), \dots, x^n(t))$ where $x^i(t) = V^i t$, $\varrho^2 \circ \gamma = \sum_{i=1}^n t^2 (V^i)^2$. Hence $\mathrm{Hess}\varrho^2(V, V) = (\varrho^2 \circ \gamma)'' = 2\sum_{i=1}^n (V^i)^2 > 0$. Therefore, there exists a neighborhood of O, $U_O \subset U$, s.t. $\mathrm{Hess}\varrho^2$ is positive definite on U_O.

□

But for "global" results, we need curvature restriction. Let us recall the first(second) variation formula for length functions.

Lemma 5.8. *Let $\gamma : [a, b] \to M$ be a normal geodeisc, and $F : [a, b] \times (-\epsilon, \epsilon) \to M$ be a variation of γ with variational field $V(t)$, $t \in [a, b]$. Then*

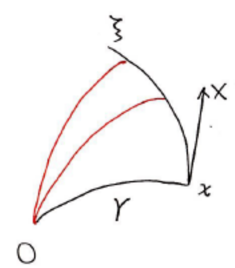
$$L'(0) = \frac{\mathrm{d}}{\mathrm{d}s}|_{s=0} L(s) = \langle V(t), \dot\gamma(t)\rangle|_a^b - \int_a^b \langle V(t), \nabla_{\dot\gamma}\dot\gamma\rangle \mathrm{d}t$$
$$L''(0) = \frac{\mathrm{d}^2}{\mathrm{d}s^2}|_{s=0} L(s) = \langle \nabla_V V, \dot\gamma\rangle|_a^b + \int_a^b \langle \nabla_{\dot\gamma} V^\perp, \nabla_{\dot\gamma} V^\perp\rangle - \langle R(V^\perp, \dot\gamma)\dot\gamma, V^\perp\rangle \mathrm{d}t$$

where $V^\perp = V - \langle V, \dot\gamma\rangle\dot\gamma$.

Proof. of Theorem 5.11.

By Cartan-Hadamard Theorem and the definition of $\exp_O$, we have $x \in M$, $\varrho^2(x) = g\left(\exp_O^{-1}(x), \exp_O^{-1}(x)\right) = |\exp_O^{-1}(x)|^2$ is a C^∞ function on M. By Lemma 5.7, it remains to show that $\mathrm{Hess}\varrho^2(V, V) > 0$ for any $x \in M$, any $0 \neq V \in T_x M$.

Let $\xi : [0, \epsilon] \to M$ be the geodesic of M with $\xi(0) = x$, $\dot\xi(0) = V$. Let γ_s, $s \in [0, \epsilon]$ be the geodesic from O to $\xi(s)$. Let us parametrize γ_s to be $\gamma_s : [0, r] \to M$ where $r = \varrho(x)$. Hence $\gamma = \gamma_0$ is a normal geodesic.



Hence, we have the following variation $F : [0, r] \times [0, \epsilon] \to M$, where $F(t, s) = \gamma_s(t)$. Notice that the corresponding variational field $V(t)$, satisfies $V(0) = O$, and

$$V(r) = \frac{\partial}{\partial s}|_{s=0} F|_{t=r} = \frac{\partial}{\partial s}|_{s=0} F(r, s)$$
$$= \frac{\partial}{\partial s}|_{s=0} \xi(s) = \dot\xi(0) = V$$

Now we compute

$$
\begin{aligned}
\mathrm{Hess}\varrho^2(V,V) =& (\varrho^2 \circ \xi)''(O) \\
=& \frac{\mathrm{d}^2}{\mathrm{d}s^2}(\varrho^2 \circ \xi(s))|_{s=0} \\
=& 2\varrho(\xi(s))\frac{\mathrm{d}^2}{\mathrm{d}s^2}\varrho(\xi(s))|_{s=0} + 2\left(\frac{\mathrm{d}}{\mathrm{d}s}\varrho(\xi(s))|_{s=0}\right)^2 \\
=& 2r\frac{\mathrm{d}^2}{\mathrm{d}s^2}\varrho(\xi(s))|_{s=0} + 2\left(\frac{\mathrm{d}}{\mathrm{d}s}\varrho(\xi(s))|_{s=0}\right)^2
\end{aligned}
$$

Recall by Cartan-Hadamard Theorem, $\varrho(\xi(s)) = d(\xi(s), O) = L(\gamma_s)$. Therefore Lemma 5.8 tells us $\mathrm{Hess}\varrho^2(V,V) = 2rL''(0) + 2(L'(0))^2 = 2rL''(0) + 2(\langle V, \dot\gamma(r)\rangle)^2$.

Notice that we have $\nabla_V V|_{t=0orr} = 0$, and hence

$$
\begin{aligned}
L''(0) =& \int_0^r \left(\langle \nabla_T V^\perp, \nabla_T V^\perp\rangle - \langle R(V^\perp, \dot\gamma)\dot\gamma, V^\perp\rangle\right)\mathrm{d}t \\
\geq& \int_0^r \langle \nabla_T V^\perp, \nabla_T V^\perp\rangle \mathrm{d}t
\end{aligned}
$$

That is, $\mathrm{Hess}\varrho^2(V,V) \geq 2r\int_a^b \langle \nabla_T V^\perp, \nabla_T V^\perp\rangle \mathrm{d}t + 2(\langle V, \dot\gamma(r)\rangle)^2$.

1. If $\langle V, \dot\gamma\rangle \neq 0$, we obtain $\mathrm{Hess}\varrho^2(V,V) \geq 2\langle V, \dot\gamma(r)\rangle^2 > 0$.

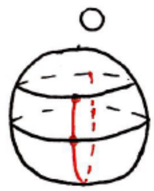
2. Otherwise if $\langle V, \dot\gamma\rangle = 0$. Since V is a Jacobi field and $\langle V, \dot\gamma(0)\rangle = \langle 0, \dot\gamma(0)\rangle = 0$, proposition (5.8) tells $\langle V(T), \dot\gamma(t)\rangle = 0\ \forall t \in [0, r]$. Therefore, $V(t) = V^\perp(t)$. We observe that $\nabla_T V \not\equiv 0$. Since otherwise $V(t)$ is parallel along γ, which contradicts to the fact $V(0) = 0$, $V(r) = V \neq 0$. That is, $\mathrm{Hess}\varrho^2(V,V) \geq \int_0^r \langle \nabla_T V^\perp, \nabla_T V^\perp\rangle \mathrm{d}t > 0$.

□

Definition 5.8 (Convex and totally convex subsets of M)**.** *Let M be a Riemannian manifold. A subset $\Omega \subset M$ is called convex, if whenever $p, q \in \Omega$ and γ is a minimizing geodesic from p to q, then $\gamma \subset \Omega$. Ω is called totally convex if whenever $p, q \in \Omega$ and γ is a geodesic from p to q, then $\gamma \subset \Omega$.*

Recall by Cartan-Hadamard Theorem, on a complete simply-connected Riemannian manifold with nonpositive sectional curvature, and geodesic is minimizing, and, hence, any convex subset is totally convex. However these two concepts do have difference.

Example. On $\mathbb{S}^2 \subset \mathbb{R}^3$ the unit sphere $\{p \in \mathbb{S}^2 | d(p, O) < r\}$ where $r \leq \frac{\pi}{2}$, is convex, but is not totally convvex. $\{p \in \mathbb{S}^2 | d(p, O) < \frac{\pi}{2}\}$ is not convex.



Convex functions and convex subsets are related by the following result.

Proposition 5.12. *Let $\tau : M \to \mathbb{R}$ be a convex function on a complete Riemannian manifod M. Then the sub-level set*

$$M_c = \{x \in M : \tau(x) < c\}$$

is totally convex.

Proof. $\forall p, q \in M_c$, and any geodesic $\gamma : [a, b] \to M$ from p to q, we have $(\tau \circ \gamma)'' \geq 0$. Therefore $\tau \circ \gamma : [a, b] \to \mathbb{R} : [a, b] \to \mathbb{R}$ attains its maximum at the two ends. Hence

$$\tau \circ \gamma(t) \leq \max\{\tau \circ \gamma((a), \tau \circ \gamma(b)\} = \max\{\tau(p), \tau(q)\} < c$$

.

This is $\gamma \subset M_c$.

□

Therefore, Theorem 5.11 tells that any (open or closed) geodesic balls

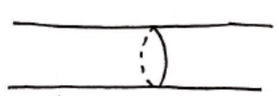
$$\{x \in M : d(x, O) < (\leq) r\}$$

is totally convex on a complete simply-connected Riemannian maniflod with nonpositive curvature. In particular, every point is totallly convex (i.e. no nontrivial geodesic $\gamma : [a, b] \to M$ with $\gamma(a) = \gamma(B) = x$ exists).

Proper totally convex sets ($\Omega \neq M$) do not exists in many manifolds. Existence of such kind of subsets has significant to pological implications.

Theorem 5.12 (The Soul Theorem, Cheeger-Gromoll 1972)**.** *If (M, g) is a complete non-compact Riemannian manifold with nonnegative sectional curvature, then M contains a closed totally convex submanifold S, such that M is diffeomorphic to the normal bundle over S.*

S is called a soul of M.



$\{(x, y, z) | z = c\}$ is a soul of the cylinder.

We also explain the geodesic meaning of the local result Lemma 5.7.

Theorem 5.13 (Whitehead 1932)**.** *Let (M, g) be a Riemannian manifold. Any $p \in M$ has a convex neighborhood.*

Proof. Recall for any $p \in M$, there exists a totally normal neighborhood, that is, a neighborhood $p \in W$ and a number $\delta > 0$ such that any two $q_1, q_2 \in W$ can be joined by a unique minimizing geodesic. However, such a geodesic may not lie completely in W.

By Lemma 5.7, there exists a neighborhood U_p s.t. $d^2(\cdot, p)$ is strictly convex in U_p. Pick r small enough s.t. $B_p(r) = \{q \in M, d(q, p) < r\} \subset U_p \cap W$. The proof of proporsition 5.12 tells $B_p(r)$ is convex.

□

Chapter 6

Comparison Theorem

Recall in our discusions about Cartan-Hadamard and Bonnet-Myers Theorems, we, in fact, have model spaces in mind.

1. Cartan-Hadamard: use $\mathbb{R}^n$ as a model, and replace the "zero curvature" of "$\mathbb{R}^n$" by "cur ≤ 0".
2. Bonnet-Myers: use $\mathbb{S}^n$ as a model, and replace "Ricci curvature $= n-1$" of S^n by "Ricci cur$\geq (n-1)$".

In this chapter, we aim at estabilshing quantitative comparison result with model spaces.

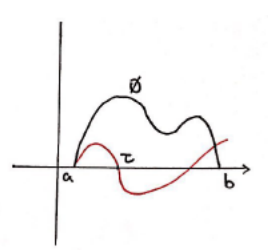
6.1 Sturm Comparison Theorem[1]

We start from a pure analysis result of Sturm.

Theorem 6.1 (Sturm)**.** *Let f and h be two continuous functions satisfying $f(t) \leq h(t)$ for all t in an interval I, and let ϕ and η be two functions satisfying the differential equations:*

$$\begin{cases} \phi'' + f\phi = 0 \\ \eta'' + h\eta = 0 \end{cases} \text{on } I. \tag{6.1.1}$$

Assume that ϕ is not the zero function and let $a, b \in I$ be two consecutive zeros of ϕ. Then:



[1]Spivak IV, Chap 8, 15-17

1. *The function η must have a zero in (a, b), unless $f = h$ everywhere on $[a, b]$ and η is a constant multiple of ϕ on $[a, b]$,*

2. *Suppose that $\eta(a) = 0$, and also $\eta'(a) = \phi'(a) > 0$. If τ is the smallest zero of η in (a, b), then*

$$\phi(t) \geq \eta(t) \text{ for } a \leq t \leq \tau$$

and equality holds for some t only if $f = h$ on $[a, t]$.

Remark: The restriction $\eta'(a) = \phi'(a) > 0$ in the theorem 6.1 case 2 can be achieved by choosing a suitable multiple of η, and changing ϕ to $-\phi$ if necessary.

Proof. (6.1.1) gives

$$\eta\phi'' - \phi\eta'' = (h - f)\phi\eta. \tag{6.1.2}$$

Suppose that η were nowhere zero on (a, b). W.o.l.g., we can assume

$$\eta, \phi > 0 \text{ on } (a, b). \tag{6.1.3}$$

Thus (6.1.2) gives

$$\eta\phi'' - \phi\eta'' = (h - f)\phi\eta \geq 0.$$

Therefore,

$$\begin{aligned} 0 &\leq \int_a^b (\eta\phi'' - \phi\eta'') = \int_a^b (\eta\phi' - \phi\eta')' \\ &= \eta(b)\phi'(b) - \eta(a)\phi'(b) - \phi(b)\eta'(b) + \phi(a)\eta'(a) \\ &= \eta(b)\phi'(b) - \eta(a)\phi'(a) \end{aligned} \tag{6.1.4}$$

On the other hand, (6.1.3) implies

$$\left.\begin{array}{r} \phi'(a) > 0, \phi'(b) < 0 \\ \eta(a) \geq 0, \eta(b) \geq 0 \end{array}\right\} \Rightarrow \eta(b)\phi'(b) - \eta(a)\phi'(a) \leq 0. \tag{6.1.5}$$

If $f \neq h$, we have

$$0 < \int_a^b (\eta\phi'' - \phi\eta'').$$

which is a contradiction to (6.1.5). Hence η must have a zero on $[a, b)$.

If $f = h$, then by (6.1.4) and (6.1.5)

$$0 = \eta(b)\phi'(b) - \eta(a)\phi'(a) \Rightarrow \eta(a) = \eta(b) = 0.$$

Now ϕ and η satisfy the same equation

$$\begin{cases} \phi'' + f\phi = 0 \\ \eta'' + f\eta = 0 \end{cases} \text{on } [a, b].$$

and $\phi(a) = \phi(b) = 0$. The solution η must be a constant multiple of ϕ on $[a, b]$.

Next suppose $\eta(a) = 0$, $\eta'(a) = \phi'(a) > 0$. (recall $\phi(a) = 0$). Let τ be the smallest zero of η in $[a, b]$. Then $\phi > 0$, $\eta > 0$ on (a, τ). Hence

$$(\phi'\eta - \eta'\phi)' = \phi''\eta - \eta''\phi = (h - f)\phi\eta \geq 0$$

on (a, τ). Recall $\phi'(a)\eta(a) - \eta'(a)\phi(a) = 0$, this implies $\phi'\eta - \eta'\phi \geq 0$ on (a, τ).

Since $\eta > 0$ on (a, τ), we obtain

$$\frac{\phi'\eta - \eta'\phi}{\eta^2} = \left(\frac{\phi}{\eta}\right)' \geq 0$$

on (a, τ).

But by L'Hôpital's Rule and our assumption, we have

$$\lim_{t \to a} \frac{\phi(t)}{\eta(t)} = \lim_{t \to a} \frac{\phi'(t)}{\eta'(t)} = 1.$$

Therefore, $\frac{\phi}{\eta} \geq 1$ on (a, τ).

This proves $\phi(t) \geq \eta(t)$ for $a \leq t \leq \tau$. If $\phi(t) = \eta(t)$ for some t, then $\frac{\phi}{\eta} = 1$ on (a, t). Hence $\left(\frac{\phi}{\eta}\right)' = 0 \Rightarrow \phi'\eta - \eta'\phi = 0$ on $(a, t) \Rightarrow \phi''\eta - \eta''\phi = 0$ on $(a, t) \Rightarrow f = h$ on (a, t). By continuity, $f = h$ on $[a, t]$.

□

Geometric translations

Theorem 6.2 (Bonnet 1855)**.** *Let M be a surface, and $\gamma : [0, L] \to M$ be a normal geodesic. Let $k > 0$.*

1. *If $K(p) \leq k$ for all $p = \gamma(t)$, and γ has length $L < \frac{\pi}{\sqrt{k}}$ then γ contains no conjugate points.*
2. *If $K(p) \geq k$ for all $p = \gamma(t)$, and γ has length $L > \frac{\pi}{\sqrt{k}}$ then there is a point $\tau \in (0, L)$ conjugate to 0, and therefore γ is not of minimal length.*

Proof. Let Y be a unit parallel vector field along γ with $\langle Y, \dot{\gamma} \rangle = 0$, $\forall t$. Any normal Jacobi field U can be written as $U = \phi Y$ for some function ϕ.

Jacobi equation $\nabla_T \nabla_T U + R(U, T)T = 0$ implies

$$\phi'' + K(Y, T)\phi(t) = 0. \tag{6.1.6}$$

The corresponding discussion on constant curved surfaes (model spaces) gives

$$\eta''(t) + k\eta(t) = 0 \tag{6.1.7}$$

with a solution $\eta(t) = sin(\sqrt{k}t)$. Note 0 and $\frac{\pi}{\sqrt{k}}$ are two consecutive zeros of η.

1. $K(Y, T) \leq k$, $\forall t$. Theorem 6.1 case (1) implies (6.1.6) cannot have a solution ϕ vanishing at 0 and at $L < \frac{\pi}{\sqrt{k}}$. Since otherwise, $sin(\sqrt{k}t)$ has to vanish at some point on $(0, L)$, which is false.

2. $K(Y,T) \geq k$, $\forall t$. Theorem 6.1 case (1) implies any Jacobi field ϕY must has a zero on $\left(0, \frac{\pi}{\sqrt{k}}\right) \subset (0, L)$. So if we choose any nonzero Jacobi field Y along γ with $\eta(0) = 0$, this Jacobi field will also vanish at some $\tau \in (0, L)$. Thus τ is conjugate to 0.

□

Theorem 6.2 case (2) is the result which Bonnet used to show his diameter estimate.

From the above proof, we observe the following facts: The Jacobi field $U = \phi Y$ where Y is a unit parallel vector field along γ with $\langle Y, \dot{\gamma} \rangle = 0$, we have

$$\begin{aligned} \phi''(t) + K(\gamma(t))\phi(t) = 0, \\ \phi(a) = 0 \Leftrightarrow \ U(a) = 0, \\ \phi'(a) = \left|\dot{U}(a)\right|, \\ |\phi(a)| = |U(a)|\,. \end{aligned}$$

So Sturm comparison theorem case(2) can be translated as:

Given two surfaces M and $\bar{M}$. Let $\gamma : [a,b] \to M$ and $\bar{\gamma} : [a,b] \to \bar{M}$ be two normal geodesics such that

$$K(\gamma(t)) \leq \bar{K}(\bar{\gamma}(t)). \tag{6.1.8}$$

Let $\tau \in (a,b]$ such that γ, $\bar{\gamma}$ have no point in $[a,\tau]$ conjugate to $\gamma(a)$, $\bar{\gamma}(a)$ respectively.

Let U, $\bar{U}$ be normal Jacobi fields along $\gamma, \bar{\gamma}$ respectively with $U(a) = \bar{U}(a) = 0$ and $\left|\dot{U}(a)\right| = \left|\dot{\bar{U}}(a)\right|$. Then $\left|\dot{U}(a)\right| \geq \left|\dot{\bar{U}}(a)\right|$, for $a \leq t \leq \tau$. And '=' holds for some t only if $K \circ \gamma = \bar{K} \circ \bar{\gamma}$ on $[a,t]$.

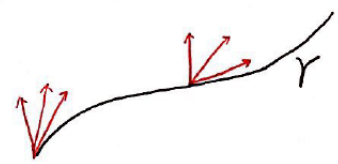
Remark The above "comparison of Jacobi fields" implies Bonnet Theorem 6.2 by choose one of M, $\bar{M}$ to be the sphere $\mathbb{S}^2\left(\frac{1}{\sqrt{k}}\right)$. In fact in theorem 6.1, case (2) $\Rightarrow$ (1) when $\eta(a) = 0$ is the case.

6.2 Morse-Schoenberg Comparison and Rauch Comparison Theorems[2]

It is natural to ask for higher-dimensional generalizations of the geometric translation of Theorem (6.1). Let (M,g) be an n-dimensional Riemann manifold $\gamma : [a,b] \to M$

[2] Spivak IV, Chap 8, 18-23

be a normal geodesic.



Now a normal Jacobi field U along γ cannot always be written as ϕY where Y is a unit normal parallel vector field along γ. In fact, let $\{Y_1, \dots, Y_n\}$ be an orthornormal parallel vector field along γ with $\dot{\gamma}(t) = Y_1(t)$. Then a normal Jacobi field U along γ can be written as

$$U(t) = \sum_{i=2}^{n} \phi_i(t) Y_i(t).$$

Jacobi equation $\Rightarrow \sum_{i=2}^{n} \phi_i''(t) Y_i(t) + \sum_{i=2}^{n} \phi_i(t) R(Y_i(t), T)T = 0$

$\Rightarrow \phi_j''(t) + \sum_{i=2}^{n} \phi_i(t) \langle R(Y_i, T)T, Y_j \rangle = 0 \ \forall 2 \leq j \leq n.$

This system of equations do not involve the sectional curvature directly.

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\phi_2(t), \dots, \phi_n(t)) + (\phi_2(t), \dots, \phi_n(t)) \begin{pmatrix} \langle R(Y_2, T)T, Y_2 \rangle & \dots & \langle R(Y_2, T)T, Y_n \rangle \\ \vdots & \ddots & \vdots \\ \langle R(Y_n, T)T, Y_2 \rangle & \dots & \langle R(Y_n, T)T, Y_n \rangle \end{pmatrix} = 0.$$

Recall the space of normal Jacobi fields along γ vanishing at $t = a$ is of dimension $n - 1$. We actually have to solve the following to solve the equation to compute Jacobi fields:

$$\begin{cases} \frac{\mathrm{d}^2}{\mathrm{d}t^2} A + AR = 0 \\ A(0) = 0 \quad \frac{\mathrm{d}A}{\mathrm{d}t}(0) = \mathrm{Id}_{n-1} \end{cases} \tag{6.2.1}$$

where $R = \left(\langle R(Y_i, T)T, Y_j \rangle \right)_{ij}$ is symmetric.

We will not discuss the generalization of Theorem (6.1) to the equation (6.2.1), but instead, will discuss the generalizaion of its geometric translations. These two are different aspect of the same result.(See[WSY, Chap8, Appendix])

From the geometirc viewpoint, we are going to compare the Jacobi fields along geodesics in two Riemann manifolds, whose sectional curvatures satisfy certain comparison estimate. For that purpose, we need "move" a vector field along a geodesic γ to a geodesic $\bar{\gamma}$ in another Riemann manifold $\bar{M}$.

Lemma 6.1. *Let (M, g), ($\bar{M}$, $\bar{g}$) be two Riemann manifolds of the same dimension n, and let γ ($\bar{\gamma}$) : $[a, b] \to M$ ($\bar{M}$) be a normal geodesic in M ($\bar{M}$). Then there is a vector space isomorphism*

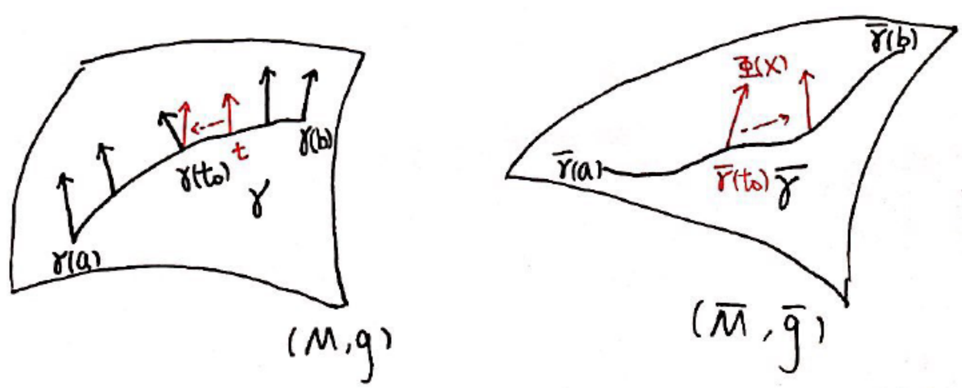
$$\Phi : \{\textit{piecewise } C^\infty \textit{ vector fields along } \bar{\gamma}\} \to \{\textit{piecewise } C^\infty \textit{ vector fields along } \gamma\}$$

such that for all $t \in [a, b]$, we have for any piecewise C^∞ vector field X along γ

1. *If $\nabla_T X := \widetilde{\nabla}_{\frac{\partial}{\partial t}} X$ is continuous at t, then $\nabla_{\bar{T}} \Phi(X) := \widetilde{\nabla}_{\frac{\partial}{\partial t}} \Phi(X)$ is continuous at t.*
2. $\langle X(t), T(t) \rangle_g = \langle \Phi(X)(t), \bar{T}(t) \rangle_{\bar{g}}$.
3. $|X(t)|_g = |\Phi(X)(t)|_{\bar{g}}$, *where* $|X(t)|_g = \sqrt{g(X(t), X(t))}$.
4. $|\nabla_T X|_g = |\nabla_{\bar{T}} \Phi(X)|_{\bar{g}}$, *it being understood that this equation refers to left and right hand limit at discontinuous points.*

where $T = \dot{\gamma}$ and $\bar{T} = \dot{\bar{\gamma}}$.

Proof.



What could be a natural choice of such a Φ?

An isomorphism between $T_{\gamma(t_0)}M$ and $T_{\bar{\gamma}(t_0)}\bar{M}$ for a fixed point is easy: Pick $\phi_{t_0} : T_{\gamma(t_0)}M \to T_{\bar{\gamma}(t_0)}\bar{M}$ be one isomorphism which preserve the inner products given by g and $\bar{g}$ respectively.

How to extend it?

For any $t \in [a, b]$, we define $\phi_t : \ T_{\gamma(t)}M \to T_{\bar{\gamma}}(t)\bar{M}$, which is given by

$$V_t \xrightarrow{\text{parallel transport along } \gamma \text{ to} \gamma(t_0)} P_{\gamma,t,t_0}(V_t) \xrightarrow{\phi_{t_0}}$$

$$\phi_{t_0}(P_{\gamma,t,t_0}(V_t)) \longmapsto P_{\bar{\gamma},t_0,t}(\phi_{t_0}(P_{\gamma,t,t_0}(V_t)))$$

Then define $\Phi(X)(t) = \phi_t(X(t))$.

Next, we give Φ an explicit expression. Let $Y_1, \ldots, Y_n$ be parallel, everywhere orthonormal vector fields along γ with $Y_1(t_0) = \dot{\gamma}(t_0)$. Let $\bar{Y}_1, \ldots, \bar{Y}_n$ be parallel, everywhere orthonormal vector fields along $\overline{\gamma}$ with $\bar{Y}_1(t_0) = \dot{\bar{\gamma}}(t_0)$.

A piecewise C^∞ vector field $X(t)$ along γ can be written as

$$X(t) = \sum_{i=1}^{n} f_i(t) Y_i(t)$$

for certain functions $f_i : [a, b] \to \mathbb{R}$.
ϕ_{t_0} is chosen such that $\phi_{t_0}(Y_i(t_0)) = \bar{Y}_i(t_0)$, for $i = 1, \dots, n$. Then

$$\Phi(X)(t) = \phi_t(X(t)) = \sum_{i=1}^{n} f_i(t)\bar{Y}_i(t).$$

This shows $\Phi(X)$ is C^∞ everywhere that X is, and that

$$\begin{aligned} \langle X(t), \dot{\gamma}(t)\rangle_g &= f_1(t) = \langle \Phi(X)(t), \dot{\bar{\gamma}}(t)\rangle_{\bar{g}}, \\ |X(t)|_g &= \sum_{i=1}^{n} f_i^2(t) = |\Phi(X)(t)|_{\bar{g}}\,, \\ |\nabla_T X|_g &= \sum_{i=1}^{n} (f_i'(t))^2 = |\nabla_{\bar{T}} \Phi(X)|_{\bar{g}}\,. \end{aligned}$$

□

Theorem 6.3. *Let M and $\bar{M}$ be two Riemann manifold of the same dimension n, and let γ ($\bar{\gamma}$) : $[a, b] \to M$ ($\bar{M}$) be a normal geodesic in M ($\bar{M}$). For each $t \in [a, b]$, suppose that for all 2-dimensional sections $\Pi_{\gamma(t)} \subset T_{\gamma(t)}M$, and all 2-dimensional sections $\bar{\Pi}_{\bar{\gamma}(t)} \subset T_{\bar{\gamma}(t)}\bar{M}$, the sectional curvatures satisfy $K(\Pi_{\gamma(t)}) \le \bar{K}(\bar{\Pi}_{\bar{\gamma}(t)})$.*

Then we have

$$\operatorname{ind}(\gamma) \le \operatorname{ind}(\bar{\gamma}).$$

In particular, if $I(W, W) < 0$ for some $W \in \mathcal{V}_0(a, b)$, then also $\bar{I}(\bar{W}, \bar{W}) < 0$ for some $\bar{W} \in \bar{\mathcal{V}}_0(a, b)$, where $\mathcal{V}_0(a, b)$ is the set of the piecewise C^∞ vector fields along γ with $W(a) = W(b) = 0$.

Proof. Let $W \in \mathcal{V}_0(a, b)$, recall that

$$I(W, W) = \int_a^b \{\langle \nabla_T W, \nabla_T W\rangle - \langle R(W, T)T, W\rangle\}\, \mathrm{d}t.$$

Let Φ be constructed as in Lemma (6.1). Then $\Phi(W) \in \bar{\mathcal{V}}_0(a, b)$ and $\langle \nabla_T W, \nabla_T W\rangle_g = \langle \nabla_{\bar{T}} \Phi(W), \nabla_{\bar{T}} \Phi(W)\rangle_{\bar{g}}$. Also we have

$$\begin{aligned} \langle R(W, T)T, W\rangle_g &= K(W, T)\left(\langle W, W\rangle\langle T, T\rangle - \langle W, T\rangle^2\right) \\ &\le \bar{K}(\Phi(W), \bar{T})\left(\langle \Phi(W), \Phi(W)\rangle\langle \bar{T}, \bar{T}\rangle - \langle \Phi(W), \bar{T}\rangle^2\right) \\ &= \langle \bar{R}(\Phi(W), \bar{T})\bar{T}, \Phi(W)\rangle. \end{aligned}$$

That is, $I(W, W) \ge \bar{I}(\Phi(W), \Phi(W))$.

So if $\mathcal{A} \subset \mathcal{V}_0(a, b)$ is a subspace on which I is negative definite, then $\Phi(\mathcal{A}) \subset \bar{\mathcal{V}}_0(a, b)$ is a subspace of the same dimension on which $\bar{I}$ is again negative definite. By definition, this means $\operatorname{ind}(\gamma) \le \operatorname{ind}(\bar{\gamma})$.

□

Corollary 6.1 (The Morse-Schoenberg Comparison Theorem)**.** *Let (M, g) be a Riemann manifold of dimension n, and let $\gamma : [0, L] \to M$ be a normal geodesic. Let $k > 0$.*

1. *If $K(\Pi_{\gamma(t)}) \leq k$ for all $\Pi_{\gamma(t)} \subset T_{\gamma(t)}M$, and γ has lenghth $L < \frac{\pi}{\sqrt{k}}$, then* $\mathrm{ind}(\gamma) = 0$ *and γ contains no cojugate point.*

2. *f $K(\Pi_{\gamma(t)}) \geq k$ for all $\Pi_{\gamma(t)} \subset T_{\gamma(t)}M$, and γ has lenghth $L > \frac{\pi}{\sqrt{k}}$, then there is a point $\tau \in (0, L)$ conjugate to 0, and γ is not of minimal length.*

Remark

1. This is a high-dimensional generalization of Bonnet Theorem (6.2).

2. Case 1 above is a generalization of proposition 5.10, which asserts M contains no conjugate points if $sec \leq 0$.

Proof. For case 1, we apply Theorem 6.3 to $M = M$, $\bar{M} = S^n\left(\frac{1}{\sqrt{k}}\right)$. Choosing $\gamma : [0, L] \to M$, $\bar{\gamma} : [0, L] \to S^n\left(\frac{1}{\sqrt{k}}\right)$ to be normal geodesic. We have $\mathrm{ind}(\gamma) \leq \mathrm{ind}(\bar{\gamma})$.

Now $\mathrm{ind}(\bar{\gamma}) = 0$ since $\bar{\gamma}$ contains no conjugate points (The Morse Index Theorem). Therefore $\mathrm{ind}(\gamma) = 0$ which implies γ contains no conjugate point.

For case 2, similar argument. Recall case 2 has already been proved when we discussed Bonnet-Myers Theorem. Now it is a good chance to understand the proof there in a more structural way: We choose $V(t) = \sin\left(\frac{\pi}{L}t\right)E(t)$ along γ in M and show $I(V, V) < 0$. Here $V(t)$ is the image of a Jacobi field on $\mathbb{S}^n$ via the isomorphism map Φ defined in Lemma 6.1.

□

Remark: Recall in the proof of Bonnet-Myers Theorem, we already show that Corollary 1 case 2 can be improved by weakening the sectional curvature restriction to Ricci curvature restriction.

We still miss the generalization of the 2^{nd} part of Sturm comparison theorem: we have not compared $|\Phi(W)|_{\bar{g}}$ with $|W|_g$ up to the first zero of $\Phi(W)$. Such information is providede by

Theorem 6.4 (Rauch Comparison Theorem)**.** *Let M, $\bar{M}$ be two Riemann manifolds of the same dimension n, and let $\gamma : [a, b] \to M$, $\bar{\gamma} : [a, b] \to \bar{M}$ be normal geodesics. Let U, $\bar{U}$ be* **normal** *Jacobi fields along γ, $\bar{\gamma}$ respectively with $U(a) = \bar{U}(a) = 0$, and $|\nabla_T U(a)|_g = \left|\nabla_{\bar{T}} \bar{U}(a)\right|_{\bar{g}}$. Suppose:*

1. *$\bar{\gamma}$ has no conjugate point on $[a, b]$.*

2. *$K(\Pi_{\gamma(t)}) \leq \bar{K}(\bar{\Pi}_{\bar{\gamma}(t)})$ for all $t \in [a, b]$, all 2-dimensional sections $\Pi_{\gamma(t)} \subset T_{\gamma(t)}M$, $\bar{\Pi}_{\bar{\gamma}(t)} \subset T_{\bar{\gamma}(t)}\bar{M}$.*

Then we have $|U(t)|_g \geq \left|\bar{U}(t)\right|_{\bar{g}}$, for all $t \in [a, b]$.

Remark

1. This is a generalization of the second part of Sturm Comparison Theorem. Notice that the Morse-Schoenberg Comparison Theorem (Cor 6.1) is also a direct consequence of theorem 6.4

2. "U, $\bar{U}$ be normal Jacobi fields": In fact, we only need require $\langle \dot{U}(a), \dot{\gamma}(a)\rangle = \langle \dot{\bar{U}}(a), \dot{\bar{\gamma}}(a)\rangle = 0$. This is because, from proposition (5.8), $U = fT + U^{\perp}$ where f is linear. And $f(a) = 0$, $f'(a) = 0$ forces $f \equiv 0$. Hence $U = U^{\perp}$.

Proof. If $\bar{U} \equiv 0$, it is trival.

If $\bar{U} \not\equiv 0$, then $\bar{U}(t) \neq 0$ for all $t \in (a, b]$, since $\bar{\gamma}$ has no conjugate points.

It suffices to prove that

$$\lim_{t\to a} \frac{|U(t)|_g}{\left|\bar{U}(t)\right|_{\bar{g}}} = 1, \tag{6.2.2}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{|U(t)|_g}{\left|\bar{U}(t)\right|_{\bar{g}}} \geq 0 \ \forall t \in (a, b]. \tag{6.2.3}$$

It turns out it is equivalent (but much easier) to consider the norm suquare.

$$\lim_{t\to a} \frac{\langle U(t), U(t)\rangle_g}{\langle \bar{U}(t), \bar{U}(t)\rangle_{\bar{g}}} = 1, \tag{6.2.4}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\langle U(t), U(t)\rangle_g}{\langle \bar{U}(t), \bar{U}(t)\rangle_{\bar{g}}} \geq 0 \ \forall t \in (a, b]. \tag{6.2.5}$$

To prove (6.2.4), we note that:

$$\begin{aligned}
&\lim_{t\to a} \frac{\langle U(t), U(t)\rangle_g}{\langle \bar{U}(t), \bar{U}(t)\rangle_{\bar{g}}} = \lim_{t\to a} \frac{\langle U(t), \nabla_T U(t)\rangle_g}{\langle \bar{U}(t), \nabla_{\bar{T}} \bar{U}(t)\rangle_{\bar{g}}} \\
=&\lim_{t\to a} \frac{\langle \nabla_T U(t), \nabla_T U(t)\rangle_g + \langle U(t), \nabla_T \nabla_T U(t)\rangle_g}{\langle \nabla_{\bar{T}} \bar{U}(t), \nabla_{\bar{T}} \bar{U}(t)\rangle_{\bar{g}} + \langle \bar{U}(t), \nabla_{\bar{T}} \nabla_{\bar{T}} \bar{U}(t)\rangle_{\bar{g}}} \\
=&1.
\end{aligned}$$

To prove (6.2.5), we note that

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\langle U(t), U(t)\rangle_g}{\langle \bar{U}(t), \bar{U}(t)\rangle_{\bar{g}}} = \frac{2\langle U(t), \nabla_T U(t)\rangle\langle \bar{U}(t), \bar{U}(t)\rangle - 2\langle U(t), U(t)\rangle\langle \bar{U}(t), \nabla_{\bar{T}} \bar{U}(t)\rangle}{\langle \bar{U}(t), \bar{U}(t)\rangle^2}. \tag{6.2.6}$$

Hence (6.2.5) $\Leftrightarrow$

$$\langle U(t), \nabla_T U(t)\rangle\langle \bar{U}(t), \bar{U}(t)\rangle \geq \langle U(t), U(t)\rangle\langle \bar{U}(t), \nabla_{\bar{T}} \bar{U}(t)\rangle. \tag{6.2.7}$$

So for each $t_0 \in [a, b]$, it suffices to show that

$$\langle U, \nabla_T U\rangle(t_0) \geq \frac{\langle U, U\rangle(t_0)}{\langle \bar{U}, \bar{U}\rangle(t_0)} \langle \bar{U}, \nabla_{\bar{T}} \bar{U}\rangle(t_0). \tag{6.2.8}$$

We denote $\frac{\langle U,U\rangle(t_0)}{\langle \bar{U},\bar{U}\rangle(t_0)} = c^2$. For any piecewise C^∞ vector field W along γ, the index form

$$
\begin{aligned}
I(W,W) &= \int_a^b \{\langle \nabla_T W, \nabla_T W\rangle - \langle (R(W,T)T, W\rangle\}\mathrm{d}t \\
&= -\int_a^b \langle (R(W,T)T + \nabla_T\nabla_T W, W\rangle \mathrm{d}t + \langle \nabla_T W, W\rangle\Big|_a^b \\
&\quad - \sum_{j=1}^k \langle \nabla_{T(t_j^+)} W - \nabla_{T(t_j^-)} W, W\rangle.
\end{aligned}
$$

Since U, $\bar{U}$ are Jacobi fields, with $U(a) = \bar{U}(a) = 0$, we have

$$
\begin{aligned}
\langle U, \nabla_T U\rangle(t_0) &= I_a^{t_0}(U,U), \\
\langle \bar{U}, \nabla_{\bar{T}} \bar{U}\rangle(t_0) &= \bar{I}_a^{t_0}(\bar{U},\bar{U}).
\end{aligned}
$$

So, it remains to show $I_a^{t_0}(U,U) \geq c^2 \bar{I}_a^{t_0}(\bar{U},\bar{U})$, or $I_a^{t_0}\left(\frac{U}{|U(t_0)|}, \frac{U}{|U(t_0)|}\right) \geq \bar{I}_a^{t_0}\left(\frac{\bar{U}}{|\bar{U}(t_0)|}, \frac{\bar{U}}{|\bar{U}(t_0)|}\right)$

Consider the map Φ constructed in Lemma 6.1, as the input, at t_0, we choose ϕ_{t_0} such that $\phi_{t_0}\left(\frac{U(t_0)}{|U(t_0)|}\right) = \frac{\bar{U}(t_0)}{|\bar{U}(t_0)|}$.

Then $\Phi\left(\frac{U}{|U(t_0)|}\right)$ is a smooth vector field along $\bar{\gamma}$, such that $\Phi\left(\frac{U}{|U(t_0)|}\right)(t_0) = \frac{\bar{U}(t_0)}{|\bar{U}(t_0)|}$.

As in the proof of Theorem 6.3, we see

$$
I_a^{t_0}\left(\frac{U}{|U(t_0)|}, \frac{U}{|U(t_0)|}\right) \geq \bar{I}_a^{t_0}\left(\Phi\left(\frac{U}{|U(t_0)|}\right), \Phi\left(\frac{U}{|U(t_0)|}\right)\right). \tag{6.2.9}
$$

Now using the minimizing property of Jacobi field in lemma (5.2), we have

$$
\bar{I}_a^{t_0}\left(\Phi\left(\frac{U}{|U(t_0)|}\right), \Phi\left(\frac{U}{|U(t_0)|}\right)\right) \geq \bar{I}_a^{t_0}\left(\frac{\bar{U}}{|\bar{U}(t_0)|}, \frac{\bar{U}}{|\bar{U}(t_0)|}\right). \tag{6.2.10}
$$

This is applicable since $\Phi\left(\frac{U}{|U(t_0)|}\right)(a) = \frac{\bar{U}(a)}{|\bar{U}(t_0)|} = 0$, and $\Phi\left(\frac{U}{|U(t_0)|}\right)(t_0) = \frac{\bar{U}(t_0)}{|\bar{U}(t_0)|} = 0$

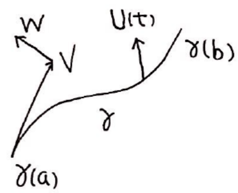
Comnining (6.2.9) and (6.2.10) yields

$$
I_a^{t_0}\left(\frac{U}{|U(t_0)|}, \frac{U}{|U(t_0)|}\right) \geq \bar{I}_a^{t_0}\left(\frac{\bar{U}}{|\bar{U}(t_0)|}, \frac{\bar{U}}{|\bar{U}(t_0)|}\right). \tag{6.2.11}
$$

□

Recall from the proof of uniqueness of simply-connected space forms (Theorem 5.10), we have used the idea of comparing the norm of Jacobi field. Therefore, we have the same sectional curvatures, and the corresponding Jacobi field has the same norm. (It definitely deserves to read through that proof again with this new perspective).

Recall for given $t \in [a, b]$, the Jacobi field $U(t)$ at t can be expressed as $U(t) = \left(\mathrm{d}\exp_{\gamma(a)}\right)_{(V)}(W)$ for some W.



Hence we have the following equivalent form of Rauch Comparison Theorem.

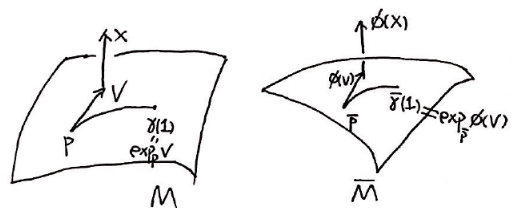
Theorem 6.5. *Let M, $\bar{M}$ be two Riemann manifolds of the same dimension n, and let $p \in M$, $\bar{p} \in \bar{M}$, $\phi : T_pM \to T_{\bar{p}}\bar{M}$ be an isometry (of inner product spaces), $V \in T_pM$, $\bar{V} = \phi(V)$.*

Let $\gamma(t) = \exp_p tV$, $t \in [0, 1]$, $\bar{\gamma} = \exp_{\bar{p}} t\bar{V}$, $t \in [0, 1]$ be geodesics in M, $\bar{M}$ respectively. Let $X \in T_V\left(T_pM\right)$, $\phi(X) \in T_{\bar{V}}\left(T_{\bar{p}}\bar{M}\right)$. Suppose

1. *$\bar{\gamma}$ has no conjugate point.*
2. *$K(\Pi_{\gamma(t)}) \leq \bar{K}(\bar{\Pi}_{\bar{\gamma}(t)})$ for all $t \in [0, 1]$, all 2-dimensional sections $\Pi_{\gamma(t)} \subset T_{\gamma(t)}M$, $\bar{\Pi}_{\bar{\gamma}(t)} \subset T_{\bar{\gamma}(t)}\bar{M}$.*

Then we have

$$\left(\mathrm{d}\exp_p\right)_{(V)}(X) \geq \left(\mathrm{d}\exp_{\bar{p}}\right)_{(\bar{V})}(\bar{X}).$$



Proof. The geodesic variation

$$F(t, s) = \exp_p t(V + sX)$$

has variational field $U(t)$ which is a Jacobi field such that

$$U(0) = 0,$$
$$\dot{U}(0) = X,$$
$$U(1) = \left(\mathrm{d}\exp_p\right)_{(V)}(X).$$

Similarly $\bar{F}(t,s) = \exp_{\bar{p}} t(\bar{V} + s\bar{X})$, gives Jacobi field $\bar{U}(t)$ such that

$$\bar{U}(0) = 0,$$
$$\dot{\bar{U}}(0) = \phi(X) = \bar{X},$$
$$\bar{U}(1) = \left(\mathrm{d}\exp_p\right)_{(\bar{V})}(\bar{X}).$$

Recall from Gauss lemma,

$$\langle X, V\rangle = \left\langle (\mathrm{d}\exp_p)_{(V)}(V), (\mathrm{d}\exp_p)_{(V)}(X)\right\rangle$$

and $\mathrm{d}\exp_p$ is an isometry along the radical direction, we only need to consider the case $\langle X, V\rangle = 0$. (and, hence, $\langle \Phi(X), \Phi(V)\rangle = 0$).

Therefore, theorem 6.5 follows from theorem 6.4.

□

A particular interesting case:

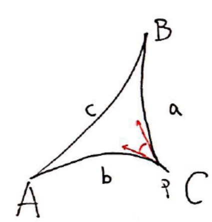
Corollary 6.2. *Let (M, g) be a complete Riemann manifold with nonpositive sectional curvature. Then $\forall p \in M$, $\exp_p : T_pM \to M$ satisfies*

$$\left(\mathrm{d}\exp_p\right)_{(V)}(X) \geq |X|$$

where the right hand side means the norm of the flat metric on T_pM. $\forall V \in T_pM$, $\forall X \in T_V\left(T_pM\right) \simeq T_pM$. In particular, for any curve $\gamma \subset T_pM$, one have $L(\gamma) \leq L(\exp_p \circ \gamma)$.

Remark: This strengthen the result Proposition 5.10 where we show $\exp_p$ has no critical points.

Corollary 6.3. *Let (M, g) be a complete simply-connected Riemann manifold with nonpositive sectional curvature. Consider a geodesic triangle in M(i.e. each side of the triangle is a minimizing geodesic). Let the side lengths are a, b, c with opposite angles A, B, C respectively.*



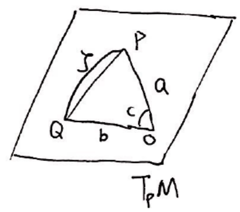
Then

1. $a^2 + b^2 - 2ab\cos C \leq c$,
2. $A + B + C \leq \pi$.

Moreover, if M has negative sectional curvature, then the inequalities are strict.

Proof. Denote the vertex at the angle C by p.

In T_pM, draw a triangle $\triangle OPQ$,



where O is the origin, such that $|OP| = a$, $|OQ| = b$, $\angle O = C$. In particular, $\exp_p \overrightarrow{OP}$, $\exp_p \overrightarrow{OQ}$ is the other two verties of the geodesic triangle. Let ξ be the preimage of the geodesic c in T_pM. Then

$$|PQ| \le L(\xi) \le c. \tag{6.2.12}$$

The Euclidean cosine law tells $|PQ|^2 = a^2 + b^2 - 2ab\cos C$.

Since a, b, c satisfy triangle inequalities, we can construct a triangle in $\mathbb{R}^2$ with side length a, b, c. Denote the corresponding opposite angles by A', B', C'. Then.

$$\begin{aligned}(6.2.12) &\Rightarrow a^2 + b^2 - 2ab\cos C' = c^2 \ge a^2 + b^2 - 2ab\cos C \\ &\Rightarrow C' \ge C.\end{aligned}$$

Similarly, we show $A' \ge A$, $B' \ge B$. Hence $\pi = A' + B' + C' \ge A + B + C$.

When $sec < 0$, the inequality in Rauch's theorem is also strict. And hence inequality in Cor (6.2) is strict. the last conclusion of this corallary then follows.

□

Final Remark: "$\bar{\gamma}$ hs no conjugate point" in Rauch Comparison theorem is necessary. For example, let us consider two spheres $M = S^2(2)$, $\bar{M} = S^2(3)$. Let $\gamma : [0, 3\pi] \to S^2(2)$, $\bar{\gamma} : [0, 3\pi] \to S^2(3)$ are normal geodesics. Let W, $\bar{W}$ be unit parallel normal vector fields along γ, $\bar{\gamma}$ respectively. Then

$$\begin{aligned}U(t) &= 2\sin\frac{t}{2}W(t), \\ \bar{U}(t) &= 3\sin\frac{t}{3}\bar{W}(t).\end{aligned}$$

are Jacobi fields such that $U(0) = \bar{U}(0) = 0$, $\left|\dot{U}(0)\right| = \left|\dot{\bar{U}}(0)\right| = 1$.

Recall $sec\left(S^2(r)\right) = \frac{1}{r^2}$. $sec\left(S^2(2)\right) = \frac{1}{4} > sec\left(S^2(3)\right) = \frac{1}{9}$. but $\left|\bar{U}(3\pi)\right| = 0 < |U(3\pi)| = \left|2\sin\frac{3\pi}{2}\right| = 2$.

6.3 Cut Point and Cut Locus

Next, we are going to discuss two more important comparison theorems:

Hessian and Laplacian comparison theorems. That is, we compare Hessϱ of the distance function $\varrho(x) := d(O, x)$ on different Riemannian manifolds whose sectional curvatures can be "well-compared". Laplacian is the trace of Hessian, so it is natural to expect a Laplacian comparison result based on Ricci curvature-comparison-assumption.

Hessϱ is closely related to the SVF. This has been shown when we discussed the convexity of ϱ^2 in (V)§(9). Recall Hess$\varrho(V, V) = \frac{d^2}{ds^2}\Big|_{s=0}\varrho(\xi(s))$, where ξ is the geodesic with $\xi(0) = X$, $\dot{\xi}(0) = V$. And we hope to calculate $\frac{d^2}{ds^2}\Big|_{s=0}\varrho(\xi(s))$ via the second variation formula of length. For that purpose, we consider the family of geodesics $\gamma_s : [0, \varrho(x)] \to M$ from 0 to $\xi(s)$, we hope

1. $\varrho(\xi(s)) = Length(\gamma_s)$,

2. $F(s, t) := \gamma_s(t)$ gives a variation.

For this purpose, we have to require that γ does not contain "cut point"! Recall in the comparison of Jacobi fields (Rauch), we have to do the comparison before the first conjugate point.

Another motivation is from the Bonnet/ Morse-Schoenberg comparison theorem.

$K \geq k$ and geodesic γ of length $> \frac{\pi}{\sqrt{k}}$ contains conjugate point, and hence γ is not minimizing $\Rightarrow$ diam$\leq \frac{\pi}{\sqrt{k}}$.

A counterexample is the projective space $\mathbb{P}^n(\mathbb{R}^n)$, with constant sectional curvature =1 and diameter only $\frac{\pi}{2}$. Hence, we may add the hypothesis of simply-connectivity.

Question: Should a complete simply-connected manifold with all sectional curvature $\leq k$ have diameter $\geq \frac{\pi}{\sqrt{k}}$?

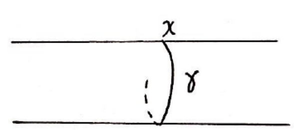
One might expect to prove as follows: Pick $p, q \in M$ maximal distance apart, and consider a minimizing geodesic

$$\gamma : [0, L] \to M$$

from p to q. If $L < \frac{\pi}{\sqrt{k}}$, then γ contains no conjugate point on $(0, L)$. Extend γ to $\gamma' : [0, L'] \to M$ where $L < L' < \frac{\pi}{\sqrt{k}}$. Thus γ' contains no conjugate point on $(0, L')$, and hence γ is a local minimum for length.

Notice, if we can conclude γ is a global minimum for length, we will get a contradictionand conclude diam$\geq \frac{\pi}{\sqrt{k}}$.

Example: consider $S^1 \times \mathbb{R}$



choose γ to be a horizontal circle. curvature=0, Cartan-Hadamard $\Rightarrow$ no conjugate point, but γ is not global minimizing, although it is loally minimizing.

There are NO general criterion to decide whether a given geodesic is minimizing.

For simplicity, we will deal only with the case of a complete Riemannian manifold (M, g). Let $\gamma : [0, \infty) \to M$ be a normal geodesic starting at a point $O = \gamma(0)$ in M. Then $\gamma|_{[0,t_0]}$ is minimizing iff $d(O, \gamma(t_0)) = t_0 = \text{Length}(\gamma|_{[0,t_0]})$.

By triangle inequality, we see if $\gamma|_{[0,t_0]}$ is minimizing so does $\gamma|_{[0,t]}$ for any $t < t_0$. On the other hand, we know for small enough t, $\gamma|_{[0,t]}$ is minimizing. So the set

$$\begin{aligned} A &= \{t > 0 : \mathrm{d}(O, \gamma(t)) = t\} \\ &= \{t > 0 : \gamma|_{[0,t]} \textit{ is a minimizing geodesic}\} \end{aligned}$$

is either $(0, \infty)$ or $(0, a]$ for some $a > 0$.(Notice that, the latter case means $\gamma|_{[0,t]}$ is not minimizing for all $t > a$).

1. If $A = (0, a]$, we say that $\gamma(a)$ is the cut point of O along the geodesic γ.

2. If $A = (0, \infty)$, we say that O has no cut point along γ.

3. The cut locus $C(O) \subset M$ of O is the set of all points which are cut points of O along some normal geodesic starting from O.

4. The cut locus $\tilde{C}(O)$ of O in T_0M is the set of all vectors $aX \in T_0M$ for which X is a unit vector and $exp_O aX$ is the cut point of O along the geodesic $\gamma_X(t) = \exp_O tX$.

Thus we have

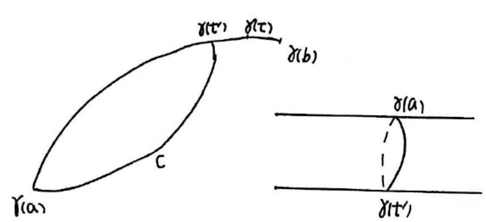
$$\exp_O\left(\tilde{C}(O)\right) = C(O).$$

The first conjugate locus of O in T_0M is the set of all vectors $aX \in T_0M$. For which X is a unit vector and a is the first conjugate value of O along γ_X.

For a geodesic $\gamma : [a, b] \to M$, recall that if $\exists \tau \in (a, b)$ conjugate to a, then γ is not local minimizing. Hence γ also contains a cut point $\gamma(t')$, with $t' \leq t$. Briefly expressed, the cut point comes before or at the first conjugate point.

What happens if $t' \neq \tau$?

In the example of a cylinder $S^1 \times \mathbb{R}$, there are no conjugate points. But when a cut point happens,



we see there are two minimizing geodesic from $\gamma(a)$ to $\gamma(t')$. In fact, if there are two distinct minimizing geodesics γ, c from $\gamma(a)$ to $\gamma(t')$, we see $\gamma|_{[a,t'+\epsilon]}$ can not be minimizing for any $\epsilon > 0$, This is because the path $\xi : [a, t' + \epsilon] \to M$,

$$\xi(t) = \begin{cases} c(t), t \in [a, t' + \epsilon] \\ \gamma(t), t \in [t't' + \epsilon] \end{cases}$$

is of the same length as $\gamma|_{[a,t'+\epsilon]}$ and is not smooth at t', and hence it can be made shorter.

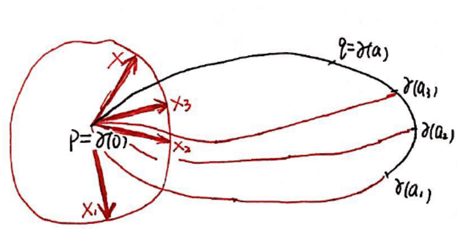
In other words, if there are two distinct minimizing geodesics from $\gamma(a)$ to $\gamma(t')$, and t' is the minimal value such that this happens, $\gamma(t')$ is a cut point of $\gamma(a)$ along γ.

Indeed, this is the only possible case.

Theorem 6.6 (cut points)**.** *Let (M, g) be complete and let $\gamma : [0, \infty) \to M$ be a normal geodesic with cut point $\gamma(a)$. Then at least one of the following holds:*

1. *The number a is first conjugate value of O along γ,*
2. *These are at least two minimal geodesic from $p = \gamma(0)$ to $q = \gamma(a)$. And a is the minimal value such that this case happens.*

Proof. Choose a sequence $a_1 > a_2 > a_3$ such that $\lim\limits_{t\to\infty} a_i = a$.



Since $\gamma(a)$ is a cut point, $b_i = \mathrm{d}(p, \gamma(a_i)) < a_i$. Let $X_i \in T_pM$ be the unit vectors such that $t \to \exp_p tX_i$, $0 \le t \le b_i$ is a minimal geodesic from p to $\gamma(a_i)$. Let $X = \dot\gamma(0)$. Since γ is normal, we have $\gamma(t) = \exp_p tX$, $0 \le t \le a$. All X_i are distinct from X. We have

$$\lim_{t\to\infty} b_i = \lim_{t\to\infty} d(p, \gamma(a_i)) = d(p, \gamma(a)) = a.$$

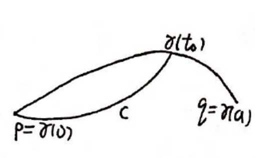
Therefore, the vectors b_iX_i are contained in a compact subset of T_pM. Choosing a subsequence if necessary, we have

$$\lim_{t\to\infty} b_iX_i = aY$$

for some unit vector $Y \in T_pM$.

Since $\exp_p(aY) = \lim\limits_{i\to\infty} \exp_p(b_iY_i) = \lim\limits_{i\to\infty} \gamma(a_i) = \gamma(a)$, the geodesic $t \mapsto \exp_p(tY)$, $0 \le t \le a$ is a minimizing geodesic from p to q.

If $X \ne Y$, we have two minimizing geodesics from p to q. If $\exists 0 < t_0 < a$, such that there is another geodesic c from p to $\gamma(t_0)$,



which is of the same length as $\gamma|_{[0,t_0]}$, we observe the path $\xi : [0,a] \to M$,

$$\xi(t) = \begin{cases} c(t), t \in [0, t_0] \\ \gamma(t), t \in [t_0, a] \end{cases}$$

can be made shorter. But $Length\,(\xi|_{[0,a]}) = Length\,(\gamma|_{[0,a]})$. This contradicts to the fact that $\gamma(a)$ is a cut point.

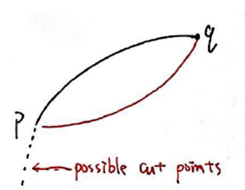
If $X = Y$, we have $\lim_{i\to\infty} b_i X_i = aY = aX = \lim_{i\to\infty} a_i X$. But $\exp_p(b_i X_i) = \gamma(a_i) = \exp_p(a_i X)$. Moreover, $a_i X$ and $b_i X_i$ are distinct for each i, since $b_i < a_i$, $X_i \neq X$. So $\exp_p$ is not 1-1 in any small neighborhood of aX in T_pM, i.e. aX is a critical point of $\exp_p$. By theorem (5.5, we conclude that a is a conjugate value of O along γ.

□

Recall along a geodesic $\gamma : [a,b] \to M$, p is a conjugate to q implies q is also conjugate to p. It turns out that this is also true for cut points.

Corollary 6.4. *In a complete manifold M, if q is the cut point of p along a geodesic γ from p to q, then p is the cut point of q along the geodesic $\overline{\gamma}$ obtained by traversing γ in the opposite direction.*

Proof. By assumption, we have that γ is a minimizing geodesic from p to q. So $\overline{\gamma}$ is also minimizing from p to q. So the cut point of q along $\overline{\gamma}$, if exists, occurs past or at p.



If q is conjugate to p along γ, then p is conjugate to q along $\overline{\gamma}$. The cut point of q along $\overline{\gamma}$ must occur before or at p. So if must occur at p.

If there is another minimizing geodesic from p to q, then again p must be the cut point.

□

Let $p \in M$. Denote by S_p the unit sphere of T_pM. Let $\mathbb{R}^* = \mathbb{R} \cup \{\infty\}$ be the real numbers together with some other set "∞". Consider the following topology of $\mathbb{R}^*$: a basis consists of all sets of the form $(a,b) \subset \mathbb{R}$ together with all sets of the form $(a,\infty] = (a,\infty) \cup \{\infty\}$.

We now define a function $\tau : S_p \to \mathbb{R}^*$ by

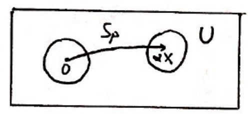
$$\tau(x) = \begin{cases} a > 0, if\ \exp_p(aX)\ is\ the\ cut\ point\ of\ p\ along\ the\ geodesic\ \gamma_X = \exp_p tX \\ \quad \infty, if\ \gamma_X\ has\ no\ cut\ point \end{cases}$$

Theorem 6.7. *If M is a complete manifold, and $p \in M$. Then the function $\tau : S_p \to \mathbb{R}^*$ is continuous.*

Proof. Let $X_1, X_2, \ldots$ be a sequence of unit vectors in S_p converging to $X \in S_p$. We have to show $\{\tau(X_i)\}$ converge to $\tau(X)$. Since the values of τ lie in the compact set $\{\alpha \in \mathbb{R}^* : \alpha \geq 0\}$, we can assume, by chooosing a subsequence, that $\tau(X_i)$ converges to some $\alpha \in \mathbb{R}^*$.

If $\alpha = \infty$, then for any given t, there exsits N s.t. $\tau(x_i) > t$, $\forall i \geq N$. hence $d(p, \exp_p tX) = d\left(p, \lim_{i\to\infty} \exp_p tX_i\right) = t$. By definition, $\tau(X) \geq t$.

If $\alpha < \infty$, then $\tau(X_i)X_i \to \alpha X$. Therefore $d(p, \exp_p \alpha X) = \alpha$. Hence $\tau(X) \geq \alpha$. So $\overline{\lim}_{i\to\infty}\tau(X_i) \leq \tau(X)(\heartsuit)$.



Next, if $\tau(X) > \alpha$, then $\exp_p(\alpha X)$ is not a conjugate point of p along $\exp_p tX$ since a conjugate point cannot come before a cut point. So the map $\exp_p$ is a diffeomorphism on some neighborhood U around αX in T_pM. W.l.o.g., we assume all $\tau(X_i)X_i$ lie in U.

Therefore $\exp_p \tau(X_i)X_i$ is not a conjugate point of p along $\exp_p tX_i$. then theorem 6.6 implies that there exists another minimizing geodesic from p to $\exp_p \tau(X_i)X_i$, i.e. $\exists Y_i \in S_p$ s,t, $\exp_p \tau(X_i)X_i = \exp_p \tau(X)Y_i$.

Since $\exp_p|_U$ is 1-1, we have $\tau(X_i)Y_i \notin U$. By choosing a subsequence, we can assume $Y_i \to Y$. Then αY also lies outside U. So

$$\exp_p(\alpha Y) = \lim_{i\to\infty} \exp_p(\tau(X_i)Y_i) = \lim_{i\to\infty}(\tau(X_i)X_i) = \exp_p(\alpha X).$$

That is, $\exp_p(tX), t \in [0, \alpha]$ and $\exp_p(tY), t \in [0, \alpha]$ are two distinct minimizing geodesics from p to $\exp_p(\alpha X)$. This contradicts to $\alpha < \tau(X)$. Hence $\tau(X) \leq \alpha$.

Combinig with $(\heartsuit)$, we obtain

$$\tau \leq \liminf_{i\to\infty} \tau(X_i) \leq \limsup_{i\to\infty} \leq \tau(X).$$

This completes the proof.

□

Corollary 6.5. *The cut locus $C(p)$ of $p \in M$ and the cut locus $\tilde{C}(p)$ of p in T_pM are closed subset of M and T_pM respectively.*

Proof. Let $q \in M$ s.t. $\exists p_i \in C(p)$ with $p_i \to q$. Let $\gamma_i(t)$ be the minimizing normal geodesic from p to p_i with $\gamma_i(t_i) = p_i$. Then $t_i = \tau(\dot{\gamma}_i(0))$. W.l.o.g., we can assume $\dot{\gamma}_i(0)$ converges to $Y \in S_p$. Then

$$q = \lim_{i\to\infty} p_i = \lim_{i\to\infty} \gamma(t_i) = \lim_{i\to\infty} \exp_p(\tau(\dot{\gamma}_i(0))\dot{\gamma}_i(0)) = \exp_p(\tau(Y)Y) \in C(p).$$

$\tilde{C}(p)$ is the preimage of $C(p)$ under $\exp_p$, and hence also closed.

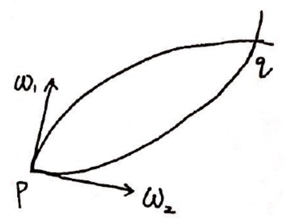
□

Theorem 6.8. *Let M be complete, $p \in M$. Define*

$$E(p) = \{tV : V \in S_p \text{ and } 0 \le t < \tau(V)\}.$$

Then $\exp_p$ *maps $E(p)$ differmorphically onto an open subset of M, and M is the disjoint union of* $\exp E(p)$ *and $C(p)$.*

Proof. Clearly, $d\exp_p$ is 1-1 on E(p), since the first conjugate point can not occur before the cut point. Next we show $\exp_p$ is 1-1 on E(p). Suppose not, $\exists \omega_1, \omega_2 \in E(p)$ with $\|\omega_1\| \le \|\omega_2\|$, say, such that $\exp_p t\omega_1 = \exp_p t\omega_2 = q$.



Then the geodesic $\exp_p t\omega_1, t \in [0,1]$, has length from p to q less than or equal to that of the geodesic $\exp_p t\omega_2, t \in [0,1]$. But these contradicts to the minimal property of $\exp_p t\omega_2, t \in [0, 1+\epsilon]$, since q comes before the cut point of γ. Therefore, $\exp_p$ is diffeomorphic on E(p) onto an open subset of M(since $E(p)$ open).

We next show that $\exp E(p)$ and $C(p)$ are disjiont. If not, $\exists \omega \in$ E(p) and $V \in \tilde{C}(p)$ with $\exp_p \omega = \exp_p V = q$

If $\|V\| \le \|\omega\|$, similar arguments as above, we have a contradiction.

If, otherwise, $\|V\| > \|\omega\|$, then the geodesic $\exp_p tV, t \in [0,1]$ is longer than the geodesic $\exp_p t\omega, t \in [0,1]$ from p to q. This contradicts to the minimizing property of $\exp_p tV, t \in [0,1]$(since $V \in \tilde{C}(p)$).

Clearly, expE(p)$\cup C(p) \subset M$. On the other hand, $\forall q \in M$ completeness implies that there is a normal minimizing geodesic $\gamma(t) = \exp_p tV$ from $p = \gamma(0)$ to $q = \gamma(a)$. Clearly $a \le \tau(V)$ so $aV \in E(p)$ or $aV \in \tilde{C}(p)$.

□

Remark(injectivity raius): Recall the injectivity radius of $p \in M$ is defined as

$$i(p) = \sup\{\rho > 0 : \exp_p \textit{ is a diffeomorphism on } B(O,\rho) \subset T_pM\}$$

Then Theorem 6.6 and theorem 6.8 together implies

$$i(p) = \sup\{\rho > 0 : B(O,\rho) \subset E(p)\}.$$

Notice each ray tX, $X \in S_p$ intersect with $\tilde{C}(p)$ at at most one point. Hence, we have

Proposition 6.1. *The cut locus $\tilde{C}(p)$ of p in T_pM is of zero measure for any $p \in M$.*

Observe that for any compact Riemannian manifold M given $p \in M$, each ray tX must intersect with $\tilde{c}(p)$. In fact, we have

Proposition 6.2. *A complete Riemannian manifold M is compact $\underline{iff}$ for any $p \in M$, the cut locus $\tilde{C}(p)$ is homeomorphisc to the unit sphere $S_p \subset T_pM$.*

Proof. If M is compact, Let diam(M)=δ. Then any normal geodesic from p of length> δ is not minimizing. Hence $\tau(X) < \delta + 1$, $\forall X \in S_p$. Define the map

$$\begin{aligned} \beta : S_p &\to \tilde{C}(p) \\ X &\mapsto \tau(X)X \end{aligned}$$

So β is a homeomorphism.

On the other hand, if $\tilde{C}(p)$ is homeomorphic to $S_p \subset T_pM$, we have in particular $\tilde{C}(p)$ is compact. Let $A \subset S_p$ be the set of $X \in S_p$ s.t. $\tau(X) < \infty$, i.e. $A = \tau^{-1}([0, \infty])$. Then the map $\beta : A \to \tilde{C}(p)$ is a homeomorphism. This tells further that A is compact. So A is closed subset. Theorem 6.6 implies that A is open: $\forall X \in A$, $\exists U$, s,t, $X \in U \subset A$.(Since otherwise, $\exists X_i \to X$ s.t. $\tau(X_i) \to \infty$. Continuity of τ tells $\tau(X) = \infty$. This contradicts to $X \in A$.)

Therefore $A = S_p$. That is, every geodesic from p has a cut point. In other words, $\forall X \in S_p$, $\tau(X) < \infty$. Hence $\max\limits_{X \in S_p} < \infty$. $\forall p, q \in M$, $\exists$ a minimizing geodesic γ from p to q, s.t. $d(p, q) = L(\gamma) \leq \max\limits_{X \in S_p} < \infty$. $\forall p, q \in M$ which implies that M is compact.

□

6.4 Hessian Comparison Theorem

Theorem 6.9 (Hessian Comparison)**.** *Let $M, \overline{M}$ be two Riemannian manifolds of the same dimension n and let $\gamma : [a, b] \to M$, $\overline{\gamma} : [a, b] \to \overline{M}$ be two normal geodesics. Denote $p = \gamma(a)$, $\overline{p} = \overline{\gamma}(a)$, and $\varrho = d(p, .)$, $\overline{\varrho} = d(\overline{p}, .)$ be the distance function resp.*

Suppose:

- *$\gamma|_{[a,b]}$ and $\overline{\gamma}|_{[a,b]}$ are minimizing and contain no cut point.*
- *$K(\Pi_{\gamma(t)}) \leq \overline{K}(\overline{\Pi}_{\overline{\gamma}})$ for all $t \in [a, b]$, all 2-dim sections.*

Then we have ϱ, $\overline{\varrho}$ are C^∞ in a neighborhood of γ, $\overline{\gamma}$ resp(except p,$\overline{p}$). and $\text{Hess}\varrho \geq \text{Hess}\overline{\varrho}$ *along γ, $\overline{\gamma}$.*

<u>Remark</u>

1. "$\text{Hess}\varrho \geq \text{Hess}\overline{\varrho}$ along $\gamma, \overline{\gamma}$" means for any $t \in (a, b]$, and for any $X \in T_{\gamma(t)}M, \overline{X} \in T_{\overline{\gamma}(t)}\overline{M}$ satisfying $|X|_g = |\overline{X}|_{\overline{g}}$, $\langle X, \dot{\gamma}(t)\rangle_g = \left\langle \overline{X}, \dot{\overline{\gamma}}(t)\right\rangle_{\overline{g}}$, and we have $\text{Hess}\varrho(X, X) \geq \text{Hess}\overline{\varrho}(\overline{X}.\overline{X})$.
2. Since $\gamma(b)$ is not a cut point, we have $(b - a)\dot{\gamma}(0) \notin \tilde{C}(p)$. By Corollary 6.5, $\tilde{C}(p)$ is closed, hence $\exists$ an open neighborhood $U = \{V \in T_pM : |V - (b - a)\dot{\gamma}(0)| < \epsilon\}$ s.t. $U \cap \tilde{C}(p) = \phi$.

Notice we can write $U = \{(b-a)V \in T_pM : |V - \dot{\gamma}(0)| < \frac{\epsilon}{b-a}\}$. Let $\mathcal{U} = \{(t-a)V \in T_pM : t \in [a,b],\ |V - \dot{\gamma}(0)| < \frac{\epsilon}{b-a}\}$. Theorem 6.8 tells that $\exp_p$ on $\mathcal{U}$ is a diffeomorphism, and $\varrho(\exp_p W) = |W|$, $\forall W \in \mathcal{U}$. So ϱ is C^∞ on $\exp_p \mathcal{U}\backslash\{p\}$. This proves the smoothness of $\varrho, \overline{\varrho}$ claimed in the theorem.

Proof. For any $X \in T_{\gamma(t)}M$ for some t, we can decompose $X = a\dot{\gamma}(t) + X^\perp$, where $\left\langle X^\perp, \dot{\gamma(t)} \right\rangle = 0$.

Observe that

$$\begin{aligned}\mathrm{Hess}\varrho\left(\dot{\gamma(t)}, \dot{\gamma(t)}\right) &= \nabla^2\varrho\left(\dot{\gamma(t)}, \dot{\gamma(t)}\right) = \nabla(\nabla\varrho)\left(\dot{\gamma(t)}, \dot{\gamma(t)}\right)\\ &= \nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}}\varrho - \nabla_{\nabla_{\dot{\gamma}}\dot{\gamma}}\varrho = \dot{\gamma}\dot{\gamma}\varrho - \nabla_{\dot{\gamma}}\dot{\gamma}\varrho = 0.\end{aligned}$$

$$\begin{aligned}\mathrm{Hess}\varrho\left(\dot{\gamma}, X^\perp\right) &= \nabla_{X^\perp}\nabla_{\dot{\gamma}}\varrho - \nabla_{\nabla_{X^\perp}\dot{\gamma(t)}}\varrho\\ &= -\left(\nabla_{X^\perp}\dot{\gamma}\right)(\varrho) = -\left\langle \nabla_{X^\perp}\dot{\gamma}, grad\varrho\right\rangle.\end{aligned}$$

(Recall $\dot{\gamma}\varrho = 1 = \langle\dot{\gamma}, grad\varrho\rangle$, and $\langle grad\varrho, E\rangle = 0$ for any $\langle E, \dot{\gamma}\rangle = 0$.)

We have

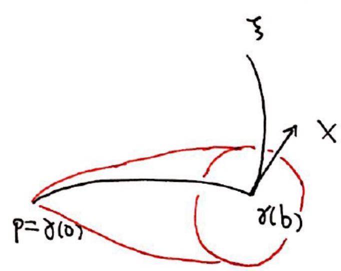
$$grad\varrho = \dot{\gamma} = -\left\langle \nabla_{X^\perp}\dot{\gamma}, \dot{\gamma}\right\rangle = -\frac{1}{2}X^\perp\langle\dot{\gamma}, \dot{\gamma}\rangle = 0.$$

That is

$$\mathrm{Hess}\varrho(X, X) = \mathrm{Hess}\varrho\left(a\dot{\gamma}(t) + X^\perp, a\dot{\gamma}(t) + X^\perp\right) = \mathrm{Hess}\varrho(X^\perp, X^\perp).$$

So we only need to consider $X \in T_{\gamma(t)}M$, $\overline{X} \in T_{\overline{\gamma}(t)}\overline{M}$, which are perpendic to $\dot{\gamma}(t)$ and $\dot{\overline{\gamma}}(t)$ resp..

W.l.o.g., we let $t = b$. $X \in T_{\gamma(b)}M$, $\overline{X} \in T_{\overline{\gamma}}\overline{M}$, $\langle X, \dot{\gamma}(b)\rangle = 0$, $\langle \overline{X}, \dot{\overline{\gamma}}(t)\rangle = 0$.

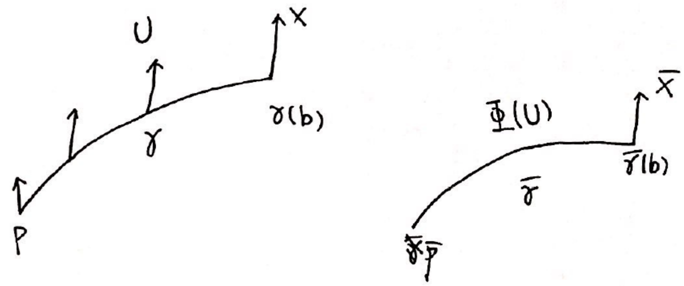


Let ξ be the geodesic s.t. $\xi(0) = \gamma(b)$, $\dot{\xi}(0) = X$. Since $\gamma(b)$ is not a cut point, by the same argument as in Remark (2), there exists a neighborhood $\mathcal{U}$ s.t. $\exp_p : \mathcal{U} \to \exp_p \mathcal{U}$ is a diffeomorphism, and $\forall W \in \mathcal{U}$, $\exp_p tW$ is minimizing. Therefore $\exists\epsilon > 0$ s.t. the minimizing geodesic $\gamma_s : [a,b] \to M$, $s \in [0,\epsilon]$ from p to $\xi(s)$ forms a well-defined variation $F(t,s) = \gamma_s(t)$, $t \in [a,b]$, $s \in [0,\epsilon]$. The corresponding variational field $U(t) = \frac{\partial}{\partial s}|_{s=0}F(t,s)$ is a Jacobi field with $U(0) = 0$, $U(b) = X$. And U is also a normal field.

Notice that $\varrho(\xi(s)) = Length(\gamma_s) = L(\gamma_s) = L(s)$. We have $\mathrm{Hess}\varrho(X, X) = L''(s)$. Recall from lemma 5.8.

$$L''(0) = \langle \nabla_U U, \dot{\gamma} \rangle \Big|_a^b + \int_a^b \langle \nabla_T U^{\perp}, \nabla_T U^{\perp} \rangle - \langle R(U^{\perp}, T)T, U^{\perp} \rangle \mathrm{d}t = I(U, U).$$

Similarly, $\mathrm{Hess}\overline{\varrho}(\overline{X}, \overline{X}) = \overline{I}(\overline{U}, \overline{U})$, where $\overline{U}$ is a normal Jacobi field along $\overline{\gamma}$ with $\overline{U}(0) = 0$, $\overline{U}(b) = \overline{X}$. It remains to show $I(U, U) \geq \overline{I}(\overline{U}, \overline{U})$.



Using the construction of Φ in Lemma 6.1. (firstly, choose $\phi_b : T_{\gamma(b)}M \to T_{\overline{\gamma}(b)}\overline{M}$ be the isometry of inner product spaces which sends X to $\overline{X}$, $\dot{\gamma}(b)$ to $\dot{\overline{\gamma}}$. This is possible since $|X| = |\overline{X}|$, $\langle X, \dot{\gamma}(b) \rangle = \langle \overline{X}, \dot{\overline{\gamma}}(b) \rangle$, $|\dot{\gamma}(b)| = |\dot{\overline{\gamma}}(b)| = 1$).

As in Theorem 6.3, we have $I(U, U) \geq \overline{I}(\Phi(U), \Phi(U))$. By minimizing property of Jacobi field $\overline{I}(\Phi(U), \Phi(U)) \geq \overline{I}(\overline{U}, \overline{U})$. So $I(U, U) \geq \overline{I}(\overline{U}, \overline{U})$.

□

Corollary 6.6. *Under the same assumption of Thm6.9, and let $f : [0, b-a] \to \mathbb{R}$ be a C^{∞} function which satisfies $f' \geq 0$. Then we have $\mathrm{Hess}f(\varrho) \geq \mathrm{Hess}f(\overline{\varrho})$ along γ, $\overline{\gamma}$.*

Proof. Note for any $X \in T_{\gamma(t)}M$ for any t,

$$\begin{aligned}
\mathrm{Hess}f(\varrho)(X, X) &= \frac{\mathrm{d}^2}{\mathrm{d}s^2}|_{s=0} f(\varrho(\xi(s))) \\
&= \frac{\mathrm{d}}{\mathrm{d}s}\left(f'(\varrho(\xi(s))) \frac{\mathrm{d}}{\mathrm{d}s} \varrho(\xi(s)) \right)|_{s=0} \\
&= f''(\varrho(t)) \left(\frac{\mathrm{d}}{\mathrm{d}s}|_{s=0} \varrho(\xi(s)) \right)^2 + f'(\varrho(t)) \frac{\mathrm{d}^2}{\mathrm{d}s^2}|_{s=0} \varrho(\xi(s)) \\
&= f''(\varrho)\,(X\varrho)^2 + f'(\varrho)\mathrm{Hess}\varrho(X, X) \\
&= f''(\varrho)\langle X, grad\varrho \rangle^2 + f'(\varrho) Hess\varrho(X, X) \\
&= f''(\varrho)\langle X, \dot{\gamma}(t) \rangle^2 + f'(\varrho)\mathrm{Hess}\varrho(X, X).
\end{aligned}$$

Hence we have $\mathrm{Hess}f(\varrho)(X, X) \geq \mathrm{Hess}f(\overline{\varrho})(\overline{X}, \overline{X})$ for X, $\overline{X}$ s.t. $|X| = |\overline{X}|$, $\langle X, \dot{\gamma}(t) \rangle = \langle \overline{X}, \dot{\overline{\gamma}}(t) \rangle$. □

Example(Hessϱ on manifolds with constant sectional curvature)

Let $\gamma : [0, b] \to M$ be a normal minimizing geodesic. From the proof of thm6.9, we see for $X \in T_{\gamma(b)}M$, $\langle X, \dot\gamma(b)\rangle = 0$,

$$\mathrm{Hess}\varrho(X, X) = I(U, U)$$

where U is a normal Jacobi field along γ with $U(0) = 0$, $U(b) = X$. Recall

$$\begin{aligned} I(U, U) &= \int_a^b \langle \nabla_T U, \nabla_T U\rangle - \langle R(U, T)T, U\rangle \mathrm{d}t \\ &= -\int_a^b \langle \nabla_T \nabla_T U + R(U, T)T, U\rangle \mathrm{d}t + \langle \nabla_T U, U\rangle|_a^b \\ &= \langle \nabla_T U(b), U(b)\rangle. \end{aligned}$$

That is $\mathrm{Hess}\varrho(X, X) = \langle \nabla_T U(b), U(b)\rangle$.

Let $\dot\gamma(t), E_2(t), \ldots, E_n(T)$ be parallel orthonormal vector fields along γ. If M has constant sectional curvature K, the Jacobi fields U with $U(0) = 0$ is given by $U(t) = f(t)\sum_{i=2}^n E_i(t)$, where

$$\begin{cases} f''(t) + Kf(t) = 0 \\ f(0) = 0 \end{cases}$$

$$\Rightarrow f(t) = \begin{cases} ct, K = 0 \\ c\sin\sqrt{K}t, K > 0 \\ c\sinh\sqrt{-K}t, K < 0 \end{cases} \quad \textit{for some constant } c.$$

Hence $\nabla_T U(b) = f'(b)\sum E_i(b)$, $\langle \nabla_T U(b), U(b)\rangle = (n-1)f'(b)f(b)$ and $|X|^2 = (n-1)f(b)^2$. Therefore

$$\begin{aligned} \mathrm{Hess}\varrho(X, X) &= (n-1)f'(b)f(b) \\ &= \frac{f'(b)}{f(b)}|X|^2. \end{aligned}$$

In particular, if $K = 0$, $\mathrm{Hess}\varrho(X, X) = \frac{1}{\varrho}g(X, X)$ for $< X, \dot\gamma >= 0$(**FACT1**).

In general, for $X \in T_{\gamma(b)}M$, we have

$$\begin{aligned} \mathrm{Hess}\varrho(X, X) = \mathrm{Hess}\varrho(X^\perp, X^\perp) &= \frac{f'(b)}{f(b)}|X^\perp|^2 \\ &= \frac{f'(b)}{f(b)}\langle X - \langle X, \dot\gamma(b)\rangle\dot\gamma(b), X - \langle X, \dot\gamma(b)\rangle\dot\gamma(b)\rangle \\ &= \frac{f'(b)}{f(b)}\left(\langle X, X\rangle - \langle X, \dot\gamma(b)\rangle^2\right). \end{aligned}$$

Again when $K = 0$, we have

$$\begin{aligned} \mathrm{Hess}\varrho^2(X, X) &= 2(X\varrho)^2 + 2\varrho\mathrm{Hess}\varrho(X, X) \\ &= 2\langle X, \dot\gamma(b)\rangle^2 + 2g(X^\perp, X^\perp) \\ &= 2g(X, X)\ (\textbf{FACT2}). \end{aligned}$$

Corollary 6.7. *Let M be a complete simply-connected Riemannian manifold with non-positive sectional curvature, $\varrho = d(p, .)$, $p \in M$. Then on $M \setminus \{p\}$, we have*

$$\mathrm{Hess}\varrho^2 \geq 2g, \tag{6.4.1}$$

$$\Delta\varrho \geq \frac{n-1}{\varrho}. \tag{6.4.2}$$

Proof. By Cartan-Hadamard, there is no cut point. Then apply Thm 6.9 and FACT2, we get (6.4.1), and apply FACT1, we get (6.4.2). □

Remark:

1. Corollary(6.7) is a stengthen result of the convexity (Theorem 5.11). We discussed in the end of last chapter.
2. Under the same assumption of thm(6.9), by taking trace, we obtain $\Delta\varrho(\gamma(t)) \geq \overline{\Delta}\overline{\varrho}(\overline{\gamma}(t))$ (★).

A natural question is: In order to obtain the comparison (★), is it enough to assume Ricci curvature comparison instead of sectional curvature comparison?

6.5 Laplacian Comparison Theorem

Theorem 6.10 (Laplacian Comparison)**.** *Let $M, \overline{M}$ be two Riemannian manifolds of the same dimension n and let $\gamma : [a, b] \to M$, $\overline{\gamma} : [a, b] \to \overline{M}$ be two normal geodesics. Denote $p = \gamma(a)$, $\overline{p} = \overline{\gamma}(a)$, and $\varrho = d(p, .)$, $\overline{\varrho} = d(\overline{p}, .)$ be the distance function resp. Ric, $\overline{Ric}$, Δ, $\overline{\Delta}$ be the Ricci curvature tensor and Laplacian of M, $\overline{M}$ resp.*

Suppose:

- *1- $\gamma|_{[a,b]}$ and $\overline{\gamma}|_{[a,b]}$ are minimizing and contain no cut point.*
- *2- $Ric(\dot{\gamma}, \dot{\gamma})(t) \leq \overline{Ric}\left(\dot{\overline{\gamma}}, \dot{\overline{\gamma}}\right)(t)$, $\forall t \in [a, b]$.*
- *3- M is a space form of sectional curvature k.*

Thus we have $\Delta\varrho(\gamma(t)) \geq \overline{\Delta}\overline{\varrho}((t))$, $\forall t \in [a, b]$. Moreover , $\Delta\varrho(\gamma(b)) = \overline{\Delta}\overline{\varrho}((b))$ iff $\forall t \in [0, b]$, any section in tangent bundle of $\overline{M}$ containing $\dot{\overline{\gamma}}$ has sectional curvature k, and any normal Jacobi field $\overline{U}(t)$ along $\overline{\gamma}$ with $\overline{U}(a) = 0$ can be represented as $\overline{U}(t) = f(t)E(t)$, where $E(t)$ is parallel along γ and $f : [a, b] \to \mathbb{R}$ is a solution of

$$\begin{cases} f'' + kf = 0 \\ f(0) = 0 \end{cases}$$

Remark:

1. On a space form of sectional curvature k, we have $\Delta\varrho\,(\gamma(t)) = (n-1)\frac{f'(t)}{f(t)}$ where

$$f(t) = \begin{cases} ct, k = 0 \\ c\sin(\sqrt{k}t), k > 0 \\ c\sinh(\sqrt{-k}t), k < 0 \end{cases}$$

2. We will explain why we have to add the assumption (3) during the proof.

Proof. W.l.o.g., we consider $t = b$, and an orthonormal basis $\{e_1, \dots, e_n\}$ of $T_{\gamma(b)}M$ with $e_1 = \dot\gamma(b)$. Then $Delta\varrho(\gamma(b)) = \sum_{i=2}^n \mathrm{Hess}\varrho(e_i, e_i)$. Same argument as in the proof of Theorem 6.9, we obtain $\mathrm{Hess}\varrho(e_i, e_i) = I(U_i, U_i)$, where U_i is a normal Jacobi field along γ with $U_i(0) = 0$, $U_i(b) = e_i$.

Do similar thing on $\overline{M}$: let $\{\overline{e}_1, \dots, \overline{e}_n\}$ be a orthonormal basis of $T_{\overline{\gamma}(b)}\overline{M}$ with $\overline{e}_1 = \dot{\overline{\gamma}}(b)$. We have $\mathrm{Hess}\overline{\varrho}(\overline{e}_i, \overline{e}_i) = \overline{I}(\overline{U}_i, \overline{U}_i)$. Then let $\phi_b : T_{\gamma(b)}M \to T_{\overline{\gamma}(b)}\overline{M}$ be the isometry with $\phi_b(e_i) = \overline{e}_i$, $i = 1, \dots, n$. Then construct Φ as in Lemma 6.1 optinal ransport. Then we obtain $(n-1)$ vector fields $\Phi(U_i)$, $i = 1, \dots, n$ along $\overline{\gamma}$ with

$$\begin{aligned} \Phi(U_i)(a) &= 0 = \overline{U}_i(a), \\ \Phi(U_i)(b) = \phi_b(U_i(b)) &= \phi_b(e_i) = e_i = \overline{U}_i(b). \end{aligned}$$

By the minimizing property of Jacobi fields, we have

$$\overline{I}\,(\Phi(U_i), \Phi(U_i)) \geq \overline{I}(\overline{U}_i, \overline{U}_i) \tag{6.5.1}$$

for $i = 2, \dots, n$

The Laplacian Comparison is reduced to show

$$\sum_{i=2}^n \overline{I}\,(U_i, U_i) \geq \sum_{i=2}^n \overline{I}(\overline{U}_i, \overline{U}_i). \tag{6.5.2}$$

Since

$$\sum_{i=2}^n \overline{I}\,(\Phi(U_i), \Phi(U_i)) \geq \sum_{i=2}^n \overline{I}(\overline{U}_i, \overline{U}_i). \tag{6.5.3}$$

it is enough to show

$$\sum_{i=2}^n \overline{I}\,(U_i, U_i) \geq \sum_{i=2}^n \overline{I}(\overline{U}_i, \overline{U}_i). \tag{6.5.4}$$

Recall, if we have ”sectional-curvature comparison”, we conclude the above inequlity immediately. If we only have ”Ricci-curvature comparison”, it turns out, we have to assume assumption 3 in order to get (6.5.4).

(6.5.4) $\Leftrightarrow$

$$\begin{aligned} &\sum_{i=2}^n \int_a^b (\langle \nabla_T U_i, \nabla_T U_i\rangle - \langle R(U_i, T)T, U_i\rangle)\,\mathrm{d}t \\ \geq &\sum_{i=2}^n \int_a^b \left(\langle \nabla_{\overline{T}}\Phi(U_i), \nabla_{\overline{T}}\Phi(U_i)\rangle - \langle R(\Phi(U_i), \overline{T})\overline{T}, \Phi(U_i)\rangle\right)\mathrm{d}t. \end{aligned}$$

Lemma 6.1 $\Rightarrow |\nabla_T U_i| = |\nabla_{\overline{T}}\Phi(U_i)|$. Hence

(6.5.4) $\Leftrightarrow$

$$\int_a^b \sum_{i=2}^n \langle R(U_i, T)T, U_i \rangle \mathrm{d}t \le \int_a^b \sum_{i=2}^n \langle R(\Phi(U_i), \overline{T})\overline{T}, \Phi(U_i) \rangle \mathrm{d}t.$$

Notice that at $\gamma(b)$, $T = e_1$, $U_i = e_i$ is an orthornomal basis and $\sum_{i=2}^n \langle R(U_i, T)T, U_i \rangle (b) = Ric\,(\dot\gamma, \dot\gamma)\,(b)$. Similarly, $\sum_{i=2}^n \langle R(\Phi(U_i), \overline{T})\overline{T}, \Phi(U_i) \rangle (b) = \overline{Ric}\left(\dot{\overline{\gamma}}, \dot{\overline{\gamma}}\right)(b)$. But in general, $\{T, U_2, \ldots, U_n\}$ is not orthonormal any more at $t \neq b$. A solution is to add the assumption. In that case, M is a space form of constant curvature k. And we know the Jacobi field $U_i(t) = f(t)e_i(t)$, where $\{e_i(t)\}$ is a parallel orthononormal vector fields along γ with $e_i(b) = e_i$ and f is a solution of

$$\begin{cases} f'' + kf = 0 \\ f(a) = 0, f(b) = 1 \end{cases}$$

Hence $\sum_{i=2}^n \langle R(U_i, T)T, U_i \rangle (t) = f^2(t) \sum_{i=2}^n \langle R(e_i, T)T, e_i \rangle = f^2(t) Ric(T, T) = f^2(t) Ric(\dot\gamma, \dot\gamma)(t)$. And by the construction of Φ, we know $\Phi(U_i) = f(t)\overline{e}_i(t)$. So $\sum_{i=2}^n \langle R(\Phi(U_i), \overline{T})\overline{T}, \Phi(U_i) \rangle = f^2(t)\overline{Ric}\left(\dot{\overline{\gamma}}, \dot{\overline{\gamma}}\right)$. The assumption (2) implies $f^2(t) Ric(\dot\gamma, \dot\gamma)(t) \le f^2(t)\overline{Ric}\left(\dot{\overline{\gamma}}, \dot{\overline{\gamma}}\right)$, $\forall t \in [a, b]$. We then prove (6.5.4) and hence the Laplacian comparison.

If $\Delta\varrho(\gamma(b)) = \overline{\Delta}\overline{\varrho}(\overline{\gamma}(b))$, we have "=" holds in (6.5.1). Recall the minimizing property of Jacobi field, this can only happen when $\forall i = 2, \ldots, n$, $\Phi(U_i) = \overline{U}_i$. By the construction, we know $\Phi(U_i) = f(t)e_i(t)$. Any normal Jacobi field $\overline{U}(t)$ along $\overline{\gamma}$ with $\overline{U}(0) = 0$ cab be expressed as a linear combination of $\Phi(U_i)$, $i = 2, \ldots, n$.

$$\begin{aligned} \overline{U} &= \sum_{i=2}^n c_i f(t) e_i(t), c_i \in \mathbb{R} \\ &= f(t) \sum_{i=2}^n c_i e_i(t). \end{aligned}$$

By Jacobi equation, $\Phi(U_i) = f(t)e_i(t)$ is a Jacobi field implies $\langle R(e_i, T)T, e_i \rangle = k$.

□

Corollary 6.8. *Under the same assumption of thm (6.10), and let $f : [a, b] \to \mathbb{R}$ be a smooth function with $f' \ge 0$. Then*

$$\Delta f(\varrho)(\gamma(t)) \ge \overline{\Delta} f(\overline{\varrho})((t))$$

$\forall t \in [a, b]$

Proof.

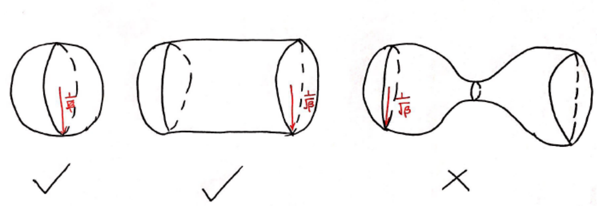
$$\begin{aligned} \Delta f(\varrho) &= f''(\varrho) + f'(\varrho)\Delta\varrho, \\ \overline{\Delta} f(\overline{\varrho}) &= f''(\varrho) + f'(\overline{\varrho})\overline{\Delta}\overline{\varrho}. \end{aligned}$$

□

6.6 Comments on Injectivity Radius Estimate and Sphere Theorems

Is it always true for a simply-connected complete two dimensional Riemannian manifold (M, g) with Gauss curvature $\leq \beta, \beta > 0$ that $\exp_O : T_O M \to M$ is a diffeomorphism on $B(0, \frac{\pi}{\sqrt{\beta}})$.

This is not always true unfortunately.



By comparison theorem, Gauss curvature $\leq \beta$ implies $\exp_O\left(B(0, \frac{\pi}{\sqrt{\beta}})\right)$ contains no conjugate point of O. But this is not enough to conclude that $\exp_O$ is a diffeomorphism. Actually, in other words, we are asked to show $inj_M \geq \frac{\pi}{\sqrt{\beta}}$. For that result, we need further restrict Gauss curvature > 0. This is actually Klingenberg's injectivity radius estimate.

Theorem 6.11 (Klingenberg,1959)**.** *Suppose (M, g) is an orientable even-dimensional manifold with $0 <$sectional curvature$\leq \beta$. Then $inj_M \geq \frac{\pi}{\sqrt{\beta}}$. If M is not orientable, then $inj_M \geq \frac{\pi}{2\sqrt{\beta}}$.*

For the proof, we refer to [Spivak IV, Chap8 34-36] or [PP,§6.2]. Here, we explain the rough ideas

Proof. We need to show $\exp_O\left(B(0, \frac{\pi}{\sqrt{\beta}})\right)$ has no cut point. If there is a cut point of q of O along γ, since q can not be a conjugate point of O, we have by 6.6, there exists exactly two minimal geodesics from O to q. (denoted by γ,γ_1)

In fact, one can further argue that when q is the closest one to O in $C(0)$, $\dot{\gamma}|_q = -\dot{\gamma}_1|_q$. Moreover, when O is a point such that it is the minimum point of the function $d(p, C(p))$, we have the two geodesics γ, γ_1 give a close geodesic.

Recall in the proof of Synge theorem, under the assumption "orientable, even-dimensional", any closed geodesic has a variation $F(t, s)$ such that the curves $\gamma_s(t) = F(t, s)$ has length $L(\gamma_s(t)) < L(\gamma_0(t))$, with $0 < s$ small. Hence the whole curve γ_s lie in the interior of the cut locus. So there exists a minimal geodesic σ_s from $\gamma_s(0)$ to the farthest point $\gamma_s(t_s)$ along γ_s.

Choose subsequence if necessary, those geodesics σ_s converge to a minimal geodesic σ from O to q which is different from γ and γ_1. This contradicts to the assumption that q is a cut point and γ, γ_1 are the only minimizing geodesic from O to q. So there is no cut point in $\exp_O\left(B(0, \frac{\pi}{\sqrt{\beta}})\right)$. ◻

A much deeper result by Klingenberg asserts that if a simply-connected manifold has all its sectional curvature in the interval $(\frac{1}{4}\beta, \beta]$, then $inj_M \geq \frac{\pi}{\beta}$. There are further improvement on the left end of the interval.

Actually, those injectivity radius estimate and Rauch comparison theorem are crucial tools to establish fascinating Sphere Theorems

Theorem 6.12 (Topological Sphere Theorem)**.** *Let (M, g) be a simply-connected complete Riemannian manifold. Suppose M has all its sectional curvatures in the interval $(\frac{1}{4}\beta, \beta]$, $\beta > 0$. Then M is homeomorphic to the sphere.*

This result is due to Rauch (prove in the case sec $\in (\frac{3}{4}\beta, \beta]$), Klingenberg, Berger.

Very brief explanation: In topology, Brown theorem tells that: if a compact manifold M is the union of two open sets, each of which is diffeomorphic to $\mathbb{R}^n$, then M is diffeomorphic to $\mathbb{S}^n$.

So let $p, q \in M$, s.t. $d(p, q) = \text{diam}(M, g) \leq \frac{2\pi}{\sqrt{\beta}}$. $\exp_p : T_pM \to M$, $\exp_q : T_qM \to M$ are diffeomorphisms on $B(0, \delta_p)$, $B(0, \delta_q)$ at least when δ_p, δ_q are small enough.

$$\exp_p(B(0, \delta_p)) \cup \exp_q(B(0, \delta_q)) \subset M$$

On the other hand, if we have δ_p, δ_q large enough, we have $\exp_p(B(0, \delta_p)) \cup \exp_q(B(0, \delta_q)) = M$.

Scaling β to be 1, $\text{diam}(M, g) \leq 2\pi$, then Klingenberg $\Rightarrow inj_M \geq \pi$. It remain to show

$$M \subset \exp_p(B(0, \delta_p)) \cup \exp_q(B(0, \delta_q))$$

That is for any $x \in M$, if $d(p, x) \geq inj_M \geq \pi$, then we need show $d(q, x) < inj_M$. For this purpose, we need a global version of the Rauch theorem:

Toponogov triangle comparison theorem. (This has been discussed in the tutorial)

6.7 Volume Coparison Theorems

Now let us come back to investigate a geometric quantity which we have discussed at the very beginning of this course: the volume.

Recall $E(p) = \{tV : V \in S_p and\ 0 \leq t < \tau(V)\}$ from 6.8. Let us denote by $E_p = \exp_p E(p)$. We have shown that

$$\exp_p : E(p) \to E_p$$

is a diffeomorhism, and $E(p)$ is diffeomorphic to an open ball. Since the cut locus is of zero measure, we have

$$Vol(M) = \int_M \mathrm{d}vol = \int_{E_p} \mathrm{d}vol.$$

Note $E_p \subset M$ can be considered as a coordinate neighborhood!

Hence

$$\begin{aligned} Vol(M) &= \int_{E_p} \sqrt{det(g_{ij})}\mathrm{d}x^1 \dots \mathrm{d}x^n \\ &= \int_{E_p} \sqrt{det(g_{ij})}\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n. \end{aligned}$$

(This is another way to see the definition of volume does not depend on orientability).

For the ball $B_p(r) = \{q \in M | d(p,q) < r\}$ we have $Vol(B_p(r)) = \int_{B_p(r)} \mathrm{d}vol = \int_{E_p \cap B_p(r)} \sqrt{det(g_{ij})}\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$.

How to calculate $vol(B_p(r))$ in a Riemannian manifold?

We first prepare some algebraic results: Let $f : \tilde{V} \to V$ be a linear transformation between two n-dimensional inner product spaces. Let $\{\tilde{e}_1, \dots, \tilde{e}_n\}$, $\{e_1, \dots, e_n\}$ be their orthonormal basis, and $\{\tilde{\omega}^1, \dots, \tilde{\omega}^n\}$, $\{\omega^1, \dots, \omega^n\}$ be their dual basis. Let $\tilde{\Omega} = \tilde{\omega}^1 \wedge \cdots \wedge \tilde{\omega}^n$, $\Omega = \omega^1 \wedge \cdots \wedge \omega^n$. Then $f^*\Omega$ is defined as

$$f^*\Omega(X_1, \dots, X_n) = \Omega\left(f(\tilde{X}_1), \dots, f(\tilde{X}_n)\right).$$

Therefore, $\exists a_0 \in \mathbb{R}$ s.t. $f^*\omega = a_0\tilde{\Omega}$.

On the other hand, let $f(\tilde{e}_i) = \sum_{j=1}^n \alpha_i^j e_j$, $i = 1, \dots, n$. Then $a_0 = det[a_i^j] = det(f)$.

<u>Claim</u>: Let $\{\tilde{A}_1, \dots, \tilde{A}_n\}$ be a basis of $\tilde{V}$. Then

$$|a_0| = |det(f)| = |\frac{f^*\Omega}{\tilde{\Omega}}| = \frac{|f(\tilde{A}_1) \wedge \cdots \wedge f(\tilde{A}_n)|}{|\tilde{A}_1 \wedge \cdots \wedge \tilde{A}_n|}.$$

Proof. Recall $\forall X_i \in V$, $i = 1, \dots, n$, we have

$$\begin{aligned} |X_1 \wedge \cdots \wedge X_n| &= \langle X_1 \wedge \cdots \wedge X_n, X_1 \wedge \cdots \wedge X_n \rangle^{\frac{1}{2}} \\ &= \sqrt{det[< X_i, X_j >]}. \end{aligned}$$

In particular, if $\{X_i\}$ are orthonormal, $|X_1 \wedge \cdots \wedge X_n| = 1$. Therefore, letting $\tilde{A}_i = \sum_{j=1}^n \beta_i^j \tilde{e}_j$, $i = 1, \dots, n$. We have

$$\begin{aligned} |f(\tilde{A}_1) \wedge \cdots \wedge f(\tilde{A}_n)| &= |det[\beta_i^j]| \times |f(\tilde{e}_1) \wedge \cdots \wedge f(\tilde{e}_j)| \\ &= |det[\beta_i^j]| \times |det[\alpha_i^j]| \times |e_1 \wedge \cdots \wedge e_n| \\ &= |det[\beta_i^j]| \times |det(f)|. \end{aligned}$$

On the other hand,

$$|\tilde{A}_1 \wedge \cdots \wedge \tilde{A}_n| = |det[\beta_i^j]| \, |\tilde{e}_1 \wedge \cdots \wedge \tilde{e}_n| = |det[\beta_i^j]|.$$

This proves the claim.

∎

Now, let $\gamma : [a, b] \to M$ be a normal geodesic with $\gamma(0) = p$. Let $\tilde{\Omega}$ be a volume n-form of the Euclidean space T_pM, and $\Omega(t)$ be a volume n-form at $\gamma(t) \in M$. Then we have

$$\exp_p : T_pM \to M,$$
$$(\mathrm{d}\exp_p)_{t\dot{\gamma}(0)} : T_{t\dot{\gamma}(0)}\left(T_pM\right) \to T_{\gamma(t)}M.$$

Define a function $\phi : [0, b] \to \mathbb{R}$ to be

$$\phi(t) = \left| \frac{(\mathrm{d}\exp_p)^*_{t\dot{\gamma}(0)}\,(\Omega(t))}{\tilde{\Omega}} \right|.$$

Note that $\Omega(t)$, $\tilde{\Omega}$ depend on the choice of different orientations, but ϕ does not. We can always choose proper orientation s.t.

$$\phi(t) = \frac{(\mathrm{d}\exp_p)^*\,(\Omega(t))}{\tilde{\Omega}}.$$

(we omit the subscript: $t\dot{\gamma}(0)$).

(because we are working in the single coordinate neighborhood E_p!!)

We in fact can define a function. $\phi : E(p) \to \mathbb{R}$ as below: $\forall \tilde{y} \in E(p)$,

$$\phi(\tilde{y}) = \left| \frac{(\mathrm{d}\exp_p)^*\,(\Omega(y))}{\tilde{\Omega}(\tilde{y})} \right| = \frac{(\mathrm{d}\exp_p)^*\,(\Omega(y))}{\tilde{\Omega}(\tilde{y})}$$

where $y = \exp_p \tilde{y}$.

Then we can rewrite the volume of M as

$$\begin{aligned} Vol(M) = Vol(E_p) &= \int_{E_p} \Omega = \int_{\exp_p(E(p))} \Omega \\ &= \int_{E(p)} (\mathrm{d}\exp_p)^*\Omega = \int_{E(p)} \phi\tilde{\Omega} \\ &= \int_{E(p)} \phi \mathrm{d}vol_{T_pM} \end{aligned}$$

and $Vol(B_p(r)) = \int_{E(p)\cap B(0,r)} \phi \mathrm{d}vol_{T_pM}$.

So the key point is to complete the funtion ϕ, for which we need employ results about Jacobi fields again.

Lemma 6.2. *Let $\gamma : [0, b] \to M$ be a normal geodesic containing no conjugate point. Let $J_1, \ldots, J_{n-1}$ be (n-1) linearly independent normal Jacobi fields along γ with $J_i(0) = 0$, $i = 1, 2, \ldots, n-1$. Thus we have*

$$\phi(t) = \frac{|J_1(t) \wedge \cdots \wedge J_{n-1}(t)|}{t^{n-1}|\dot{J}_1(0) \wedge \cdots \wedge \dot{J}_{n-1}(0)|}$$

$t \in (0, b]$

Proof.

$$\mathrm{d}\exp_p : T_{\gamma(t)}T_pM \to T_{\gamma(t)}M.$$

Note that $J_1(t), \ldots, J_{n-1}(t), \dot{\gamma}$ are n linearly independent vectors in $T_\gamma(t)M$, and hence form a basis of $T_{\gamma(t)}M$. By "dimension consideration",, this also implies $\dot{J}_1(0) \ldots \dot{J}_{n-1}(0)$ are linearly independent.

$\langle J_i(t), \dot{\gamma}(t)\rangle = 0$, $t \in [0, b]$ implies $\langle \dot{J}_i(0), \dot{\gamma}(0)\rangle = 0$. Hence $\dot{J}_1(0) \ldots \dot{J}_{n-1}(0), \dot{\gamma}(0)$ form a basis of T_pM. Recall for the variation $F(t, s) = \exp_p t(\dot{\gamma}(0) + sW)$, we have its variational field $U(t)$ is a Jacobi field with $U(0) = 0$, $\dot{U}(0) = W$, and $U(t) = \left(\mathrm{d}\exp_p\right)_{t\dot{\gamma}(0)}(tW)$. Pick $W = \dot{J}_i(0)$, we then obtain $J_i(t) = \left(\mathrm{d}\exp_p\right)_{t\dot{\gamma}(0)}(t\dot{J}_i(0))$. Notice that for any $t \in (0, b]$, $t\dot{J}_1(0) \ldots t\dot{J}_{n-1}(0), \dot{\gamma}(0)$ also form a basis of T_pM.

Hence

$$\begin{aligned}\phi(t) &= \frac{|\dot{\gamma}(t) \wedge J_1(t) \wedge \cdots \wedge J_{n-1}(t)|}{\left|\dot{\gamma}(0) \wedge t\dot{J}_1(0) \wedge \cdots \wedge t\dot{J}_{n-1}(0)\right|} \\ &= \frac{|J_1(t) \wedge \cdots \wedge J_{n-1}(t)|}{t^{n-1}\left|\dot{J}_1(0) \wedge \cdots \wedge \dot{J}_{n-1}(0)\right|}.\end{aligned}$$

□

Theorem 6.13 (Bishop)**.** *Let M be a Riemannian maninfold with $Ric \geq (n-1)k$. Let $\gamma : [0, b] \to M$ be a normal geodesic containing no cut point, then $\frac{\phi(t)}{\phi_k(t)}$ is nonincreasing, $\forall t \in (0, b]$. (ϕ_k is the funcion on the simply-connected space form M^k of sectional curvature k).*

By lemma 6.2, we can check $\phi_k(t) = \left(\frac{f_k(t)}{t}\right)^{n-1}$, where

$$f_k(t) = \begin{cases} f, k = 0 \\ \dfrac{1}{\sqrt{k}} \sin \sqrt{k}t, k > 0 \\ \dfrac{1}{\sqrt{-k}} \sinh \sqrt{-k}t, k < 0 \end{cases}$$

We will show this result by reduce it to the Laplacian comparison via the following lemma.

Lemma 6.3. *Let $\gamma : [0, b] \to M$ be a normal geodesic with no cut point of $\gamma(0)$. Let $\varrho(x) = d(x, \gamma(0))$. Then*

$$\frac{\phi'}{\phi}(t) = \left(\Delta\varrho - \frac{n-1}{\varrho}\right)(\gamma(t)) \tag{6.7.1}$$

$t \in (0, b]$

Proof. We only need prove (6.7.1) at $\gamma(b)$. Let $J_1, \ldots, J_{n-1}$ be Jacobi fields along γ with $J_i(0) = 0$, $i = 1, \ldots, n-1$ s.t. $\langle J_i(b), J_j(b)\rangle = \delta_{ij}$, $1 \leq i, j \leq n-1$. Recall

$\mathrm{Hess}\rho(\dot\gamma,\dot\gamma)=0$. Hence we have

$$\Delta\varrho(\gamma(b)) = \sum_{i=1}^{n-1}\mathrm{Hess}\varrho(J_i(b),J_i(b)) \tag{6.7.2}$$

$$= \sum_{i=1}^{n-1} I(J_i(b),J_i(b)) = \sum_{i=1}^{n-1}\langle\nabla_T\dot J_i(b),J_i(b)\rangle. \tag{6.7.3}$$

On the other hand,

$$\frac{\phi'}{\phi}(b) = \frac{\frac{\mathrm{d}}{\mathrm{d}t}\phi^2}{2\phi^2}(b)$$

where

$$\phi(b) = \frac{|J_1(b)\wedge\cdots\wedge J_{n-1}(b)|}{b^{n-1}\left|\dot J_1(0)\wedge\cdots\wedge\dot J_{n-1}(0)\right|} = \frac{1}{b^{n-1}\left|\dot J_1(0)\wedge\cdots\wedge\dot J_{n-1}(0)\right|}.$$

We have

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi^2(b) = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=b}\frac{\langle J_1(t)\wedge\cdots\wedge J_{n-1}(t), J_1(t)\wedge\cdots\wedge J_{n-1}(t)\rangle}{t^{2n-2}\left|\dot J_1(0)\wedge\cdots\wedge\dot J_{n-1}(0)\right|^2}$$

$$= \frac{2\sum_{i=1}^{n-1}\langle J_1(t)\wedge\cdots\wedge\dot J_i(t)\wedge\cdots\wedge J_{n-1}(t), J_1(t)\wedge\cdots\wedge J_{n-1}(t)\rangle}{t^{2n-2}\left|\dot J_1(0)\wedge\cdots\wedge\dot J_{n-1}(0)\right|^2} - 2(n-1)\frac{|J_1(t)\wedge\cdots\wedge J_{n-1}(t)|^2}{t^{2n-1}\left|\dot J_1(0)\wedge\cdots\wedge\dot J_{n-1}(0)\right|^2}|_{t=}$$

$$= \frac{2}{\left|\dot J_1(0)\wedge\cdots\wedge\dot J_{n-1}(0)\right|^2}\left(\frac{\sum_{i=1}^{n-1}\langle\dot J_i(b),J_i(b)\rangle}{b^{2n-2}} - \frac{n-1}{b^{2n-1}}\right).$$

Hecne, we calculate

$$\frac{\phi'(b)}{\phi(b)} = \frac{\frac{\mathrm{d}}{\mathrm{d}t}\phi^2}{2\phi^2}(b) = \sum_{i=1}^{n-1}\langle\dot J_i(b),J_i(b)\rangle - \frac{n-1}{b}. \tag{6.7.4}$$

Combining (6.7.2), (6.7.3) and (6.7.4), we obtain

$$\frac{\phi'}{\phi}(b) = \Delta\varrho(\gamma(b)) - \frac{n-1}{\varrho(\gamma(b))} = \left(\Delta\varrho - \frac{n-1}{\varrho}\right)(\gamma(b)).$$

□

Proof. of Theorem 6.13. Let $\gamma : [0,b] \to M^k$ be a normal geodesic in the simply-connected space form M^k of sectional curvature k.

Note that $\overline{\gamma}$ has no cut point iff $b < \frac{\pi}{\sqrt{k}}$ ($\frac{\pi}{\sqrt{k}} = \infty$ if $k < 0$). By our assumption $\gamma : [0, b] \to M$ is a normal geodesic in M without cut point, hence $b < \frac{\pi}{\sqrt{k}}$, by Bonnet-Myers Theorem. Therefore, the above assertion implies $\overline{\gamma} : [0, b] \to M^k$ has no cut point too. So the Laplacian comparison Theorem is applicable. Hence Lemma 6.3 tells

$$\frac{\phi'}{\phi}(t) \geq \frac{\phi_k'}{\phi_k}(t)$$

Hence $(\ln\phi)'(t) \geq (\ln\phi_k)'(t)$, which implies $(\ln\phi - \ln\phi_k)'(t) \geq 0$. This means $\left(\frac{\phi}{\phi_k}\right)'(t) \geq 0$. This completes the proof.

□

Theorem 6.14 (Bishop-Gromov)**.** *If (M, g) is a complete Riemannian manifold with $Ric \geq (n-1)k$, $k \in \mathbb{R}$. Let $p \in M$ be an arbitrary point. Then the function*

$$r \mapsto \frac{vol(B_p(r))}{vol(B^k(r))}$$

is nondecreasing, where $B^k(r)$ is a geodesic ball of radius r in the simply-connected space form M^k.

Corollary 6.9 (Bishop)**.** *If (M, g) is a complete Riemannian manifold with $Ric \geq (n-1)k > 0$. Then $Vol(M) \leq Vol\left(S^n(\frac{1}{\sqrt{k}})\right)$. The equality holds iff M is isometric to $S^k(\frac{1}{\sqrt{k}})$.*

Proof. of Theorem 6.14.

What is $Vol(B_p(r))$? First note that when $B_p(r) \subset E_p$, we have

$$\begin{aligned} Vol(B_p(r)) &= \int_{B_p(r)} \mathrm{d}vol_M \\ &= \int_{B(0,r)} \phi \mathrm{d}vol_{T_pM} \\ &= \int_0^r \int_{\mathbb{S}^{n-1}} \phi(t,\theta) t^{n-1} \mathrm{d}t \mathrm{d}\theta. \end{aligned}$$

The assumption $B_p(r) \subset E_p$ means $r\theta \in E(p)$ for any $\theta \in \mathbb{S}^{n-1}$.

How to go beyond cut point?

Let χ be the characteristic function of $E(p) \subset T_M$, i.e.

$$\chi(r,\theta) = \begin{cases} 1, when (r,\theta) \in E(p) \\ 0, otherwise \end{cases}$$

Then for any $B_p(r) \subset M$,

$$\begin{aligned} vol(B_p(r)) &= \int_{B(0,r)\cap E_p} \phi \mathrm{d}vol_{T_pM} \\ &= \int_{B(0,r)} \chi\phi \mathrm{d}vol_{T_pM} \\ &= \int_0^r \int_{\mathbb{S}^{n-1}} \chi(t,\theta)\phi(t,\theta) t^{n-1} \mathrm{d}t \mathrm{d}\theta. \end{aligned}$$

Remark: Recall that for $(t,0)\in E(p)$, $\phi(t,\theta)t^{n-1} = \frac{|J_1(t)\wedge\cdots\wedge J_{n-1}(t)|}{|\dot{J}_1(0)\wedge\cdots\wedge\dot{J}_{n-1}(0)|}$. On the simply-connected space form M^k, we have $\phi_k(t,\theta)t^{n-1} = \phi_k(t)t^{n-1} = (f_k(t))^{n-1}$, where

$$f_k(t) = \begin{cases} f, k=0 \\ \dfrac{1}{\sqrt{k}}\sin\sqrt{k}t, k>0 \\ \dfrac{1}{\sqrt{-k}}\sinh\sqrt{-k}t, k<0 \end{cases}$$

We also define the characteristi funtion χ_k on M^k. Recall, when $k\le 0, \chi_k \equiv 1$, when $k>0, \chi_k = 0$ only at the one point. Since $\forall p\in M^k$, $C(p) = -p$ and $d(p, C(p)) = \frac{\pi}{\sqrt{k}}$.
Therefore we have $\chi(r,\theta)\le\chi_k(r,\theta) = \chi_k(r)$. That is, the function

$$r\mapsto\frac{\chi(r,\theta)}{\chi_k(r)}$$

is non-increasing (where we use $\frac{0}{0}=0$).
Recall Theorem 6.13 tells $r\mapsto\frac{\phi(r,\theta)}{\phi_k(r)}$ is non-increasing ($r<\tau(0)$). Hence

$$r\mapsto\frac{\chi(r,\theta)\phi(r,\theta)}{\chi_k(r)\phi_k(r)} \tag{6.7.5}$$

is non-creasing.
Consider the function

$$a(t) = \int_{\mathbb{S}^{n-1}}\chi\phi(t,\theta)t^{n-1}\mathrm{d}\theta,$$
$$a_k(t) = \int_{\mathbb{S}^{n-1}}\chi_k\phi_k(t,\theta)t^{n-1}\mathrm{d}\theta.$$

First observe $a_k(t) = \chi_k\phi_k(t)t^{n-1}vol(\mathbb{S}^{n-1})$. Hence we have $\frac{a(t)}{a_k(t)} = \frac{1}{vol(\mathbb{S}^{n-1})}\int_{\mathbb{S}^{n-1}}\frac{\chi\phi(t,\theta)}{\chi_k\phi_k(t)}\mathrm{d}\theta$.
(6.7.5) implies immediately that $t\mapsto\frac{a(t)}{a_k(t)}$ is non-increasing. This tells $r\mapsto\frac{Vol(B_p(r))}{Vol(B^k(r))} = \frac{\int_0^r a(t)\mathrm{d}t}{\int_0^r a_k(t)\mathrm{d}t}$ is non-increasing due to the following Lemma 6.4.

□

Lemma 6.4. *Let $f,g:[0,\infty)\to(0,\infty)$ be two positive function and the funcion $t\mapsto\frac{f(t)}{g(t)}$ is non-increasing. Then the funcion $t\mapsto\frac{\int_0^t f}{\int_0^t g}$ is also non-increasing.*

Proof. Let us denote $h=\frac{f}{g}$. For $t_1\le t_2$, we hope to show

$$\frac{\int_0^{t_1} f}{\int_0^{t_1} g}\ge\frac{\int_0^{t_2} f}{\int_0^{t_2} g}$$
$$i.e. \int_0^{t_1} f\int_0^{t_2} g\ge\int_0^{t_2} f\int_0^{t_1} g.$$

We observe

$$\int_0^{t_1} f \int_0^{t_2} g = \int_0^{t_1} f \int_0^{t_1} g + \int_0^{t_1} f \int_{t_1}^{t_2} g,$$
$$\int_0^{t_2} f \int_0^{t_1} g = \int_{t_1}^{t_2} f \int_0^{t_1} g + \int_0^{t_1} f \int_0^{t_1} g.$$

Hence it remains to show $\int_0^{t_1} f \int_{t_1}^{t_2} g \geq \int_{t_1}^{t_2} f \int_0^{t_1} g$.
This follows from the calculation:

$$\int_0^{t_1} f \int_{t_1}^{t_2} g = \int_0^{t_1} gh \int_{t_1}^{t_2} g \geq \left(\int_0^{t_1} g\right) h(t_1) \int_{t_1}^{t_2} g$$
$$\geq \left(\int_0^{t_1} g\right)\left(\int_{t_1}^{t_2} hg\right) = \int_{t_1}^{t_2} f \int_0^{t_1} g.$$

□

Remark: Recall we actually have $\phi(r,\theta) = \sqrt{det(g_{ij}) \circ x^{-1}}(r,\theta)$, $r < \tau(\theta)$, and $\phi_k(r) = \left(\frac{f_k(r)}{r}\right)^{n-1}$. $\lim_{r\to 0} \phi(r,\theta) = 1 \Rightarrow \lim_{r\to 0} \frac{\phi(r,\theta)}{\phi_k(r)} = 1$. Hence Theorem 6.13 $\Rightarrow$ $\phi(r,\theta) \leq \phi_k(r)$, when $r < \tau(\theta) \Rightarrow \chi\phi(r,\theta) \leq \chi_k\phi_k(r) \Rightarrow$

$$Vol\left(B_p(\frac{\pi}{\sqrt{k}})\right) = Vol(M) \leq Vol(M_k). \tag{6.7.6}$$

In fact, noe have $\lim_{r\to 0} \frac{Vol(B_p(r))}{Vol(B^k(r))} = 1$. So (6.7.6) can be derived from Theorem 6.14.

Proof. of Corollary 6.9.
Recall the sphere of radius $\frac{1}{\sqrt{k}}$ has constant sectional curvature k. Hence Theorem 6.13 $\Rightarrow Vol(M) \leq Vol\left(\mathbb{S}^n(\frac{1}{\sqrt{k}})\right)$. If "=" holds, then all inequalities in the proof of Theorem 6.13 should be "=". Particularly, $\Delta\varrho(\gamma(t)) = \Delta\varrho_k(\overline{\gamma}(t))$ for any t s.t. $t\dot{\gamma} \in E(\gamma(0)) \subset T_{\gamma(0)}M\ \forall\gamma$.

Recall from the Laplacian comparison theorem. this means any secion in $T_{\gamma(t)}M$ containing $\dot{\gamma}(t)$ has sectional curvature k. Since γ is arbitrary, we have M has constant sectional curvature k. So its universal covering space is isometric to $\mathbb{S}\left(\frac{1}{\sqrt{k}}\right)$ (by the uniqueness of simply-connected space forms). But since $Vol(M) = Vol\left(\mathbb{S}(\frac{1}{\sqrt{k}})\right)$, we have M is isometric to $\mathbb{S}\left(\frac{1}{\sqrt{k}}\right)$.

□

Next we explore two applications of Volume Comparison Theorem.

6.7.1 Maximal Diameter Theorem

Theorem 6.15 (Maximal Diameter Theorem, Shiu-Yuen Cheng 1975)**.** *Let M be a complete Riemannian manifold with $Ric \geq (n-1)k > 0$ and $diam_M = \frac{\pi}{\sqrt{k}}$. Then M is isometric to $\mathbb{S}\left(\frac{1}{\sqrt{k}}\right)$.*

Remark

1. This is a good complement of Bonnet-Myers Diameter Estimate: the "=" holds in Bonnet-Myers iff M is isometric to $\mathbb{S}\left(\frac{1}{\sqrt{k}}\right)$.
2. When assuming $sec \geq k > 0$, this result has been proved by Toponogov in 1959. Cheng's original proof uses his comparison theorems for first eigenvalues. Shioham (Trans. AMS. 1983) gives a much more elementary proof using the Volume Comparison .

Proof. of Theorem 6.15.
By scaling, we only need deal with the case $k = 1$. Let $p, q \in M$ be two points such that $d(p,q) = \text{diam}_M = \pi$. Then $B_p(r) \cap B_q(\pi - r) = \phi$, $\forall r \in [0.\pi]$. Hence $Vol(B_p(r)) + Vol(B_q(\pi - r)) \leq Vol(M)$, $\forall r \in [0.\pi]$.

Using Theorem 6.14, we have

$$\begin{aligned}
Vol(M) &\geq Vol(B_p(r)) + Vol(B_q(\pi - r)) \\
&= \frac{Vol(B_p(r))}{Vol(B'(r))} Vol(B'(r)) + \frac{Vol(B_q(\pi - r))}{Vol(B'(\pi - r))} Vol(B'(\pi - r)) \\
&\geq \frac{Vol(B_p(\pi))}{Vol(B'(\pi))} Vol(B'(r)) + \frac{Vol(B_q(\pi))}{Vol(B'(\pi))} Vol(B'(\pi - r)) \\
&= \frac{Vol(M)}{Vol(B'(\pi))} \left(Vol(B'(r)) + Vol(B'(\pi - r))\right) \\
&= Vol(M).
\end{aligned}$$

Hence all "≤" are "=". Particulary,

$$\frac{Vol(B_p(\pi))}{Vol(B'(\pi))} = \frac{Vol(B_p(r))}{Vol(B'(r))}$$

$\forall r \in (0, \pi]$.

Let $r \to 0$, we have $1 = \frac{Vol(B_p(\pi))}{Vol(B'(\pi))} = \frac{Vol(M)}{Vol(\mathbb{S}^n(1))}$. Then corollary 6.9 implies that M is isometric to $\mathbb{S}^n(1)$.

□

6.7.2 Volume Growth Rate Estimate

Theorem 6.16. *Let (M^n, g) be a complete Riemannian manifold with $Ric \geq 0$.*

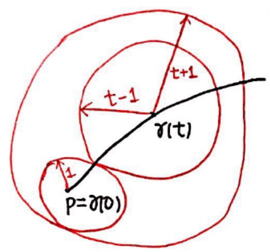
1. *we have $Vol(B_p(r)) \leq Vol(B^0(r)) = \omega_n r^n$ and $=$ holds iff M is isometric to $\mathbb{R}^n$.*

2. *If, furthermore, M^n is non-compact, then there exists a positive constant c depending only on p and n such that $Vol(B_p(r) \geq cr$, for any $r > 2$. (Calabi, Notices AMS 1975, ST Yau, Indiana Univ. Math J 1976. independently.)*

Proof. (1). follows directly from Theorem 6.13, and $\lim_{r\to 0} \frac{Vol(B_p(r))}{Vol(B^0(r))} = 1$. Then "=" case again following from that in Laplacian Comparison Theorem.

(2). (Following Gromov). From the following proof, we see again that $\frac{Vol(B_p(r))}{Vol(B^0(r))}$ decreases tells much more than only $Vol(B_p(r)) \leq Vol(B^0(r))$!!

Since M is non-compact complete, for any $p \in M$, there exists a ray, i.e. a geodesic $\gamma : [0, \infty) \to M$ with $\gamma(0) = p$, and $d(p, \gamma(t)) = t$, $\forall t \geq 0$.



Let $t > \frac{3}{2}$, Theorem 6.13 tells

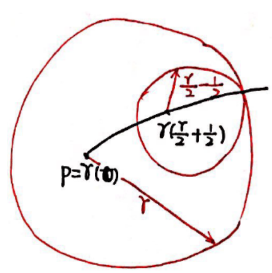
$$\frac{Vol\Big(B_{\gamma(t)}(t+1)\Big)}{Vol\Big(B_{\gamma(t)}(t-1)\Big)} \leq \frac{\omega_n(t+1)^n}{\omega_n(t-1)^n} = \frac{(t+1)^n}{(t-1)^n}. \tag{6.7.7}$$

On the other hand,

$$\frac{Vol\Big(B_p(1)\Big)}{Vol\Big(B_{\gamma(t)}(t-1)\Big)} \leq \frac{Vol\Big(B_{\gamma(t)}(t+1)\Big) - Vol\Big(B_{\gamma(t)}(t-1)\Big)}{Vol\Big(B_{\gamma(t)}(t-1)\Big)} \leq \frac{(t+1)^n - (t-1)^n}{(t-1)^n}. \tag{6.7.8}$$

i.e. $Vol\Big(B_{\gamma(t)}(t-1)\Big) \geq \frac{1}{t}\frac{(t-1)^n}{(t+1)^n-(t-1)^n} Vol(B_p(t))t$.

Observe that $\exists C_n > 0$ s.t. $\frac{(t-1)^n}{(t+1)^n-(t-1)^n} \geq C_n$ on $[\frac{3}{2}, \infty)$. Since $B_{\gamma(\frac{r}{2}+\frac{1}{2})}\Big(\frac{r}{2}-\frac{1}{2}\Big) \subset B_p(r)$,



$\forall r > 2. \Rightarrow$

$$Vol(B_p(r)) \geq Vol\left(B_{\gamma(\frac{r}{2}+\frac{1}{2})}\left(\frac{r}{2}-\frac{1}{2}\right)\right)$$
$$\geq C_n Vol(B_p(1))(\frac{r}{2}+\frac{1}{2}).$$

□

Final Remark: Recall we used Lapkacian Comparison to prove the volume comparison: $Ric \geq (n-1)k \Rightarrow Vol(B_p(r)) \leq Vol(B^k(r))$. (Inside the cut locus, this is due to Bishop. Gromov made the crucial step to show it for any r, and to a "full" use of the fact that $\frac{Vol(B_p(r))}{Vol(B^k(r))}$ decrease.) It is natural to ask the "other direction". Recall we do not have the "other direction" in Laplacian Comparison, but we do have it in Hessian Laplacian.

Excercise(Gunther, 1960) Let (M, g) be a complete Riemannian manifold, with sectional curvature $\leq k$. Let $B_p(r)$ be a ball in M which does not meet the cut locus of p. Then $Vol(B_p(r)) \geq Vol(B^k(r))$.

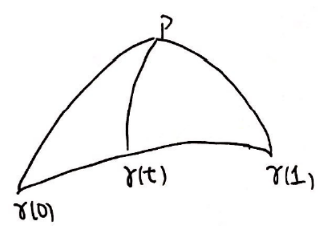
Chapter 7

Candidates for Synthetic Curvature Conditions

In the last part of our course , we discuss properties of a Riemannian manifold which does not necessarily depend on the smooth structure of the underlying spaces. Those properties may be taken to be definition of a general (metric, measure) space with aurvature restrictions.

7.1 Nonpositive Sectional Curvature and Convexity

Theorem 7.1. *Let (M, g) be a complete, simply-connected Riemannian manifold with nonpositive curvature. Let $p \in M$, $\gamma : [0, 1] \to M$ be a geodesic. Then*



$$d^2(p, \gamma(t)) \leq (1 - t)d^2(p, \gamma(0)) + td^2(p, \gamma(1)) - t(1 - t)d^2(\gamma(0), \gamma(1)). \tag{7.1.1}$$

Remark: Actually, on a complete Riemannian manifold with sec ≤ 0, 7.1.1 holds whenever $\gamma(t) \subset E_p$. For simplicity, we suppose M is simply-connected, then $E_p = \phi$ and $\exp_p : T_pM \to M$ is a diffeomorphism.

Proof. Let $k_0 : [0, 1] \to$ R be given by

$$k_0(t) = (1 - t)d^2(p, \gamma(0)) + td^2(p, \gamma(1)) - t(1 - t)d^2(\gamma(0), \gamma(1)).$$

We have

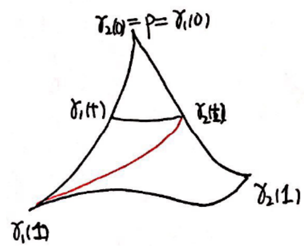
$$\begin{aligned} k_0(0) &= d^2(p, \gamma(0)), \\ k_0(1) &= d^2(p, \gamma(1)), \\ k_0''(t) &= -2d^2(\gamma(0), \gamma(1)) = -2|\dot\gamma(t)|^2. \end{aligned}$$

Let $\varrho(x) = d^2(x, p)$, then $\varrho \circ \gamma(t)$ satisfies

$$\begin{aligned} \varrho \circ \gamma(0) &= d^2(p, \gamma(0)) = k_0(0), \\ \varrho \circ \gamma(1) &= d^2(p, \gamma(1)) = k_0(1), \\ \varrho \circ \gamma''(t) &= \mathrm{Hess}\varrho\,(\dot\gamma(t), \dot\gamma(t)) \geq 2|\dot\gamma(t)|^2 = k_0''(t). \end{aligned}$$

Therefore the function $h : [0, 1] \to \mathbb{R}$ given by $h(t) = (\varrho \circ \gamma - k_0)(t)$ satisfies that $h(0) = h(1) = 0$, $h''(t) \geq 0$, $\forall t \in [0, 1]$. Therefore $h(t) \leq 0$. (Convex functions attains maximum on the boundary). That is $\varrho \circ \gamma(t) \leq k_0(t)$, $\forall t \in [0, 1]$. □

Corollary 7.1. *Let (M, g) be a compact simply-connected Riemannian manifold with $sec \leq 0$. Let $\gamma_1, \gamma_2 : [0, 1] \to M$ be geodesics with $\gamma_1(0) = p = \gamma_2(0)$.*



Then for $0 \leq t \leq 1$,

$$d(\gamma_1(t), \gamma_2(t)) \leq t d(\gamma_1(1), \gamma_2(1)).$$

Proof. Appling Theorem 7.1 twice:

$$d^2(\gamma_1(1), \gamma_2(t)) \leq td^2(\gamma_1(1), \gamma_2(1)) + (1-t)d^2(\gamma_1(1), p) - t(1-t)d^2(p, \gamma_2(1)), \quad (7.1.2)$$
$$d^2(\gamma_2(1), \gamma_1(t)) \leq td^2(\gamma_1(1), \gamma_2(t)) + (1-t)d^2(\gamma_2(t), p) - t(1-t)d^2(p, \gamma_1(1)). \quad (7.1.3)$$

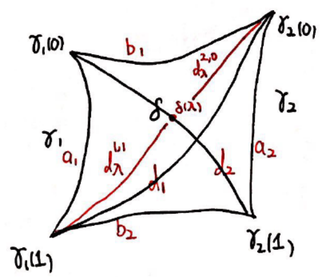
Inserting 7.1.2 into 7.1.3, and observing $d^2(\gamma_2(t), p) = t^2 d^2(\gamma_1(t), p)$, we complete the proof. □

<u>Remark</u>: The property (7.1.1) for all p and all geodesic γ is actually equivalent to the nonpositive sectional curvature of M. Namely, if the sectional curvature $\geq k > 0$ in a neighborhood of p, then locally $(\varrho \circ \gamma)''(t) = \mathrm{Hess}\varrho(\dot\gamma(t), \dot\gamma(t)) \leq \mathrm{Hess}\overline{\varrho}(\dot{\overline{\gamma}}(t), \dot{\overline{\gamma}}(t))$.

Then one can show ">" in (7.1.1). In fact, this is taken to be the definition of a length space with nonpositive sectional curvature in the sense of Alexandrow.

Corollary 7.1 is also equivalent to nonpositive sectional curvature, and is taken as a general curvature bound notion by Busemann. see[Chap2, Jost, Nonpositive Curvature: Geometric and Analytic Aspects, Birkhāuser].

Theorem 7.2 (Reshetnyak's quadrilateral comparison theorem)**.** *Let (M, g) be a compact simply-connected Riemannian manifold with $\sec \le 0$. Let $\gamma_1, \gamma_2 : [0, 1] \to M$.*



Then

$$\begin{aligned}
&d^2(\gamma_1(0), \gamma_2(1)) + d^2(\gamma_2(0), \gamma_1(1))\\
\le &d^2(\gamma_1(0), \gamma_2(0)) + d^2(\gamma_1(1), \gamma_2(1)) + d^2(\gamma_1(0), \gamma_1(1))\\
+ &d^2(\gamma_2(0), \gamma_2(1)) - (d(\gamma_1(0), \gamma_2(0)) - d(\gamma_1(1), \gamma_2(1)))^2 .
\end{aligned}$$

Proof. For simplicity, denote the distance as $a_1, a_2, b_1, b_2, d_1, d_2$ as in the above figure.

Let δ be the geodesic from $\gamma_1(0)$ to $\gamma_2(1)$ whose length is d_2. Consider

$$\begin{aligned}
d_\lambda^{2,0} &= d(\gamma_2(0), \delta(\lambda)),\\
d_\lambda^{1,1} &= d(\gamma_1(1), \delta(\lambda)).
\end{aligned}$$

Then theorem 7.1 $\Rightarrow$

$$\begin{aligned}
(d_\lambda^{2,0})^2 &\le (1-\lambda)b_1^2 + \lambda a_2^2 - \lambda(1-\lambda)d_2^2,\\
(d_\lambda^{1,1})^2 &\le (1-\lambda)a_1^2 + \lambda b_2^2 - \lambda(1-\lambda)d_2^2.
\end{aligned}$$

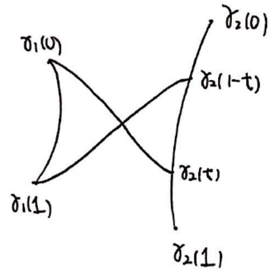
Therefore, for $\epsilon > 0$,

$$\begin{aligned}
d_1^2 &\le \left(d_\lambda^{0,2} + d_\lambda^{1,1}\right)^2 \le (1+\epsilon)\left(d_\lambda^{2,0}\right)^2 + (1 + \frac{1}{\epsilon})\left(d_\lambda^{1,1}\right)^2\\
&\le (1+\epsilon)(1-\lambda)b_1^2 + (1+\epsilon)\lambda a_2^2 + (1+\frac{1}{\epsilon})(1-\lambda)a_1^2\\
&+ (1+\frac{1}{\epsilon})\lambda b_2^2 - (2 + \epsilon + \frac{1}{\epsilon})\lambda(1-\lambda)d_2^2.
\end{aligned}$$

Set $\epsilon = \frac{1-\lambda}{\lambda}$ so that the coefficent of d_2^2 becomes $\left(2 + \frac{1-\lambda}{\lambda} + \frac{\lambda}{1-\lambda}\right)\lambda(1-\lambda) = 2\lambda(1-\lambda) + (1-\lambda)^2 + \lambda^2 = 1$. This yeilds $d_1^2 + d_2^2 \le \frac{1-\lambda}{\lambda}b_1^2 + a_2^2 + a_1^2 + \frac{\lambda}{1-\lambda}b_2^2$. Set $\lambda = \frac{b_1}{b_1+b_2} \Rightarrow d_1^2 + d_2^2 \le a_1^2 + a_2^2 + 2b_1b_2 = a_1^2 + a_2^2 + b_1^2 + b_2^2 - (b_1 - b_2)^2$. □

Corollary 7.2. *Let (M, g) be as in Theorem 7.2, and $\gamma_1, \gamma_2 : [0, 1] \to M$ be geodesics.*

Then $\forall 0 \le t \le 1, 0 \le s \le 1$



$$d^2(\gamma_1(0), \gamma_2(t)) + d^2(\gamma_1(1), \gamma_2(1-t)) \le d^2(\gamma_1(0), \gamma_2(0))$$
$$+ d^2(\gamma_1(1), \gamma_2(1)) + 2t^2 d^2(\gamma_2(0), \gamma_2(1)) + t\left(d^2(\gamma_1(0), \gamma_1(1)) - d^2(\gamma_2(0), \gamma_2(1))\right)$$
$$- ts\left(d(\gamma_1(0), \gamma_1(1)) - d(\gamma_2(0), \gamma_2(1))\right)^2$$
$$- t(1-s)\left(d(\gamma_1(0), \gamma_2(0)) - d(\gamma_1(1), \gamma_2(1))\right)^2.$$

Proof. Notice that Theorem 7.2 is the case $t = 1, s = 0$. By symmetry, we also have the above inequation holds for $t = 1, s = 1$. i.e.

$$d_1^2 + d_2^2 \le a_1^2 + a_1^2 + b_1^2 + b_2^2 - (a_1 - a_2)^2.$$

Taking convex combinations yields the inequlity for $t = 1, 0 \le s \le 1$,

$$d_1^2 + d_2^2 \le a_1^2 + a_1^2 + b_1^2 + b_2^2 - s(a_1 - a_2)^2 - (1-s)(b_1 - b_2)^2.$$

Therefore, for $0 \le t \le 1$, Theorem 7.1 implies

$$d^2(\gamma_1(0), \gamma_2(t)) + d^2(\gamma(1), \gamma_2(1-t)) \le (1-t)b_1^2 + td_2^2 - t(1-t)a_2^2 + td_1^2 + (1-t)b_2^2 - t(1-t)a_2^2$$
$$\le t\left(a_1^2 + a_2^2 + b_1^2 + b_2^2 - s(a_1 - a_2)^2 - (1-s)(b_1 - b_2)^2\right) + (1-t)\left(b_1^2 + b_2^2\right) - 2t(1-t)a_2^2$$
$$= b_1^2 + b_2^2 + 2t^2 a_2^2 + t\left(a_1^2 - a_2^2\right) - ts(a_1 - a_2)^2 - t(1-s)(b_1 - b_2)^2.$$

□

Exercise: Let (M, g) be as above, and $\gamma_1, \gamma_2 : [0, 1] \to M$ be geodesics. Then we have $\forall 0 \le t \le 1, 0 \le s \le 1$.

$$d^2(\gamma_1(t), \gamma_2(t)) \le (1-t)d^2(\gamma_1(0), \gamma_2(0)) + td^2(\gamma_1(1), \gamma_2(1))$$
$$- t(1-t)\left\{s[d(\gamma_1(0), \gamma_1(1)) - d(\gamma_2(0), \gamma_2(1))]^2 + (1-s)[d(\gamma_1(0), \gamma_2(0)) - d(\gamma_1(1), \gamma_2(1))]^2\right\}.$$

Hint: Using the above corollary.

Remark: The above exercise tells particular that $d^2 : M \times M \to \mathbb{R}$ where M is a compact simply-connected Riemannian manifold with sec≤ 0, is a convex function. This is because a geodesic γ on $M \times M$ is given as (γ_1, γ_2) where γ_1, γ_2 are geodesics in M. Exercise tells that $d^2 \circ \gamma = d^2(\gamma_1(t), \gamma_2(t))$ is a convex function. ([JJ,Corollary 4.8.2]).

7.2 Bochner technique and Bakry-Émery Γ-calculus

7.2.1 A computation trick

When we verify tensor equalities or tensor inequalities, we can pick a proper local coordinate or a proper local frame at a point, and check the equality or inequality at two points.

1. normal coordinate system around $p \in M$ $x : U \to \mathbb{R}^n$ such that

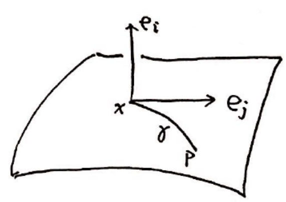
$$\begin{cases} x^i(p) = 0 \\ g_{ij}(p) = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})(p) = \delta_{ij} \\ \Gamma^i_{jk}(p) = 0 \end{cases}$$

2. (local) coordinate frame: $\{\frac{\partial}{\partial x^i}\}$

3. (local) orthonormal frame: $\{e_1\}$ with $\langle e_i, e_j\rangle = \delta_{ij}$.

4. local normal frame at x: $\{E_i\}$ with $\nabla_{E_i} E_j(x) = 0$, $\forall i, j$.

Exercise: Pick an orthonormal basis $\{e_1, \ldots, e_n\}$ for the vector space $T_x M$.



Choose any $p \in M$ near to x, let γ be the geodesic from x to p. Let $E_i(p)$ be the vector in $T_p M$ which is obtained by transport e_i parallelly along γ. The local frame obtained in this way is what we want.

Let us discuss an exomple.

Lemma 7.1. *Choosing any local frame $\{V_i\}_{i=1}^n$ on M and its dual coframe $\{\omega^i\}_{i=1}^n$, then we have*

$$\mathrm{d} = \sum_i \omega^i \wedge \nabla_{V_i}. \tag{7.2.1}$$

Recall $\mathrm{d} : A^p(M) \to A^{p+1}(M)$ is the exterior derivative where $A^p(M)$ stands for the vector space of smooth p-forms on M.

Proof. Let us denote by $\overline{\mathrm{d}} = \sum_i \omega^i \wedge \nabla_{V_i}$. Notice this does not depend on the choice of different frames. Indeed, for the frame $X_k = c^i_k V_i$ and its dual $\eta^k = d^k_i \omega^i$, where

$\sum_k c^i_k d^k_j = \delta^i_j$. Hence

$$\sum_k \eta^k \wedge \nabla_{X_k} = \sum_{k,j,i} d^k_j \omega^j \wedge \nabla_{c^i_k V_i}$$
$$= \sum_{k,j,i} d^k_j c^i_k \omega^j \wedge \nabla_{V_i} = \sum_{j,i} \delta^i_j \omega^j \wedge \nabla_{V_i} = \sum_i \omega^i \wedge \nabla_{V_i}.$$

As both sides of (7.2.1) are independent of the choice of frames, we can prove it pointwise, and choose the normal coordinate $\{x^i\}$ around a fixed point $p \in M$, and consider the local coordinate frame $\{\frac{\partial}{\partial x^i}\}$ and its dual $\{\mathrm{d}x^i\}$.

First, we observe

$$\left(\nabla_{\frac{\partial}{\partial x^i}} \mathrm{d}x^j\right)\left(\frac{\partial}{\partial x^k}\right) = \nabla_{\frac{\partial}{\partial x^i}} \left(\mathrm{d}x^j(\frac{\partial}{\partial x^k})\right) - \mathrm{d}x^j \left(\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^k}\right)$$
$$= - dx^j \left(\Gamma^l_{ik} \frac{\partial}{\partial x^l}\right) = -\Gamma^l_{ik}\delta^j_k = -\Gamma^j_{ik}.$$

This means that $\nabla_{\frac{\partial}{\partial x^i}} \mathrm{d}x^j = -\Gamma^j_{ik}\mathrm{d}x^k$. Hence at p, we have

$$\left(\nabla_{\frac{\partial}{\partial x^i}} \mathrm{d}x^j\right)(p) = 0. \tag{7.2.2}$$

By the linear property of d and RHS of (7.2.1), we only need to verify (7.2.1) when applying to any q-form $\eta = f\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^q$. Then

$$\left(\sum_i \mathrm{d}x^i \wedge \nabla_{\frac{\partial}{\partial x^i}}\right)\eta = \sum_i \mathrm{d}x^i \wedge \nabla_{\frac{\partial}{\partial x^i}} \left(f\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^q\right)$$
$$= \sum_i \mathrm{d}x^i \wedge \frac{\partial f}{\partial x^i}\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^q = \sum_i \frac{\partial f}{\partial x^i}\mathrm{d}x^i \wedge \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^q$$
$$= \mathrm{d}\eta.$$

□

7.2.2 The Hodge Laplacian

On a Riemannian manifold (M, g), g induces an inner product on T_xM for each $x \in M$: Let us denote $g(X, Y) = \langle X, Y\rangle$, $\forall X, Y \in T_xM$. Choose a local orthonomal frames $\{E_i\}_{i=1}^n$ on M, i.e. $\langle E_i, E_j\rangle = \delta_{ij}$. Let $\{\omega^i\}_{i=1}^n$ be the dual coframe of $\{E_i\}_{i=1}^n$, i.e. $\omega^i(X_j) = \delta^i_j$. We stipulate that these 1-forms are orthonormal pairwise, i.e. we define $\langle \omega^i, \omega^j\rangle = \delta_{ij}$.

Extend it by linearly: $\forall \phi, \psi \in A^1(M)$, suppose $\phi = \sum_i \phi_i \omega^i$, $\psi = \sum_j \psi_j \omega^j$. Then $\langle \phi, \psi\rangle = \sum_i \phi_i \psi$. We can check that the above definition is independent of the choice of frames. In this way, each cotangent space T^*_xM becomes an inner product space.

We can continue to assign a natural inner proudct on the space $\wedge^p T^*_xM$: we stipulate that the p-forms are $\{\omega^{i_1} \wedge \cdots \wedge \omega^{i_p} : i_1 < \cdots < i_p\}$ are orthonormal. For $\forall \phi, \psi \in A^p(M)$, suppose

$$\phi = \sum_{i_1<\cdots<i_p} \phi_{i_1\ldots i_p}\omega^{i_1} \wedge \cdots \wedge \omega^{i_p},$$
$$\psi = \sum_{i_1<\cdots<i_p} \psi_{i_1\ldots i_p}\omega^{i_1} \wedge \cdots \wedge \omega^{i_p}.$$

Then define

$$\langle\phi,\psi\rangle(x) = \sum_{i_1<\cdots<i_p} \phi_{i_1\ldots i_p}\psi_{i_1\ldots i_p}(x).$$

We can check that this definition is independent of the choice of frame.

Now let M be an n-dimensional orientable manifold and we suppose $\{\omega^i\}_{i=1}^n$ is a locally orthonormal coframe such that

$$\omega^1 \wedge \cdots \wedge \omega^n$$

is the volume form Ω on M. Recall Ω determines an orientaion on M. Then we define the Hodge star operator $\star : A^p(M) \to A^{n-p}(M)$ by pairing $\phi \wedge \star\psi = \langle\phi,\psi\rangle\Omega$, $\forall\phi,\psi \in A^p(M)$.

For that purpose, we obeserve that we have to define

$$\star\left(\omega^1 \wedge \cdots \wedge \omega^p\right) = \omega^{p+1} \wedge \cdots \wedge \omega^n. \tag{7.2.3}$$

If $1 \le i_1 < \cdots < i_p \le n$, $1 \le i_{p+1} < \cdots < i_n \le n$ and $\{i_{p+1},\ldots,i_n\} = \{1,\ldots,n\} \setminus \{i_1,\ldots,i_p\}$, then $\{\omega^{i_1},\ldots,\omega^{i_p},\omega^{i_{p+1}},\ldots,\omega^{i_{n-1}},\epsilon_{i_1\ldots i_n}\omega^{i_n}\}$ is also an orthonormal coframe determining the same orientaion, where $\epsilon_{i_1,\ldots,i_n}$ is the sign of the permutaion

$$(1,\ldots,n) \to (i_1,\ldots,1_n)$$

$$\begin{pmatrix}\omega^{i_1} \wedge \cdots \wedge \omega^{i_p} \wedge \omega^{i_{p+1}} \wedge \cdots \wedge \omega^{i_{n-1}} \wedge \epsilon_{i_1\ldots i_n}\omega^{i_n} \\ =\omega^1 \wedge \cdots \wedge^n = \Omega\end{pmatrix}$$

Hence, by the definition (7.2.3), we have

$$\star\left(\omega^{i_1} \wedge \cdots \wedge \omega^{i_p}\right) = \epsilon_{i_1,\ldots,i_n}\omega^{i_{p+1}} \wedge \cdots \wedge \omega^{i_n}.$$

Extendind $\star$ as an $A^0(M)$-linear operator, for $f = \sum_{i_1<\cdots<i_p} f_{i_1\ldots i_p}\omega^{i_1} \wedge \cdots \wedge \omega^{i_p}$ we have

$$\begin{aligned}\star f &= \sum_{i_1<\cdots<i_p} f_{i_1\ldots i_p} \star\left(\omega^{i_1} \wedge \cdots \wedge \omega^{i_p}\right) \\ &= \sum_{i_1<\cdots<i_p,i_{p+1}<\cdots<i_n} \epsilon_{i_1\ldots i_n} f_{i_1\ldots i_p}\omega^{i_{p+1}} \wedge \cdots \wedge \omega^{i_n}.\end{aligned}$$

We can check that this definition is independent of the choice of an orthonormal frame. Moreover we have for

$$\phi = \sum_{i_1<\cdots<i_p} \phi_{i_1\ldots i_p}\omega^{i_1}\wedge\cdots\wedge\omega^{i_p},$$
$$\psi = \sum_{i_1<\cdots<i_p} \psi_{i_1\ldots i_p}\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}.$$

that

$$\phi\wedge\star\psi = \left(\sum_{i_1<\cdots<i_p}\phi_{i_1\ldots i_p}\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}\right)\wedge\left(\sum_{j_1<\cdots<j_p,j_{p+1}<\cdots<j_n}\epsilon_{j_1\ldots j_n}\psi_{j_1\ldots j_p}\omega^{j_{p+1}}\wedge\cdots\wedge\omega^{j_n}\right)$$
$$= \sum_{i_1<\cdots<i_p,i_{p+1}<\cdots<i_n}\epsilon_{i_1\ldots i_n}\phi_{i_1\ldots i_p}\psi_{i_1\ldots i_p}\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}\wedge\omega^{j_{p+1}}\wedge\cdots\wedge\omega^{j_n}$$
$$= \left(\sum_{i_1<\cdots<i_p}\phi_{i_1\ldots i_p}\psi_{i_1\ldots i_p}\right)\omega^1\wedge\cdots\wedge\omega^n = \langle\phi,\psi\rangle\Omega.$$

Proposition 7.1. *We have*

$$\star\Omega = 1,\ \star 1 = \Omega,$$
$$\star\star\phi = (-1)^{p(n-p)},\ \forall\phi\in A^p(M),$$
$$\langle\phi,\psi\rangle = \langle\star\phi,\star\psi\rangle,\ \forall\phi,\psi\in A^p(M).$$

Proof. We only show that the last:

$$\begin{aligned}\star\star\phi &= \star\left(\sum_{i_1<\cdots<i_p,i_{p+1}<\cdots<i_n}\epsilon_{i_1\ldots i_n}\phi_{i_1\ldots i_p}\omega^{i_{p+1}}\wedge\cdots\wedge\omega^{i_n}\right)\\ &= \sum_{i_1<\cdots<i_p,i_{p+1}<\cdots<i_n}\epsilon_{i_1\ldots i_n}\epsilon_{i_{p+1}\ldots i_n i_1\ldots i_p}\phi_{i_1\ldots i_p}\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}\\ &= \sum_{i_1<\cdots<i_p}\phi_{i_1\ldots i_p}(-1)^{p(n-p)}\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}\\ &= (-1)^{p(n-p)}\phi.\end{aligned}$$

Then $\langle\star\phi,\star\psi\rangle\Omega = \star\phi\wedge\star(\star\psi) = (-1)^{p(n-p)}\star\phi\wedge\psi = \phi\wedge\star\psi = \langle\psi,\phi\rangle\Omega = \langle\phi,\psi\rangle\Omega$. Hence $\langle\star\phi,\star\psi\rangle = \langle\phi,\psi\rangle$.

□

Definition 7.1 (Hodge Laplacian)**.** *We define the operator*

$$\delta : A^p(M)\to A^{p-1}(M)$$

by $\delta = (-1)^{np+n+1}\star\mathrm{d}\star$. *Then Hodge Laplacian* Δ *is defined as* $\Delta = \delta\mathrm{d}+\mathrm{d}\delta : A^p(M)\to A^p(M)$.

Let us explain why we define δ like that, especially with such a complicated sign. Let M be an n-dimensional, closed orientable Riemannian manifold. We can introduce an inner product in the space of the whole exteror Algebra $A^*(M) = \oplus_{p=0}^n A^p(M)$ as below:

$\forall \phi, \psi \in A^p(M)$, set

$$(\phi, \psi) = \int_M \langle \phi, \psi \rangle \Omega = \int_M \phi \wedge \star\psi$$

$\forall \phi \in A^p(M)$ $\forall \psi \in A^q(M)$ with $p \neq q$ set

$$(\phi, \psi) = 0.$$

Excercise: Check this definition gives an inner product in A^*.

Proposition 7.1 implies $(\star\phi, \star\psi) = (\psi, \phi)$. That is, $\star$ is an isometry tansformation beween A^* and itself. The definition of δ is carefully given to ensure the following property:

Proposition 7.2. *$\forall \alpha \in A^{p-1}(M), \beta \in A^p(M)$, we have* $(\mathrm{d}\alpha, \beta) = (\alpha, \delta\beta)$.

Proof.

$$\begin{aligned}
(\mathrm{d}\alpha, \beta) &= \int_M \mathrm{d}\alpha \wedge \star\beta \\
&= \int_M \mathrm{d}(\alpha \wedge \star\beta) - (-1)^{p-1}\alpha \wedge \mathrm{d} \star \beta \\
&= -(-1)^{p-1} \int_M (-1)^{(n-p+1)(p-1)} \alpha \wedge \star \star (\mathrm{d} \star \beta) \\
&= (-1)^{(n-p+2)(p-1)+1} \int_M \alpha \wedge \star(\star \mathrm{d} \star \beta) \\
&= (-1)^{np+n+1} \int_M \alpha \wedge \star(\star \mathrm{d} \star \beta).
\end{aligned}$$

Hence $(\mathrm{d}\alpha, \beta) = \int_M \alpha \wedge \star(\delta\beta) = (\alpha, \delta\beta)$.

□

Now the definition of δ is justified. Notice that $\delta^2 = 0$ following from $\mathrm{d}^2 = 0$. Hence $\Delta = (\delta + \mathrm{d})(\delta + \mathrm{d})$.

Proposition 7.3. *The Hodge Laplacian Δ is a self-adjoint operator.*

Proof. $\forall \alpha, \beta \in A^*$, we have

$$\begin{aligned}
(\Delta\alpha, \beta) &= ((\delta + \mathrm{d})(\delta + \mathrm{d})\alpha, \beta) \\
&= (\mathrm{d}(\delta + \mathrm{d})\alpha, \beta) + (\delta(\delta + \mathrm{d})\alpha, \beta) \\
&= ((\delta + \mathrm{d})\alpha, \delta\beta) + ((\delta + \mathrm{d})\alpha, \mathrm{d}\beta) \\
&= ((\delta + \mathrm{d})\alpha, (\delta + \mathrm{d})\beta) \\
&= (\alpha, (\delta + \mathrm{d})(\delta + \mathrm{d})\beta) = (\alpha, \Delta\beta)
\end{aligned}$$

.

□

As a direct corollary, Δ is a positive operator on $A^*(M)$. That is, $\forall f \in A^*$, $(\Delta f, f) = ((\delta + \mathrm{d})f, (\delta + \mathrm{d})f) = (\mathrm{d}f, \mathrm{d}f) + (\delta f, \delta f) \geq 0$.

Suppose λ is any eigenvalue of Δ, i.e., $\Delta f = \lambda f$ for some nontrival f, then $\lambda(f, f) = (\Delta f, f) \geq 0$. $\Rightarrow \lambda \geq 0$. Furthermore, $f \in A^*$ is harmonic ($\Delta f = 0$) iff $\mathrm{d}f = \delta f = 0$.

Next, we aim at establishing an expression for δ similar to lemma 7.1 for d. Recall for any vector field X on M, the inerior product

$$i(X) : A^p(M) \to A^{p-1}(M)$$

by $(i(X)\phi)(Y_1, \ldots, Y_{p-1}) = \phi(X, Y_1, \ldots, Y_{p-1})$, $\forall \phi \in A^p(M)$, and any vector fields $Y_1, \ldots, Y_{p-1}$.

Proposition 7.4. *For any $\phi \in A^p(M)$, $\psi \in A^q(M)$, we have*

1. $i(X)(\phi \wedge \psi) = (i(X)\phi) \wedge \psi + (-1)^p \phi \wedge (i(X)\psi)$,
2. $i(X)(f\phi) = f\,(i(X)\phi)$,
3. $i(X) \circ i(X) = 0$.

Proof. Direct proof. We check 3. $\forall \phi \in A^p(M)$, we have

$$((i(X) \circ i(X) = 0)\phi)(Y_1, \ldots, Y_{p-2}) = \phi(X, X, Y_1, \ldots, Y_{p-2}) = 0.$$

□

Lemma 7.2. *Choosing any local orthonomal frame $\{E_i\}_{i=1}^n$ on M, we have*

$$\delta = -\sum_{j=1}^{n} i(E_j)\nabla_{E_j}. \tag{7.2.4}$$

Proof. Denote $\overline{\delta} = -\sum_{j=1}^n i(E_j)\nabla_{E_j}$. We can check $\overline{\delta}$ is independent of the choice of local orthonormal frames. So we only need to prove (7.2.4) at a fixed point $x \in M$. We pick a local normal frame $\{E_i\}$ at x. Let $\{\omega^i\}$ be the dual coframe. Since $\nabla_{E_i} E_j = \sum_k \Gamma_{ij}^k E_k$, we have

$$\begin{aligned}\left(\nabla_{E_i}\omega^j\right)(E_k) &= \nabla_{E_i}\left(\omega^j(E_k)\right) - \omega^j\left(\nabla_{E_i}E_j\right) \\ &= -\omega^j(\Gamma_{ik}^l E_l) = -\Gamma_{ik}^j.\end{aligned}$$

$\Rightarrow \nabla_{E_i}\omega^j = -\Gamma^j_{ik}\omega^k$. Therefore at $x \in M$, we have $\nabla_{E_i}\omega^j(x) = 0$, $1 \le i, j \le n$. By linearity, we only need to verify $\delta\eta = \overline{\delta}\eta$ for $\eta = f\omega^1 \wedge \cdots \wedge \omega^p$. We compute

$$\begin{aligned}
\overline{\delta}\eta &= -\sum_{j=1}^{p} i(E_j)\nabla_{E_j}(f\omega^1 \wedge \cdots \wedge \omega^p) \\
&= -\sum_{j=1}^{p} i(E_j)\left(E_j(f)\right)\omega^1 \wedge \cdots \wedge \omega^p \\
&= -\sum_{j=1}^{p}\left(E_j(f)\sum_{k=1}^{p}(-1)^{k-1}\omega^1 \wedge \cdots \wedge \omega^{k-1} \wedge i(E_j)\omega^k \wedge \cdots \wedge \omega^p\right) \\
&= -\sum_{j=1}^{p} E_j(f)(-1)^{j-1}\omega^1 \wedge \cdots \wedge \widehat{\omega}^j \wedge \cdots \wedge \omega^p \\
&= \sum_{j=1}^{p}(-1)^j E_j(f)\omega^1 \wedge \cdots \wedge \widehat{\omega}^j \wedge \cdots \wedge \omega^p.
\end{aligned}$$

On the other hand, note that for $j = 1, \ldots, p$.

$$\begin{aligned}
\star\left(\omega^j \wedge \omega^{p+1} \wedge \cdots \wedge \omega^n\right) &= (-1)^{(n-p)(p-1)} \times (-1)^{j-1}\omega^1 \wedge \cdots \wedge \widehat{\omega}^j \wedge \cdots \wedge \omega^p \\
&= (-1)^{np+n+1+j}\omega^1 \wedge \cdots \wedge \widehat{\omega}^j \wedge \cdots \wedge \omega^p.
\end{aligned}$$

Therefore

$$\begin{aligned}
(\delta\eta)(p) &= (-1)^{np+n+n+} \star \mathrm{d} \star (f\omega^1 \wedge \cdots \wedge \omega^p) \\
&= (-1)^{np+n+1} \star \mathrm{d}(f\omega^{p+1} \wedge \cdots \wedge \omega^n) \\
&= (-1)^{np+n+1} \star \left(\sum_{j=1}^{n} \omega^j \wedge \nabla_{E_j}(f\omega^{p+1} \wedge \cdots \wedge \omega^n)\right) \\
&= (-1)^{np+n+1} \star \left(\sum_{j=1}^{n} E_j(f)\omega^j \wedge \omega^{p+1} \wedge \cdots \wedge \omega^n\right) \\
&= (-1)^{np+n+1} \sum_{j=1}^{n} E_j(f) \star (\omega^j \wedge \omega^{p+1} \wedge \cdots \wedge \omega^n) \\
&= \sum_{j_1}^{n}(-1)^j E_j(f)\omega^1 \wedge \cdots \wedge \widehat{\omega}^j \wedge \cdots \wedge \omega^p \\
&= (\overline{\delta}\eta)(p).
\end{aligned}$$

This completes the proof. □

Observation: for $f \in C^\infty(M)$, we have

$$\begin{aligned}\Delta f = -\delta \mathrm{d} f &= -\sum_j i(E_j)\nabla_{E_j}\left(\sum_i \omega^i \wedge \nabla_{E_i} f\right) \\ &= -\sum_j i(E_j)\nabla_{E_j}\left(E_i(f)\omega^i\right) \\ &= -\sum_{i,j} E_j\left(E_i(f)\right)\omega^i(E_j) = -\sum_i E_i((E_j)(f)) \\ &= \mathrm{trHess} f.\end{aligned}$$

So Hodge Lapalacian is the negative of the Laplace-Beltrani operator we defined before.

7.2.3 Weitzenböck formula

For $\omega \in A^p(M)$, which can be considered as a $(0, p)$ tensor, recall $\nabla^2\omega$ is a $(0, p+2)$-tensor. Let $\{e_i\}$ be a local orthonormal frame. We define

$$\mathrm{tr}(\nabla^2\omega)(\dots) = \sum_i \nabla^2\omega(\dots, e_i, e_j).$$

One can check this definition is independent of the choice of an orthonormal frame, and

$$\mathrm{tr}(\nabla^2\omega)(\dots) = \sum_i \left(\nabla_{e_i}\nabla_{e_i}\omega - \nabla_{\nabla_{e_i}e_i}\omega\right).$$

Theorem 7.3 (Weitzenböck formula). *For any $\omega \in A^p(M)$, let $\{e_i\}$ be a local orthonormal frame and $\{\omega^i\}$ its dual, then*

$$\Delta\omega = -\mathrm{tr}(\nabla^2\omega) - \omega^i \wedge i(e_j)R(e_i, e_j)\omega. \tag{7.2.5}$$

Proof. We can check that the RHS is independent of the choice of othonormal frame. So we will prove the W-formula at a point $x \in M$, and pick a local normal frame $\{E_i\}$. We again use $\{\omega^i\}$ for its dual.

Recall the covariant derivative is commutable with the contraction, i.e.

$$\begin{aligned}\nabla_{E_i}\left(C(E_j \otimes \nabla_{E_j}\omega)\right) &= C\nabla_{E_i}(E_j \otimes \nabla_{E_j}\omega) \\ &= C\left(E_j \otimes \nabla_{E_i}\nabla_{E_j}\omega\right).\end{aligned}$$

Hence $\nabla_{E_i}\left(i(E_j)\nabla_{E_j}\omega\right) = i(E_j)\nabla_{E_i}\nabla_{E_j}\omega$. Recall $\nabla_{E_i}E_j = 0$, $\forall i, j \Rightarrow \nabla_{E_j}\omega^j = 0$,

$\forall i, j$. Therefore, we compute

$$\begin{aligned}
\Delta\omega = \mathrm{d}\delta\omega + \delta\mathrm{d}\omega &= \sum_i \omega^i \wedge \nabla_{E_i}(\delta\omega) - \sum_j i(E_j)\nabla_{E_j}(\mathrm{d}\omega) \\
&= -\sum_i \omega^i \wedge \nabla_{E_i}\left(\sum_j i(E_i)\nabla_{E_j}\omega\right) - \sum_j i(E_j)\nabla_{E_j}\left(\sum_i \omega^i \wedge \nabla_{E_i}\omega\right) \\
&= -\sum_{ij} \omega^i \wedge \nabla_{E_i}\left(i(E_i)\nabla_{E_j}\omega\right) - \sum_{j,i} i(E_j)\nabla_{E_j}\left(\omega^i \wedge \nabla_{E_i}\omega\right) \\
&= -\sum_{ij} \omega^i \wedge i(E_i)\nabla_{E_i}\nabla_{E_j}\omega - \sum_{i,j} i(E_j)\left(\omega^i \wedge \nabla_{E_j}\nabla_{E_i}\omega\right) \\
&= -\sum_{ij} \omega^i \wedge i(E_i)\nabla_{E_i}\nabla_{E_j}\omega - \sum_{i,j} \delta^i_j \nabla_{E_j}\nabla_{E_i}\omega + \sum_{i,j} \omega^i \wedge i(E_j)\nabla_{E_j}\nabla_{E_i}\omega \\
&= -\sum_i \nabla_{E_i}\nabla_{E_i}\omega - \sum_{i,j} \omega^i \wedge i(E_j)\left(\nabla_{E_i}\nabla_{E_j}\omega - \nabla_{E_j}\nabla_{E_i}\omega\right) \\
&= -\mathrm{tr}(\nabla^2\omega) - \sum_{i,j} \omega^i \wedge i(E_j)R(E_i, E_j)\omega.
\end{aligned}$$

□

Corollary 7.3 (Bochner)**.** *For any $\omega \in A^p(M)$, we have*

$$-\frac{1}{2}\Delta|\omega|^2 = -\langle\Delta\omega, \omega\rangle + |\nabla\omega|^2 + F(\omega) \tag{7.2.6}$$

where $F(\omega) = -\langle\omega^i \wedge i(E_j)R(e_i, e_j)\omega, \omega\rangle$

Remark: Here we are using Hodge Laplacian.

Proof. We complete in local normal frame $\{E_i\}$, we have

$$\begin{aligned}
-\langle\Delta\omega, \omega\rangle + F(\omega) &= \langle -\Delta\omega - \omega^i \wedge i(E_j)R(E_i, E_j)\omega, \omega\rangle \\
&= \langle \mathrm{tr}(\nabla^2\omega), \omega\rangle \\
&= \langle \sum_i \nabla_{E_i}\nabla_{E_i}\omega, \omega\rangle \\
&= \sum_i \left(\nabla_{E_i}\langle\nabla_{E_i}\omega, \omega\rangle - \langle\nabla_{E_i}\omega, \nabla_{E_i}\omega\rangle\right) \\
&= \frac{1}{2}\sum_i \nabla_{E_i}\nabla_{E_i}\langle\omega, \omega\rangle - |\nabla\omega|^2 \\
&= -\frac{1}{2}\Delta|\omega|^2 - |\nabla\omega|^2.
\end{aligned}$$

□

In particular, when $\omega \in A^1(M)$, the curvature term becomes simpler. Let $\sharp\omega = \langle\omega, \omega^i\rangle E_i$, we have

$$\begin{aligned} F(\omega) &= -\langle \omega^i \wedge i(E_j)R(E_i, E_j)\omega, \omega\rangle \\ &= -i(E_j)R(E_i, E_j)\omega\langle \omega^i, \omega\rangle \\ &= -\Big(R(E_i, E_j)\omega\Big)(E_j)\langle \omega^i, \omega\rangle. \end{aligned}$$

<u>Claim</u>:

$$\begin{aligned} (R(X, Y)(\omega))(Z) &= -\omega\,(\,R(X, Y)Z\,) \\ &= (\nabla_X\nabla_Y\omega - \nabla_Y\nabla_X\omega - \nabla_{[X,Y]}\omega)(Z), \\ (\nabla_X\nabla_Y\omega)(Z) &= (\nabla_X(\nabla_Y\omega))(Z) = X\,((\nabla_Y\omega)(Z)) - (\nabla_Y\omega)(\nabla_X Z) \\ &= X\{Y(\omega(Z)) - \omega(\nabla_Y Z)\} - Y(\omega(\nabla_X Z)) + \omega(\nabla_Y\nabla_X Z) \\ &= X(Y(\omega(Z))) - X(\omega(\nabla_Y Z)) - Y(\omega(\nabla_X Z)) + \omega(\nabla_Y\nabla_X Z), \\ (\nabla_{[X,Y]}\omega)(Z) &= [X, Y](\omega(Z)) - \omega(\nabla_{[X,Y]}(Z)). \end{aligned}$$

Hence $(R(X, Y)(\omega))(Z) = \omega(\nabla_Y\nabla_X Z) - \omega(\nabla_X\nabla_Y Z) + \omega(\nabla_{[X,Y]}(Z)) = -\omega(R(X, Y)Z)$. Now we continue our computation:

$$\begin{aligned} F(\omega) &= -\Big(R(E_i, E_j)\omega\Big)(E_j)\langle \omega^i, \omega\rangle \\ &= \omega\Big(R(E_i, E_j)E_j\Big)\langle \omega^i, \omega\rangle \\ &= \langle \sharp\omega, R(E_i, E_j)\omega E_j\rangle\langle \omega^i, \omega\rangle \\ &= \langle \sharp\omega, R(\langle \omega^i, \omega\rangle E_i, E_j)E_j\rangle \\ &= \langle \sharp\omega, R(\sharp\omega, E_j)E_j\rangle = \langle R(\sharp\omega, E_j)E_j, \sharp\omega\rangle \\ &= \sum_j R(\sharp\omega, E_j, \sharp\omega, E_j) = \sum_j R(E_j, \sharp\omega, E_j, \sharp\omega) \\ &= \mathrm{tr}R(\cdot, \sharp\omega, \cdot, \sharp\omega) = Ric(\sharp\omega, \sharp\omega). \end{aligned}$$

Corollary 7.4. *For any $\omega \in A^1(M)$, we have*

$$-\frac{1}{2}\Delta|\omega|^2 = -\langle \Delta\omega, \omega\rangle + |\nabla\omega|^2 + Ric(\sharp\omega, \sharp\omega). \tag{7.2.7}$$

Theorem 7.4 (Bochner)**.** *Let (M, g) be a closed oriented Riemannian manifold.*

1. *If $Ric \geq 0$, then any harmonic 1-form ω is parallel, i.e. $\nabla\omega = 0$.*
2. *If $Ric \geq 0$ on M and $Ric > 0$ at one point, then there is no non-trival harmonic 1-form.*

Proof. Recall $\int_M -\Delta|\omega| \mathrm{d}vol_M = 0$. Hence we have

$$\begin{aligned} 0 &= -\int_M \langle \Delta\omega, \omega\rangle + |\nabla\omega|^2 + Ric(\sharp\omega, \sharp\omega)\mathrm{d}vol_M \\ &= \int_M |\nabla\omega|^2 + Ric(\sharp\omega, \sharp\omega)\mathrm{d}vol_M \geq 0. \end{aligned}$$

$\Rightarrow \nabla\omega = 0$, i.e. ω is parallel. If $Ric > 0$ at some point, we must have $\sharp\omega = 0$, i.e.$\omega = 0$.

□

Corollary 7.5. *For any $f \in C^\infty(M)$, we have*

$$\frac{1}{2}\Delta_{LB}|\mathrm{grad}f|^2 = |\mathrm{Hess}f|^2 + \langle \mathrm{grad}(\Delta_{LB}f), \mathrm{grad}f\rangle + Ric(\mathrm{grad}f, \mathrm{grad}f). \tag{7.2.8}$$

Proof. We see $\mathrm{d}f \in A^1(M)$, and $|\mathrm{d}f|^2$, $\sharp(\mathrm{d}f) = \mathrm{grad}f$.

$$\begin{aligned} -\langle \Delta \mathrm{d}f, \mathrm{d}f\rangle &= -\langle (\mathrm{d}\delta + \delta\mathrm{d})\mathrm{d}f, \mathrm{d}f\rangle = -\langle \mathrm{d}\delta\mathrm{d}f, \mathrm{d}f\rangle \\ &= -\langle \mathrm{d}(\delta\mathrm{d}f), \mathrm{d}f\rangle = -\langle \mathrm{d}(\Delta f), \mathrm{d}f\rangle \\ &= -\langle \mathrm{grad}(\Delta f), \mathrm{grad}f\rangle = \langle \mathrm{grad}(\Delta_{LB}f), \mathrm{d}f\rangle. \end{aligned}$$

$$\begin{aligned} |\nabla \mathrm{d}f|^2 &= \sum_i \langle \nabla_{E_i}\mathrm{d}f, \nabla_{E_i}\mathrm{d}f\rangle \\ &= \sum_i \langle \nabla_{E_i}\mathrm{grad}f, \nabla_{E_i}\mathrm{grad}f\rangle \\ &= \sum_i \left\langle \sum_j \langle \nabla_{E_i}\mathrm{grad}f, E_j\rangle E_j, \sum_k \langle \nabla_{E_i}\mathrm{grad}f, E_k\rangle E_k \right\rangle \\ &= \sum_{ij} \langle \nabla_{E_i}\mathrm{grad}f, E_j\rangle^2 = \sum_{ij} \mathrm{Hess}f(E_i, E_j)^2 \\ &= |\mathrm{Hess}f|^2. \end{aligned}$$

□

Let (M, g) be a closed Riemannian manifold. We say $\lambda \in \mathbb{R}$ is an eigenvalue of Δ_{LB} if $\exists$ a smooth function $u \neq 0$ such that $\Delta_{LB}u + \lambda u = 0$. It is konwm that the eigenvalues can be listed as $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots \nearrow \infty$.

Theorem 7.5 (Lichnerowicz)**.** *Let (M^n, g) be a closed Riemannian manifold with Ric $\geq k > 0$. Then we have $\lambda_1 \geq \frac{n}{n-1}k$.*

Proof. Integrate the Bochner formula in corallary 7.5, we have

$$\begin{aligned} 0 &= \int_M |\mathrm{Hess}f|^2 + \langle \mathrm{grad}(\Delta_{LB}f), \mathrm{grad}f\rangle + Ric(\mathrm{grad}f, \mathrm{grad}f) \\ &= \int_M |\mathrm{Hess}f|^2 - \lambda_1 \int_M \langle \mathrm{grad}f, \mathrm{grad}f\rangle + \int_M Ric(\mathrm{grad}f, \mathrm{grad}f) \\ &\geq \int_M |\mathrm{Hess}f|^2 - \lambda_1 \int_M |\mathrm{grad}f|^2 + \int_M |\mathrm{grad}f|^2. \end{aligned}$$

$\Rightarrow \lambda_1 \geq k$.

We can make more use of $|\mathrm{Hess} f|^2$ term.

$$\begin{aligned}
|\mathrm{Hess} f|^2 &= \sum_{ij} \mathrm{Hess} f(E_i, E_j)^2 \geq \sum_i \mathrm{Hess} f(E_i, E_i)^2 \\
&\geq \frac{1}{n}\left(\sum_i \mathrm{Hess} f(E_i, E_i)\right)^2 = \frac{1}{n}(\Delta_{LB} f) = \frac{1}{n}\lambda_1^2 f^2 \\
\Rightarrow \int_M |\mathrm{Hess} f|^2 &\geq \frac{1}{n}\lambda_1 \int_M \lambda_1 f^2 = \frac{\lambda_1}{n}\int_M \langle f, -\Delta_{LB} f\rangle \\
&= \frac{\lambda_1}{n}\int_M \langle \mathrm{grad} f, \mathrm{grad} f\rangle
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow 0 \geq \left(\frac{\lambda_1}{n} - \lambda_1 + k\right)\int_M |\mathrm{grad} f|^2 \\
&\Rightarrow \lambda_1 \geq \frac{nk}{n-1}.
\end{aligned}$$

□

Like for the Bonnet-Myers Theorem, we have the following RIGIDITY result due to Obata.

Theorem 7.6 (Obata)**.** *Let (M^n, g) be a closed Riemannian manifold with $Ric \geq (n-1)k$, $k > 0$. Then $\lambda_1 = nk$ iff (M^n, g) is isometric to the space $\mathbb{S}\left(\frac{1}{\sqrt{k}}\right)$.*

Proof. W.l.o.g., we can suppose $k = 1$. If $\lambda_1 = n$, then the proof of 7.5 implies

$$Ric(\mathrm{grad} f, \mathrm{grad} f) = (n-1)|\mathrm{grad} f|^2.$$

Since $\Delta_{LB} u^2 = 2|\mathrm{grad} f|^2 + 2u\Delta_{LB} u$.

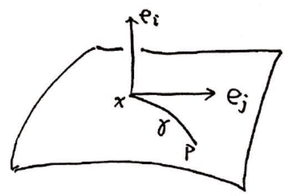
$$\begin{aligned}
\Rightarrow \frac{1}{2}\Delta_{LB}\left(|\nabla f|^2 + f^2\right) =& \frac{1}{2}\Delta_{LB}|\nabla f|^2 + |\mathrm{grad} f|^2 + f\Delta_{LB} f \\
\geq& \frac{\lambda_1}{n}\langle f, -\Delta_{LB} f\rangle - n|\mathrm{grad} f|^2 + (n-1)|\mathrm{grad} f|^2 \\
&+ |\mathrm{grad} f|^2 + f\Delta_{LB} f \\
=& 0.
\end{aligned}$$

Recall $\int_M \frac{1}{2}\Delta_{LB}\left(|\nabla f|^2 + f^2\right) = 0$. Hence $\Delta_{LB}\left(|\nabla f|^2 + f^2\right) = 0$. That is,

$$|\mathrm{grad} f|^2 + f^2 \equiv const.$$

Normalize f so that $\max_M f^2 = 1$. Since at the maximum/minimum points of f, we have $\mathrm{grad} f = 0$. Therefore we have $|\mathrm{grad} f|^2 + f^2 = 1$ and $\max_M f = -\min_M f = 1$.

Let $p, q \in M$ be points s.t. $f(p) = -1, f(q) = 1$.



Let γ be a normal minimizing geodesic from p to q. Note that

$$\frac{\frac{\mathrm{d}}{\mathrm{d}t} f \circ \gamma(t)}{\sqrt{1-(f\circ\gamma(t))^2}} \leq \frac{|\mathrm{grad} f(\gamma(t))|^2}{\sqrt{1-(f\circ\gamma(t))^2}} = 1.$$

Integrating over t,

$$\begin{aligned}\left|\int_0^{d(p,q)} \frac{\frac{\mathrm{d}}{\mathrm{d}t} f\circ\gamma(t)}{\sqrt{1-f\circ\gamma(t)^2}}\right| &= \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\mathrm{d}x = \pi \\ &\leq \int_0^{d(p,q)} \frac{\left|\frac{\mathrm{d}}{\mathrm{d}t} f\circ\gamma(t)\right|}{\sqrt{1-f\circ\gamma(t)^2}} \leq d(p,q).\end{aligned}$$

On the other hand, $Ric \geq (n-1) \Rightarrow d(p,q) \leq \pi \Rightarrow diam = \pi \Rightarrow M$ is isometric to $\mathbb{S}(1)$.

□

7.2.4 Bakry-Émery Γ-calculus:A systematic way of understanding the Bochner formula

For any $f, g \in C^\infty(M)$, define

$$\Gamma(f,g) = \frac{1}{2}\left(\Delta_{LB}(fg) - f\Delta_{LB}g - g\Delta_{LB}f\right)$$

Observe $\Gamma(f,f) = |\mathrm{grad} f|^2$.

Interatively, define $\Gamma_2(f,g) = \frac{1}{2}\left(\Delta(\Gamma(f,g)) - \Gamma(f, \Delta_{LB}g) - \Gamma(\Delta_{LB}f, g)\right)$.

Observe that $\Gamma_2(f,f) = \frac{1}{2}\Delta_{LB}|\mathrm{grad} f|^2 - \langle \mathrm{grad} f, \mathrm{grad}(\Delta_{LB}f)\rangle$. So the Bochner formula in Corollary 7.5 implies

$$\begin{aligned}\Gamma_2(f,f) =& |\mathrm{Hess} f|^2 + Ric(\mathrm{grad} f, \mathrm{grad} f) \\ \geq& \frac{1}{n}(\Delta_{LB}f)^2 + Ric(\mathrm{grad} f, \mathrm{grad} f).\end{aligned}$$

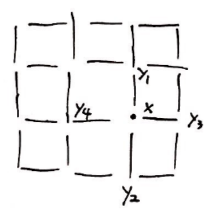
Moreover $Ric \geq k$ implies $\Gamma_2(f,f) \geq \frac{1}{n}(\Delta_{LB}f)^2 + k\Gamma(f,f), \forall f \in C^\infty(M)$.

The above property enables us to define "Ricci curvature lower bound" for a general operator, which can be operators on more general spaces. Here we discuss possibilites on discrete metric spaces: a combinatorial graph $G = (V, E)$, where

1. V is the set of vertices (points),
2. E is the set of edges,
3. metric: combinatorial distance (length of shortest path).

For example, a discrete set $\{p_1, \ldots, p_n\}$ with the metric $d(p_i, p_j) = \delta_{ij}$ can be represented by a complete graph K_n. Define the degree of a vertex p to be $\deg(p) = \sum_{q \in V, d(p,q)=1} 1$.

We can consider the graph Laplacian Δ defined via $\Delta f(x) = \sum_{y \in V, d(y,x)=1} (f(y) - f(x))$,



for $f : V \to \mathbb{R}$. We say λ is eigenvalue of Δ if $\exists f \neq 0$ s.t. $\Delta f + \lambda f = 0$. We can list $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{|V|-1}$.

Definition 7.2. *A graph $G = (V, E)$ is said to satisfy the curvature dimension inequality $CD(K, n)$ for some $K \in \mathbb{R}$, $n \in (0, \infty]$ if for all $f : V \to \mathbb{R}$, it holds*

$$\Gamma_2(f, f)(x) \geq \frac{1}{n}(\Delta f)^2(x) + K\Gamma(f, f)(x)$$

$\forall x \in V$.

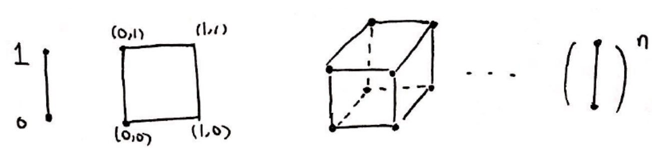
Since we do not have a proper understanding about the "dimension" of a graph, quite oftenly we assume $CD(K, \infty)$ conditions.

Theorem 7.7 (L.-Münch-Peyerimhoff,arXiv:1608.09998,arXiv:1705.08119)**.** *Let $G = (V, E)$ be a connected graph satisfying $CD(K, \infty)$, and $\deg_{\max} < \infty$. Then*

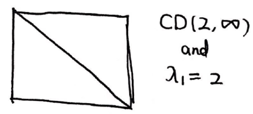
$$diam_d(G) \leq \frac{2 \deg_{\max}}{K}.$$

Moreover, "=" holds iff G is a $\deg_{\max}$-dimensional hypercube. Under the same assumption, (By standard argument, we have $\lambda_1 \geq K$) $\lambda_{\deg_{\max}} = K$ iff G is a $\deg_{\max}$-dimensional hypercube.

Remark: (1)Hypercube:



(2) $\lambda_1 = K$ is not strong enough to conclude the Rigidity Theorem. counterexample:



Open question: Let $G = (V, E)$ be a connected graph satisgying $CD(0, \infty)$. What is the volume growth rate? Polynomial? This is equivalent to ask for the (non-)existence of a family of expanders in the class of Graphs satisfying $CD(0, \infty)$.

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