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Multi-way dual Cheeger constants and hostile spectral clustering

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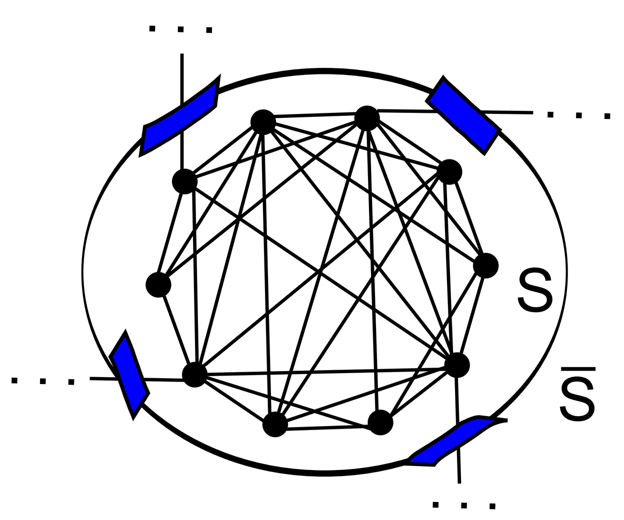
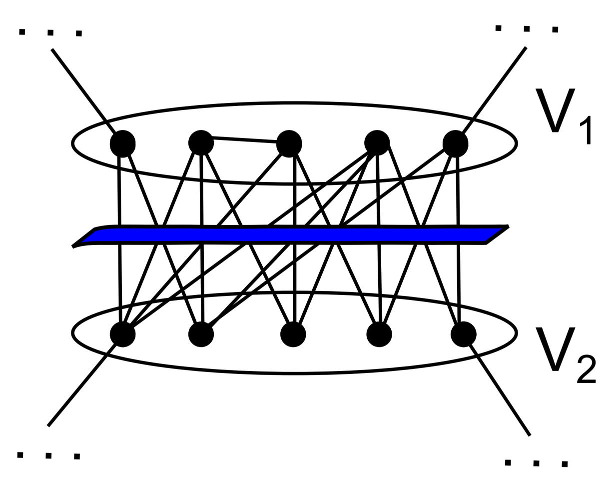
Problem: Min Expansion and Max Cut



Expansion of S

$$\phi(S) = \frac{|E(S, \overline{S})|}{\mathrm{vol}(S)}$$



Cut ratio of V_1, V_2

$$\overline{\phi}(V_1, V_2) = \frac{2|E(V_1, V_2)|}{\mathrm{vol}(V_1 \cup V_2)}$$

Multi-way dual Cheeger constants

Given an undirected weighted finite graph $G = (V, E, w)$,

- k-way Cheeger constant (Miclo '08, $k = 2$: extended to discrete setting from Cheeger '69)

$$h(k) = \min_{S_1, S_2, \ldots, S_k} \max_{1 \le i \le k} \phi(S_i),$$

  where the minimum is taken over all k-subpartition of V;

- **k-way dual Cheeger constant** ($k = 1$: Bauer-Jost [1], bipartiteness ratio of Trevisan [3])

$$\overline{h}(k) = \max_{(V_1, V_2), \ldots, (V_{2k-1}, V_{2k})} \min_{1 \le i \le k} \overline{\phi}(V_{2i-1}, V_{2i}),$$

  where the maximum is taken over all k-sub-bipartition of V. We only require $V_{2i-1} \cup V_{2i} \neq \emptyset$, $\forall\ 1 \le i \le k$.

Intuitions of $\overline{h}(k)$

$\{\overline{h}(k)\}$ describe how far/close a graph is from being a bipartite one.

$$\begin{aligned} \text{connected } G \text{ is bipartite} \ &\Leftrightarrow\ \overline{h}(1) = 1; \\ G \text{ is bipartite} \ &\Leftrightarrow\ h(k) + \overline{h}(k) = 1,\ 1 \le k \le N; \\ G \text{ is } \textbf{an odd cycle} \ &\Rightarrow\ h(k) + \overline{h}(k) = 1,\ \ 2 \le k \le N. \end{aligned}$$

Any proper subgraph of an odd cycle is bipartite. Roughly speaking, if a graph can satisfy $h(k) + \overline{h}(k) = 1$ for a small k, then it possess a "large" bipartite subgraph.

Spectrum of normalized Laplace operator

Normalized graph Laplacian Δ : for any $f : V \to \mathbb{R}$, and $u \in V$

$$\Delta f(u) := \frac{1}{d_u} \sum_{v, v \sim u} w_{uv}(f(u) - f(v)).$$

In matrix form, $\Delta = I - D^{-1}A$.
List of its spectrum: $0 = \lambda_1 \le \lambda_2 \le \cdots \le \lambda_{N-1} \le \lambda_N \le 2$.

- $\lambda_2 = 0 \Leftrightarrow G$ is disconnected (Cheeger inequality),

$$\lambda_2/2 \le h(2) \le \sqrt{2\lambda_2};$$

- $\lambda_N = 2 \Leftrightarrow G$ is bipartite (dual Cheeger inequality, Bauer-Jost [1], independently Trevisan [3] for regular graph case),

$$\frac{2 - \lambda_N}{2} \le 1 - \overline{h}(1) \le \sqrt{2(1 - \overline{h}(1))};$$

- Higher-order Cheeger inequalities (Lee-Gharan-Trevisan [2])

$$\lambda_k/2 \le h(k) \le Ck^2\sqrt{\lambda_k},$$

  where C is a universal constant.

Higher-order dual Cheeger inequalities

Theorem 1. *For every graph G, and each natural number $1 \le k \le N$, we have*

$$\frac{2 - \lambda_{N-k+1}}{2} \le 1 - \overline{h}(k) \le Ck^3\sqrt{2 - \lambda_{N-k+1}},$$

where C is a universal constant.

Remark: $\lambda_{N-k+1} = 2 \Leftrightarrow \overline{h}(k) = 1 \Leftrightarrow G$ has at least k bipartite connected components.
Example: For any unweighted cycle, we have for each $1 \le k \le N$

$$\frac{\sqrt{2}}{\pi}\sqrt{2 - \lambda_{N-k+1}} \le 1 - \overline{h}(k) \le \frac{3}{\sqrt{\pi}}\sqrt{2 - \lambda_{N+k-1}}.$$

Proof: Hostile spectral clustering

Goal: finding k disjoint subsets each of which has a bipartition such that $1 - \overline{\phi}$ is small, i.e. each is close to be bipartite.

- Spectral representation: using the top k eigenfunctions $f_{N-k+1}, \ldots, f_N$ of Δ to represent V as points in $\mathbb{R}^k$

$$F : V \to \mathbb{R}^k, v \mapsto (f_{N-k+1}(v), \ldots, f_N(v)).$$

  Ignoring those vertices on which F vanishes, we further consider,

$$\widetilde{F} : V \to \mathbb{S}^{k-1}, v \mapsto \frac{F(v)}{\|F(v)\|}.$$

- **Projective space with a rough metric**:
  The canonical antipodal projection,

$$\mathrm{P} : \mathbb{S}^{k-1} \to P^{k-1}\mathbb{R}, \{x, -x\} \mapsto [x].$$

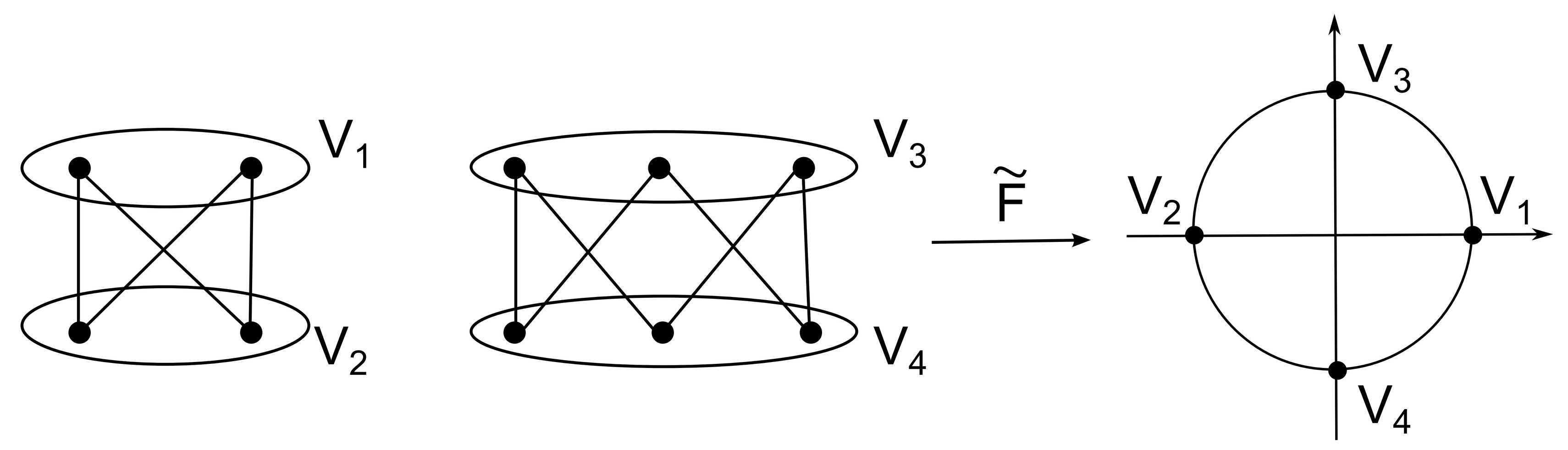
  Assign the following metric,

$$\overline{d}([x],[y]) := \min\{\|x + y\|, \|x - y\|\}.$$

- Partition points in $(P^{k-1}\mathbb{R}, \overline{d})$ that represent V via $\mathrm{P} \circ \widetilde{F}$.

Remark: The classical spectral clustering algorithm verified by Lee-Gharan-Trevisan employs the bottom k eigenfunctions and the spherical distance.

An inspiring example



- Cluster on $\mathbb{S}^1 \Rightarrow V_1 \cup V_3$ and $V_2 \cup V_4$;
- Cluster on $P^1\mathbb{R} \Rightarrow V_1 = V_2$ and $V_3 = V_4$.

Goal: obtain the 2-sub-bipartition $\{(V_1, V_2), (V_3, V_4)\}$.

Remark: Think of edges as "hostile" relations. Vertices are clustered because of sharing common enemies. Compare the classical spectral clustering, edges are considered as "friendly" relations. Vertices are clustered because of being friends.

Acknowledgements and References

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[1] F. Bauer and J. Jost, Bipartite and neighborhood graphs and the spectrum of the normalized graph Laplacian, Comm. Anal. Geom. 21 (2013), no. 4, 787-845.

[2] J. R. Lee, S. Oveis Gharan and L. Trevisan, Multi-way spectral partitioning and higher-order Cheeger inequalities, STOC'12, 1117-1130, ACM, New York, 2012.

[3] L. Trevisan, Max cut and the smallest eigenvalue. SIAM J. Comput. 41 (2012), no. 6, 1769-1786.