Proceedings of Symposia in Pure Mathematics

Discrete Geometric Analysis

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ABSTRACT. This is an expository article on discrete geometric analysis based on the lectures which the author gave at Gregynog Hall, University of Wales, as an activity of the Project "Analysis on graphs and its applications" in the Issac Newton Institute. Topics are selected to illustrate the nature of the field; that is, we focus upon analysis on graphs with a geometric flavor such as graph versions of harmonic theory and spectral geometry of Laplacians. Zeta functions and random walks are also discussed from a geometric viewpoint.

1. Intorduction

Discrete geometric analysis is a hybrid field of several traditional disciplines; say, graph theory, geometry, theory of discrete groups, and probability. As indicated by the title, this field concerns solely analysis on *graphs*, a synonym of "1-dimensional cell complex". Edges in a graph as 1-dimensional objects, however, do not play a substantial role except when we discuss geometry of graphs. Therefore our view is different from, for instance, the case of quantum graphs where differential operators on edges are vital. Actually the role of edges in analysis is just to give a neighboring relation among vertices, and difference operators linked with this relation stand in for differential operators. We thus do not need to worry about irregularity of functions which, for differential operators, may cause some trouble, if not serious. Instead the combinatorial aspect of graphs creates a different kind of technical complication. Furthermore, analysis on both infinite graphs and non-compact manifolds involves much the same degree of difficulty.

The "protagonist" in our story is a discrete analogue of Laplacians on manifolds (and related operators), which appears in many parts of mathematical sciences. In the nature of things, ideas cultivated in global analysis provide us a good guiding principle on the one hand, and the practical motivation is the moving force of theoretical progress on the other. Thus, what a Russian luminary P. L. Chebyshev (1815-1897) maintained in a general context applies to our field; that is, the agreement of theory and practice brings most beneficial results. The aim of this introductory article is to summarize an overview of what has been done in discrete geometric analysis for the past few decades. For more comprehensive accounts of some specific subjects and related matters, the reader may refer to Colin de Verdière

1991 *Mathematics Subject Classification.* Primary 05-02; Secondary 31C20, 39A12.
Key words and phrases. Graph, Discrete Laplacian, Zeta function.

[**31**], W. Woess [**137**] and F. R. K. Chung [**24**]. See also N. Biggs [**13**] which deals with aspect of this field from a similar viewpoint.

2. Discrete harmonic integrals

We shall start with a short historical note on " harmonic integrals" on graphs which gives us rudimental ideas involved in the development of discrete geometric analysis. The story begins with Kirchhoff's famous work (1845) on *resistive circuits*, electric networks consisting of resistors alone. What were established by G. Kirchhoff at the age of twenty-one are the fundamental laws of electric circuits[1]; namely the current law and the voltage law, which, in conjunction with Ohm's law, lead to the existence and uniqueness of steady currents in circuits. His argument can be effectively interpreted by modern terminology such as chains and cochains on a graph. And so Kirchhoff's theory is, from a mathematical viewpoint, regarded as a prototype of algebraic topology for cell complexes which flourished after the fundamental work of Poincaré (1895). Years later came H. Weyl on the stage, a founder of the harmonic theory of manifolds. He observed in 1920s that the proto-idea of his far-reaching "method of orthogonal projections" could be applied to the problem of circuits. This, in fact, reduces the problem to the discrete Poisson equation $\Delta f = g$ (f being an electric potential and g being an external current to flow into the circuit)[2].

Leaving the historical remark aside for a while, let us give several basic definitions. The operator Δ in the discrete Poisson equation above is a special case of what we call the *discrete Laplacian* on a *weighted graph*. Here a graph[3] $X = (V, E)$ with a set of vertices V and a set of all oriented edges E is said to be *weighted* if we are given positive-valued functions m_V on V and m_E on E satisfying $m_E(\overline{e}) = m_E(e)$ ($\overline{e}$ being the inverse edge of e). We denote by $o(e)$ (resp. $t(e)$) the *origin* (resp. *terminus*) of an edge $e \in E$. The discrete Laplacian $\Delta = \Delta_X$ is defined as $d^*d : C^0(X,\mathbb{R}) \longrightarrow C^0(X,\mathbb{R})$ where $d : C^0(X,\mathbb{R}) \longrightarrow C^1(X,\mathbb{R})$ is the coboundary operator of cochain groups; namely $C^0(X,\mathbb{R}) = \{f : V \longrightarrow \mathbb{R}\}$, $C^1(X,\mathbb{R}) = \{\omega : E \longrightarrow \mathbb{R};\ \omega(\overline{e}) = -\omega(e)\}$, and $df(e) = f\big(t(e)\big) - f\big(o(e)\big)$. We are thus thinking of d as a graph version of the exterior differentiation. The adjoint d^* is the one associated with the inner products on $C^k(X,\mathbb{R})$ ($k = 0, 1$) defined by

$$\langle f_1, f_2\rangle = \sum_{x\in V} f_1(x)f_2(x)m_V(x), \quad \langle \omega_1, \omega_2\rangle = \frac{1}{2}\sum_{e\in E}\omega_1(e)\omega_2(e)m_E(e).$$

Obviously the operator Δ is symmetric and nonnegative. Its explicit shape is

$$\Delta f(x) = -\frac{1}{m_V(x)}\sum_{e\in E_x} m_E(e)\big[f\big(t(e)\big) - f\big(o(e)\big)\big] \quad (x \in V,\ f \in C^0(X,\mathbb{R})),$$

where $E_x = \{e \in E;\ o(e) = x\}$. The Poisson equation $\Delta f = g$ has a solution f if and only if $\sum_{x\in V} g(x)m_V(x) = 0$ since Ker Δ consists of constant functions, and Image $\Delta = (\text{Ker } \Delta)^{\perp}$. In what follows, we occasionally write $C(V)$ for $C^0(X,\mathbb{R})$.

The discrete Laplacian for a resistive electric circuit is the case when $m_V \equiv 1$ and $m_E(e)$ is the conductance (the reciprocal of the resistance) of the resistor

[1] His work was based on a report to the exercises given by his university teacher.

[2] The same idea applies to circuits consisting of capacitors.

[3] Graphs are allowed to have loop edges and multiple edges. Unless otherwise stated, graphs are always supposed to be connected and locally finite.

represented by e. To see why so, note that both current $\mathbf{i}$ and voltage $\mathbf{v}$ are regarded as 1-cochains which, in view of Ohm's law, are related to each other by $m_E(e)\mathbf{v}(e) = \mathbf{i}(e)$. The current law says $\sum_{e\in E_x} \mathbf{i}(e) - g(x) = 0$, while the voltage law claims that $\sum_{i=1}^{n} \mathbf{v}(e_i) = 0$ for every closed path $c = (e_1, \ldots, e_n)$ $(e_i \in E)$ which is equivalent to say that there exists a $f \in C(V)$ with $\mathbf{v} = -df$ (f being called the electric potential). Thus combining Ohm's law and Kirchhoff's laws, we get the discrete Poisson equation $\Delta f = g$, from which it follows that a steady flow of electricity takes place if and only if the total amount of currents entering the circuit from outside is zero. Since the potential f is uniquely determined up to additive constants, the current (and hence the voltage) is also uniquely determined.

Just for fun, we suggest the reader to prove the following claim due to M. Dehn (1903) by reducing it to a problem involving the discrete Poisson equation. *If a rectangle K can be divided into finitely many rectangles with rational ratio of two adjacent sides, then the ratio of two adjacent sides of K is rational* (the converse is trivial). Though the claim itself is hardly more than a puzzle, one can still see some essence of procedures for transforming from a problem seemingly irrelevant to graphs to a problem of graphs. A hint will be given at the end of this section.

The notion of discrete Laplacians shows up in the "data points modeling" in CG and CAD (Computer Aided Design)[4]. The idea is to measure the shape of a real object by using devices such as a laser scanner or a charge-coupled device (CCD) camera. Digital images from CCD camera are then transmitted to the "digital space" in a computer for reconstruction, storage and analyzation using image processing technique. More specifically, by measuring, we obtain an enormous number of data points realized in $\mathbb{R}^3$ which form a PL (piecewise linear) surface Σ in $\mathbb{R}^3$ consisting of triangular meshes. What we need to do is to give an explicit parametrization of Σ by two parameters, say (u, v). We do this by a PL map $\mathbf{f}$, with the least "distortion", from Σ into a region in the (u, v)-plane with fixed boundary values. If we consider the graph $X = (V, E)$ obtained by taking the 1-skeleton of Σ, and think of $\mathbf{f}$ as a $\mathbb{R}^2$-valued function on V, then the magnitude of distortion is expressed as $\langle \Delta\mathbf{f}, \mathbf{f}\rangle$ where Δ is the discrete Laplacian associated with suitable weight functions. Thus a discrete analogue of the Dirichlet principle allows us to reduce the problem to the boundary value problem for the discrete Laplace equation.

We shall formulate the boundary value problem in a slightly general setting. A subset $S \subset V$ is said to be *thin* if any two vertices $x, y \in V\backslash S$ can be joined by a path not passing through S, and, for any $x \in S$, there exists $e \in E$ such that $o(e) \in S$, $t(e) \in V\backslash S$. For a thin subset S, which is going to play the role of boundary, one can prove that, if f satisfies $\Delta f = 0$ on $V\backslash S$, and if f attains its maximum (or minimum) on $V\backslash S$, then f is constant (*Maximum principle*). From this, it follows that, if f satisfies $\Delta f = 0$ on $V\backslash S$ and $f = 0$ on S, then $f = 0$. This implies that the linear map $T : C(V) \longrightarrow C(V)$ defined by $(Tf)(x) = (\Delta f)(x)$ (on $V\backslash S$), $(Tf)(x) = f(x)$ (on S) is injective so that T is an isomorphism. Therefore, for every $g \in C(V\backslash S)$ and $h \in C(S)$, the boundary value problem $\Delta f = g$ (on $V\backslash S$), $f = h$ (on S) has a unique solution f. Moreover, for a given $h \in C(S)$, critical functions for the functional $\mathbf{E}(f) = \|df\|^2$ on the space $F = \{f \in C(V); f|_S = h\}$ are those $f \in F$ satisfying $\Delta f = 0$ on $V\backslash S$ (*Dirichlet principle*).

The set-up above of discrete harmonic theory leads us right away to an analogue of the Hodge-Kodaira theorem, which claims that the space $\mathbb{H}^1(X)$ of "harmonic

[4]Due to Yasuhiro Shimizu, Nihon Unisys, Ltd.

forms" on X is isomorphic to $H^1(X,\mathbb{R}) = C^1(X,\mathbb{R})/\mathrm{Image}\ d$. Here, imitating harmonic forms on a manifold, we say that a 1-cochain $\omega \in C^1(X,\mathbb{R})$ is a *harmonic form* if $d^*\omega = 0$. Although this is again an easy exercise in linear algebra, it is still useful in some scenes.

Weyl's idea for electric circuits has been rediscovered by many applied mathematicians even after 1950. Meanwhile, algebraic techniques have been developed under the name of "algebraic graph theory", especially since 1974, when the first edition of N. L. Biggs' book [**14**] was published. This concise book is still an informative source of the field.

One of the main objects in this field is the *adjacency operator*[5] defined by $(\mathcal{A}f)(x) = \sum_{e\in E_x} f(t(e))$, which shows up as a principal part of a discrete Laplacian on a graph with $m_V \equiv 1,\ m_E \equiv 1$; say, $\Delta = \mathcal{D}-\mathcal{A}$, where $\mathcal{D}f(x) = (\deg x)f(x)$. As a sample of results in algebraic graph theory, we bring up the following remarkable identity linking the *tree number* $\kappa(X)$ (the number of spanning trees[6]) and eigenvalues $0 = \lambda_0 < \lambda_1 \leq \cdots \leq \lambda_{N-1}$ $(N = |V|)$ of $\Delta = \mathcal{D} - \mathcal{A}$

$$\kappa(X) = \lambda_1 \cdots \lambda_{N-1}/N. \tag{2.1}$$

The proof, which can be found in [**14**], exhibits an indigenous character of algebraic graph theory.

The notion of tree numbers originated in the work of A. Cayley (1889) who tried to count all trees with given N vertices (the answer is N^{N-2}, which is nothing but the tree number of the complete graph K_N over N vertices). An interesting aspect of $\kappa(X)$ is that it appears in various forms in graph theory. For instance, if $\{c_1,\ldots,c_n\}$ is a $\mathbb{Z}$-basis of $H_1(X,\mathbb{Z})$, then $\kappa(X) = \det(c_i \cdot c_j)_{ij}$ where the symbol "$\cdot$" stands for the inner product in $C_1(X,\mathbb{R})$ defined by $e \cdot e' = 1$ $(e = e')$, $e \cdot e' = -1$ $(e = \overline{e'})$, $e \cdot e' = 0$ (otherwise). In other words, the volume of the torus $\mathbb{A}(X) = H_1(X,\mathbb{R})/H_1(X,\mathbb{Z})$ with the flat metric induced from this inner product is equal to $\kappa(X)^{1/2}$.

Another sample is the claim that $\mathcal{A}$ *has at least* $\mathrm{diam}(X)+1$ *distinct eigenvalues*, where $\mathrm{diam}(X)$ is the *diameter* of X, the maximal value of the distance function[7] $d(x,y)$. This is interesting since, in the claim, we see a naive relation between length of paths and eigenvalues. The proof actually resorts to a path of the shortest length joining x and y with $d(x,y) = \mathrm{Diam}(X)$ and employs the fact that the matrix element $\langle \mathcal{A}^n\delta_y, \delta_x\rangle$ is non-zero if and only if there exists a path of length n joining x and y, where δ_x is the characteristic function on V with support $\{x\}$. Such a relation is also seen in the simple fact that $\mathrm{tr}\ \mathcal{A}^n$ is equal to the number of closed paths of length n. In Section 8, we treat a counting problem of closed paths without backtracking[8], which is more subtle, and is related to $\mathcal{A}$ via the trace formula or the Ihara zeta function. In this connection, the auxiliary operator $\mathcal{A}_n$ defined by

$$(\mathcal{A}_n f)(x) = \sum_{\substack{c;\,geodesic\\ |c|=n,\,o(c)=x}} f(t(c))$$

[5]When we talk about $\mathcal{A}$, we always assume $m_V \equiv 1$.

[6]A spanning tree of X is a subgraph which has no cycles and contains all vertices of X.

[7]The *distance* between two vertices x, y is defined to be the the number of edges in the shortest path joining x and y, which we denote by $d(x,y)$.

[8]A path $c = (e_1,\ldots,e_n)$ is called a path without backtracking (or *geodesic*) if $\overline{e_{i+1}} \neq e_i$ $(i = 1,\ldots,n-1)$. If, in addition, c is closed and $\overline{e_1} \neq e_n$, then c is said to be a closed path without backtracking (or *closed geodesic*).

is useful in the case of regular graphs, where $o(c)$ (resp. $t(c)$) denotes the origin (resp. terminus) of a path c, and $|c|$ denotes the length of c, i.e., the number of edges in c. Incidentally, when X is a regular graph[9] of degree $q+1$, we have the recursion formula $\mathcal{A}_1^2 = \mathcal{A}_2 + (q+1)\mathcal{A}_0$, $\mathcal{A}_1\mathcal{A}_n = \mathcal{A}_{n+1} + q\mathcal{A}_{n-1}$ $(n \geq 2)$. We also find that the operators $T_0 = I$, $T_1 = \mathcal{A}$, $T_n = \sum_{0 \leq i \leq n/2} \mathcal{A}_{n-2i}$ $(n \geq 2)$ satisfy $T_1T_n = T_{n+1} + qT_{n-1}$ $(n \geq 1)$. We thus have the following identities which will play an important role later in our discussion (cf. [**110**]):

$$\sum_{n=0}^{\infty} \mathcal{A}_n z^n = \frac{1-z^2}{1-\mathcal{A}z+qz^2}, \tag{2.2}$$

$$\sum_{n=0}^{\infty} T_n z^n = \frac{1}{1-\mathcal{A}z+qz^2}. \tag{2.3}$$

The discrete Laplacian $\mathcal{D}-\mathcal{A}$ is canonical in the sense that it depends only on the graph structure. Another canonical one showing up frequently is $I-\mathcal{D}^{-1}\mathcal{A}$, the discrete Laplacian associated with the weight functions $m_V(x) = \deg x$, $m_E \equiv 1$. This is closely related to simple random walks on graphs (see the last section). In what follows, $\mathcal{D}-\mathcal{A}$ is said to be the *combinatorial Laplacian*, and $I-\mathcal{D}^{-1}\mathcal{A}$ is said to be the *canonical Laplacian.*

The reader already could conceive that discrete geometric analysis forms a small realm in the "world" of mathematics, which somehow reminds us of *Bonsai*, the famous Japanese traditional art of botanical miniature[10]. But this does not mean frivolousness of the field since, as will be seen subsequently, it has produced results of a surprising variety and depth[11].

A Hint to prove Dehn's theorem[12]: To construct a weighted graph $X=(V,E)$, we first orient K by choosing one of its two sides to be "horizontal". We may assume that its height= 1. Let a be its width. By setting V =the set of all vertical line segments appearing in the dissection, E^o =the set of all small rectangles, $o(e)$ =the line segment containing the left side of e. $t(e)$ =the line segment containing the right side of e, we get an oriented graph (V,E^o). Forget the orientation, and put $m_E(e)$=(height of e)/(width of e), $m_V \equiv 1$. Let x_0 and y_0 be two vertices representing the left and right sides of K, respectively, and define the function $g \in C(V)$ by $g(x_0)=1, g(y_0)=-1$, and $g(x)=0$ otherwise. Then $\Delta f = g$ has the solution f defined by $f(x)$ =distance between x_0 and the vertical line segment x. By the assumption, $\Delta f = g$ is transformed to a linear equation $A\mathbf{x} = \mathbf{b}$ with $A, \mathbf{b}$ having rational entries The existence of a real solution assures that this equation has a rational solution, and hence $\Delta f = g$ has a rational-valued solution f_0. Since $f - f_0$ is constant, we conclude that $a = f(y_0) - f(x_0)$ is rational.

[9] $|E_x|$ is called the degree of x, and is denoted by deg x. If deg x does not depend on x, then X is said to be regular.

[10] In this art, real plants such as pine and ume (apricot) trees are beautifully and skillfully miniaturized in a terse style, and their conciseness stirs up our imagination on the real beauty of their original appearances.

[11] See the prologue of N. Biggs [**13**].

[12] The proof here is a modification of the one due to R. L. Brooks, C. A. B. Smith, A. H. Stone and W. T. Thutte [**15**] who reduced the problem to that of electric circuits. The original proof due to M. Dehn [**36**] is much more complicated.

3. Abel-Jacobi maps in graph theory

The tree number we have mentioned has a group theoretic meaning. Indeed, if we put $C^0(X,\mathbb{Z})_0 = \{f \in C^0(X,\mathbb{Z});\ \sum_{x\in V} f(x) = 0\}$, then the equality (2.1) implies that $\kappa(X)$ is the order of the finite group $C^0(X,\mathbb{Z})_0/\Delta\big(C^0(X,\mathbb{Z})\big)$, where Δ is the combinatorial Laplacian. For example, when $X = K_N$, the group is isomorphic to $(\mathbb{Z}_N)^{N-2}$. This view tempts us to pursue the analogy further. This time, we consider graph theoretic analogues of algebraic-geometric concepts.

The group $C^0(X,\mathbb{Z})_0/\Delta\big(C^0(X,\mathbb{Z})\big)$ reminds us of the *Picard group* for a complete algebraic curve[13] In algebraic geometry, the Picard group is an abelian variety which is the target of the Abel-Jacobi map, a canonical regular map from the curve into Pic. In the case of graphs, taking a reference point $x_0 \in V$, we define $\Phi : V \longrightarrow \mathrm{Pic}(X) = C^0(X,\mathbb{Z})_0/\Delta\big(C^0(X,\mathbb{Z})\big)$ by $\Phi(x) = \delta_x - \delta_{x_0}$, which is entitled to be called a discrete analogue of Abel-Jacobi maps. It is easily checked that Φ is an abelian-group-valued harmonic function in the sense that $\Delta\Phi(x) = \sum_{e\in E_x}\big[\Phi\big(t(e)\big) - \Phi\big(o(e)\big)\big] = 0$, and is characterized by the following universal property: *For any harmonic function φ on V with values in an abelian group G, there exists a unique homomorphism $f : \mathrm{Pic}(X) \longrightarrow G$ such that $\varphi = f \circ \Phi$.*

The notion of graph theoretic Abel-Jacobi maps above was introduced by R. Bacher, P. De La Harpe, and T. Nagnibeda [**7**] in 1997. See also N. L. Biggs [**13**] for related topics. In connection with Abel-Jacobi maps, M. Baker and S. Norine [**8**] recently gave a discrete analogue of the classical Riemann-Roch theorem together with its application to a certain chip-firing game played on the vertices of a graph.

There is another version of Abel-Jacobi maps in graph theory ([**80**],[**81**]). Recall Abel's theorem which claims that the Picard group of a curve S is identified with the *Jacobian* (torus) $J(S) = \big(\Omega^1(S)\big)^*/H_1(S,\mathbb{Z})$ where $\Omega^1(S)$ denotes the space of holomorphic 1-forms on S, and we think of $H_1(S,\mathbb{Z})$ as a subgroup of $(\Omega^1(S))^*$ by using the paring map $([\alpha],\omega) = \displaystyle\int_\alpha \omega$. In terms of the Jacobian, the Abel-Jacobi map $\Phi : S \longrightarrow J(S)$ turns out to be the map defined by the paring

$$(\Phi(x),\omega) = \int_{x_0}^{x} \omega \pmod{H_1(S,\mathbb{Z})}. \tag{3.1}$$

Having this in mind, we define a map $\Phi : V \longrightarrow H_1(X,\mathbb{R})/H_1(X,\mathbb{Z})$ in the following way. Let $X^{ab} = (V^{ab}, E^{ab})$ be the *maximal abelian covering graph*[14] over X; in other words, X^{ab} is the covering graph over X with the covering transformation group $H_1(X,\mathbb{Z})$. Let $P : C_1(X,\mathbb{R}) \longrightarrow H_1(X,\mathbb{R}) = \mathrm{Ker}\ \partial$ be the orthogonal projection. Fix a reference point $x_0 \in V^{ab}$, and let $\widetilde{c} = (\widetilde{e_1},\dots,\widetilde{e_n})$ be a path in X^{ab} with $o(\widetilde{c}) = x_0, t(\widetilde{c}) = x$. Then put $\widetilde{\Phi}(x_0) = \mathbf{0}$ and

$$\widetilde{\Phi}(x) = P(e_1 + \cdots + e_n) = P(e_1) + \cdots + P(e_n),$$

where $e_i \in E$ is the image of $\widetilde{e_i}$ by the covering map. The $\widetilde{\Phi}$ is well-defined; namely, $\widetilde{\Phi}(x)$ is independent of the choice of $\widetilde{c}$. It should be noted that $\widetilde{\Phi}$ is completely determined by $\{P(e)\}_{e\in E}$. We extend $\widetilde{\Phi}$ to X^{ab} as a piecewise linear map. Then $\widetilde{\Phi}$

[13] For an algebraic curve S, the Picard group is defined as $\mathrm{Div}^0(S)/\mathrm{Prin}(S)$ where $\mathrm{Div}^0(S)$ is the group of divisors with degree 0, and $\mathrm{Prin}(S)$ is the group of principal divisors which are divisors of non-zero meromorphic functions on S. Note that $\Delta\log|f| = 2\pi\sum \mathrm{ord}_p(f)\delta_p$ for a meromorphic function f.

[14] Also called the *universal homology covering graph.*

satisfies $\widetilde{\Phi}(\sigma x) = \widetilde{\Phi}(x) + \sigma$ for $\sigma \in H_1(X, \mathbb{Z})$. Therefore $\widetilde{\Phi}$ induces a map $\Phi : X \longrightarrow H_1(X, \mathbb{R})/H_1(X, \mathbb{Z})$. We may check that, for $\omega \in \mathbb{H}^1(X)$,

$$(\Phi(x), \omega) = \omega(e_1) + \cdots + \omega(e_n) \pmod{H_1(X, \mathbb{Z})},$$

in which the reader may feel a flavor of Abel-Jacobi maps defined by (3.1).

The flat torus $\mathbb{A}(X) = H_1(X, \mathbb{R})/H_1(X, \mathbb{Z})$ introduced in the previous section is called the *Albanese torus*, and $\Phi : X \longrightarrow \mathbb{A}(X)$ is said to be the *Albanese map*. To see a relation between Abel-Jacobi maps and Albanese maps, note that the homology group $H_1(X, \mathbb{Z})$ is an integral lattice in $H_1(X, \mathbb{R})$ in the sense that $\alpha \cdot \beta \in \mathbb{Z}$ for $\alpha, \beta \in H_1(X, \mathbb{Z})$. Therefore the dual lattice $H_1(X, \mathbb{Z})^{\#} = \{\alpha \in H_1(X, \mathbb{R});\ \alpha \cdot \beta \in \mathbb{Z}$ for every $\beta \in H_1(X, \mathbb{Z})\}$ contains $H_1(X, \mathbb{Z})$. The *discrete Albanese torus* $A(X)$, a finite subgroup of $\mathbb{A}(X)$, is defined to be the quotient group $H_1(X, \mathbb{Z})^{\#}/H_1(X, \mathbb{Z})$. It is verified that $\Phi(V) \subset A(X)$, and that $A(X)$ is isomorphic to $\mathrm{Pic}(X)$ in a canonical way (an analogue of Abel's theorem). Under the identification $A(X) = \mathrm{Pic}(X)$, the Albanese map as a map of V into $A(X)$ coincides with the Abel-Jacobi map.

The inner product on $H_1(X, \mathbb{R})$ induces a nondegenerate symmetric bilinear form on $A(X)$ with values in $\mathbb{Q}/\mathbb{Z}$. Thinking of this form as an analogue of "polarization", one may ask whether the Torelli type theorem holds in the discrete realm. More specifically, one asks whether two regular graphs X_1 and X_2 with the same degree are isomorphic when there exists a group isomorphism between $A(X_1)$ and $A(X_2)$ preserving polarizations[15].

The Albanese map is a special harmonic map of the graph X (as a singular Riemannian space) into the flat torus $\mathbb{A}(X)$, which, *together with the flat metric*, is characterized by the minimizing property for a certain energy functional (see [**82**] for the detail, and [**74**] for harmonic maps of graphs into metric spaces). As we will see in the last section, the Albanese map (and its generalization) ties in with an asymptotic property of simple random walks on a crystal lattice.

Let us give two examples of $\widetilde{\Phi}$ to illustrate that Albanese maps are related to "crystals with big symmetry". In general, a *crystal* in mathematical sense is a graph realized in space having *periodicity* with respect to a lattice group action by translations (thus ignoring the physical characters of atoms and atomic forces in a crystal which may be different one by one). We consider the graphs (A) and (B) in Figure 1. The graph (B) is nothing but the complete graph K_4.

(A) Take three paths $c_1 = (e_0, \overline{e_1}), c_2 = (e_0, \overline{e_2}), c_3 = (e_0, \overline{e_3})$, which constitute a $\mathbb{Z}$-basis of $H_1(X, \mathbb{Z})$ satisfying $\|c_1\|^2 = \|c_2\|^2 = \|c_3\|^2 = 2$, $c_i \cdot c_j = 1 \quad (i \neq j)$. Then

$$P(e_0) = \frac{1}{4}c_1 + \frac{1}{4}c_2 + \frac{1}{4}c_3, \quad P(e_1) = -\frac{3}{4}c_1 + \frac{1}{4}c_2 + \frac{1}{4}c_3,$$
$$P(e_2) = \frac{1}{4}c_1 - \frac{3}{4}c_2 + \frac{1}{4}c_3, \quad P(e_3) = \frac{1}{4}c_1 + \frac{1}{4}c_2 - \frac{3}{4}c_3.$$

The image $\widetilde{\Phi}(X^{ab})$ turns out to be the *diamond crystal*, which describes the arrangement of carbon atoms (corresponding to vertices) together with the bonding (corresponding to edges) in diamond. The diamond crystal is a join of the same hexagonal rings (the chair conformation in chemical terminology). One observes that 12 hexagonal rings gather at each vertex.

[15]This problem was suggested by Kenichi Yoshikawa. If we would remove the conditions on degree and polarizations, there are many examples of X_1, X_2 with $A(X_1) \cong A(X_2)$ ([**7**]).

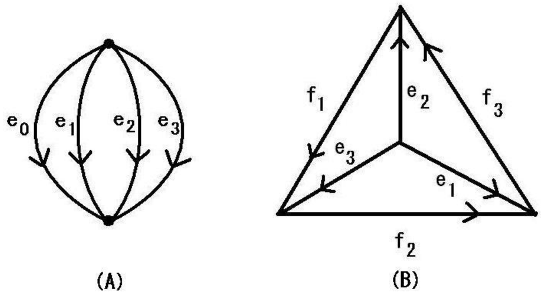

FIGURE 1

(B) Take three closed paths $c_1 = (e_2, f_1, \overline{e_3})$, $c_2 = (e_3, f_2, \overline{e_1})$, $c_3 = (e_1, f_3, \overline{e_2})$ in K_4, which constitute a $\mathbb{Z}$-basis of $H_1(K_4, \mathbb{Z})$ satisfying $\|c_1\|^2 = \|c_2\|^2 = \|c_3\|^2 = 3$, $c_i \cdot c_j = -1 \ \ (i \neq j)$. Then

$$\begin{aligned} &P(e_1) = -\frac{1}{4}c_2 + \frac{1}{4}c_3, \quad P(e_2) = \frac{1}{4}c_1 - \frac{1}{4}c_3, \quad P(e_3) = -\frac{1}{4}c_1 + \frac{1}{4}c_2, \\ &P(f_1) = \frac{1}{2}c_1 + \frac{1}{4}c_2 + \frac{1}{4}c_3, \quad P(f_2) = \frac{1}{4}c_1 + \frac{1}{2}c_2 + \frac{1}{4}c_3, \\ &P(f_3) = \frac{1}{4}c_1 + \frac{1}{4}c_2 + \frac{1}{2}c_3. \end{aligned}$$

The image $\widetilde{\Phi}(X^{ab})$ is what we call the K_4 *crystal* (see Remark below). The K_4 crystal is a web of the same decagonal rings, and the number of decagonal rings passing through each vertex is 15.

The two crystals above share two remarkable properties. Firstly, they have *maximal symmetry* in the sense that every automorphism as abstract graphs extends to a congruent transformation (actually $\widetilde{\Phi}(X^{ab})$ has always maximal symmetry for every X). Secondly they have the *strong isotropic property.* Here a graph is said to have the strong isotropic property if it is regular as a graph, say, of degree n, and if, for any $x, y \in V$ and for any permutation σ of $\{1, 2, \ldots, n\}$, there exists an automorphism g such that $gx = y$ and $ge_i = f_{\sigma(i)}$ where $E_x = \{e_1, \ldots, e_n\}, E_y = \{f_1, \ldots, f_n\}$. One can prove that *a crystal in space* ($\mathbb{R}^3$) *with these two properties is either the diamond crystal or the* K_4 *crystal* (*and its mirror image*)[16] ([**127**]).

Remark. The author learned from Stephen T. Hyde, just after the publication of [**127**], that the structure of the K_4 crystal has been known in crystallography, and that the first description of the structure goes back to a pioneer of the area, A. Wells [**135**], who called it "(10,3)-a". M. O'Keeffe and his colleagues [**37**] have discussed this structure in some details and renamed it "srs" due to its chemical relevance. See also [**108**], [**72**] and [**38**] for related matters.

4. Discrete spectral geometry

So far we have traced the history along one stream in discrete geometric analysis. Meanwhile, steady progress has been made in spectral geometry since 1960s,

[16] The K_4 crystal has *chirality* while the diamond crystal does not.

which gave rise to new motivation in graph theory, and provided us with many concepts in finite and infinite graphs ([**40**], [**41**]).

A typical example of such concepts is *Cheeger constants*, a sort of "isoperimetric" constants introduced by J. Cheeger in 1970 ([**23**]). The Cheeger constant for a weighted finite graph $X = (V, E)$ is defined by

$$h = \inf\left\{\frac{m_E(\partial_E A)}{m_V(A)};\ A \subset V,\ A \neq \emptyset,\ m_V(A) \leq (1/2)m_V(V)\right\}, \tag{4.1}$$

where $\partial_E A = \{e \in E; o(e) \in A, t(e) \notin A\}$. We further put

$$k = \sup_{x\in V} \frac{1}{m_V(x)} \sum_{e\in E_x} m_E(e).$$

Then we have $\lambda_1 \geq h^2/2k$ which is mimicry of the *Cheeger inequality* (J. Dodziuk [**40**], N. Alon [**3**]). Actually, the discrete setting allows us to prove the following sharper inequality (F. R. K. Chung [**26**], K. Fujiwara [**46**]).

$$\lambda_1 \geq k - \sqrt{k^2 - h^2}. \tag{4.2}$$

To outline the proof, we start with the following inequality

$$\frac{1}{2}\sum_{e\in E} m_E(e)|f(t(e))^2 - f(o(e))^2| \leq \|df\|(2k\|f\|^2 - \|df\|^2)^{1/2}. \tag{4.3}$$

This is an easy consequence of the Schwartz inequality, and holds for any function f. The crucial part of the proof is in showing that, if $V_+ = \operatorname{supp} f^2$ satisfies $m_V(V_+) \leq (1/2)m_V(V)$, then

$$\sum_{e\in E} m_E(e)|f(t(e))^2 - f(o(e))^2| \geq 2h\|f\|^2. \tag{4.4}$$

For this sake, we put

$$\begin{aligned}
E_0 &= \{e \in E;\ f(o(e))^2 = f(t(e))^2\}, \quad E_1 = \{e \in E;\ f(o(e))^2 > f(t(e))^2\},\\
g(V) &= \{\beta_0, \beta_1, \dots, \beta_r\} \text{ with } 0 = \beta_0 < \beta_1 < \cdots < \beta_r,\\
E_{ik} &= \{e \in E_1; f(o(e))^2 = \beta_i \text{ and } f(t(e))^2 = \beta_k\} \quad (i > k).
\end{aligned}$$

Then $E = E_1 \coprod \bar{E}_1 \coprod E_0$, $E_1 = \coprod_{i>k} E_{ik}$ and

$$\frac{1}{2}\sum_{e\in E} m_E(e)|f(t(e))^2 - f(o(e))^2| = \sum_{l=1}^{r}(\beta_l - \beta_{l-1}) \sum_{k<l\leq i}\ \sum_{e\in E_{ik}} m_E(e).$$

Put $L_i = \{x \in V; f(x)^2 \geq \beta_i\}$ $(0 \leq i \leq r)$, and note that $V = L_0 \supset L_1 \supset \cdots \supset L_r$. Since $\bigcup_{k<l\leq i} E_{ik} = \{e \in E; o(e) \in L_l, t(e) \notin L_l\} = \partial_E L_l$, we have $\sum_{k<l\leq i}\sum_{e\in E_{ik}} m_E(e) = m_E(\partial_E L_l)$. Noting that $m_V(L_l) \leq m_V(L_1) = m_V(V_+) \leq (1/2)m_V(V)$ for all $l \geq 1$, we have $m_E(\partial_E L_l) \geq h \cdot m_V(L_l)$. Therefore

$$\frac{1}{2}\sum_{e\in E} m_E(e)|f(o(e))^2 - f(t(e))^2| \geq h\sum_{l=1}^{r}(\beta_l - \beta_{l-1})m_V(L_l).$$

Since $f(x)^2 = \beta_i$ for $x \in L_i - L_{i+1}$, and $V = L_r \coprod (L_{r-1} - L_r) \coprod \dots \coprod (L_1 - L_2) \coprod (L_0 - L_1)$, we find $\sum_{l=1}^{r}(\beta_l - \beta_{l-1})m_V(L_l) = \sum_{x\in V} f(x)^2 m_V(x)$. Hence

$$\frac{1}{2}\sum_{e\in E} m_E(e)|f(o(e))^2 - f(t(e))^2| \geq h\sum_{x\in V} f(x)^2 m_V(x) = h\|f\|^2.$$

The next step is to construct a specific function f satisfying the conditon $m_V(V_+) \le (1/2)m_V(V)$. For this end, take a nonzero eigenfunction ψ for λ_1, and put $V^+ = \{x \in V;\ \psi(x) > 0\}$, $V^- = \{x \in V;\ \psi(x) < 0\}$. Then $V^+ \neq \emptyset$ and $V^- \neq \emptyset$. Without loss of generality, we may assume $m_V(V^+) \le (1/2)m_V(V)$. Then the function f defined by $f(x) = \psi(x)$ $(x \in V^+)$, $f(x) = 0$ $(x \notin V^+)$ satisfies obviously our condition so that we may put (4.3) and (4.4) together to obtain $h\|f\|^2 \le \|df\|(2k\|f\|^2 - \|df\|^2)^{1/2}$. Hence $\alpha = \|df\|^2/\|f\|^2$ satisfies $h^2 \le \alpha(2k-\alpha)$, from which the inequality $\alpha \ge k - (k^2-h^2)^{1/2}$ follows.

The final step is to see that the function f above satisfies the inequality $\|df\|^2 \le \lambda_1\|f\|^2$. This is shown by a tedious but elementary computation. We have thus obtained the desired inequality $\lambda_1 \ge \alpha \ge k - (k^2-h^2)^{1/2}$.

We also have the inequality $\lambda_1 \le 2h$ which is viewed as an analogue of Buser's inequality in spectral geometry ([**21**]), and is much easier to show than the lower bound above. In fact, if we take a subset A such that $h = m_E(\partial_E A)/m_V(A)$ and $m_V(A) \le (1/2)m_V(V)$, and put $m_V(A) = a$ and $m_V(A^c) = b$, then the function defined by $f(x) = b$ $(x \in A)$ and $f(x) = -a$ $(x \in A^c)$ satisfies $\sum_{x\in V} f(x)m_V(x) = 0$, $\|f\|^2 = m_V(A)m_V(A^c)m_V(V)$, and $\|df\|^2 = m_E(\partial_E A)\ (m_V(V))^2$. Therefore, using the following characterization of λ_1

$$\lambda_1 = \inf\left\{\frac{\|df\|^2}{\|f\|^2};\ f \in C(V) \text{ such that} \sum_{x\in V} f(x)m_V(x) = 0\right\}, \tag{4.5}$$

we obtain

$$\lambda_1 \le \frac{\|df\|^2}{\|f\|^2} = \frac{m_E(\partial_E A)m_V(V)}{m_V(A)m_V(A^c)} \le 2\frac{m_E(\partial_E A)}{m_V(A)} = 2h.$$

The Cheeger inequality for the case $m_V \equiv 1, m_E \equiv 1$ is linked to the theory of communication networks and computer science. A network is a system consisting of an enormous number of nodes (each node represents an individual) and cables (such as telephone lines). Foremost among problems in this theory is to construct (in an explicit way) a graph with the least number of edges such that every subset of vertices has the most number of distinct neighbors. This is because we would like to minimize the cost of the network by reducing the number of cables and to minimize the transmission time needed for all individuals to receive information from a few individuals. Cheeger's and Buser's inequalities tell us that the bigger the first positive eigenvalue λ_1 of the combinatorial Laplacian Δ is, the more efficient the network is, and vice versa[17]. This observation leads us to the notions of *expanders* and *Ramanujan graphs*.

An infinite family of finite graphs $\{X_n\}_{n=1}^\infty$ with $|V_n| \to \infty$ as $n \to \infty$ is said to be a family of expanders if X_n's have bounded degree and there exists a positive constant c with $\lambda_{1,n} \ge c$ for the first positive eigenvalue $\lambda_{1,n}$ of Δ_{X_n}. A regular graph of degree $q+1$ is said to be a Ramanujan graph if $\Lambda(X) \le 2\sqrt{q}$ where $\Lambda(X) = \max\{|\mu_i|;\ \mu_i$ is an eigenvalue of the adjacency operator and $|\mu_i| \neq q+1\}$[18]. Obviously an infinite family of Ramanujan graphs with a fixed degree

[17] This was pointed out first by N. Alon [**3**] and N. Alon and V. D. Milman [**4**]. The Cheeger constant is essentially the same as the *magnifying constant* which represents the degree of efficiency of a network.

[18] $q+1$ is always an eigenvalue, while $-(q+1)$ is an eigenvalue if and only if X is *bipartite*. Here a graph is said to be bipartite if one can paint the vertices by two colours in such a way that any adjacent vertices have different colours.

is a family of expanders. Thus our task is to construct a family of expanders or preferably a family of Ramanujan graphs with a fixed degree[19]. The reason why the quantity $2\sqrt{q}$ comes up in the definition of Ramanujan graphs is that, for any infinite family $\{X_n\}$ of regular graphs of degree $q+1$, we have

$$\liminf_{n\to\infty} \Lambda(X_n) \geq 2\sqrt{q}, \tag{4.6}$$

which implies that it is impossible to construct an infinite family $\{X_n\}$ with $\Lambda(X_n) \leq 2\sqrt{q}-\epsilon$ ($\epsilon > 0$). An explicit construction of a family of Ramanujan graphs, which is by no means trivial, was given by A. Lubotzky, R. Phillips and P. Sarnak [**90**]. They showed that the Cayley graph associated with the group $PGL_2(\mathbb{Z}/p\mathbb{Z})$ (or $PSL_2(\mathbb{Z}/p\mathbb{Z})$) and a suitable set of generators is Ramanujan. Their proof relies heavily on the theory of automorphic forms and the classical Ramanujan conjecture (see the end of Section 7, and [**105**]). Lubotzky [**91**] gives a recipe relying on the representation theory of $SL_2(\mathbb{Q}_p)$. See also A. Terras [**129**] for other examples in which the notion of finite symmetric spaces is introduced. As for expanders, see J. Bourgain and A. Gamburd [**20**] for instance[20]. They showed, among other things, that, given a subset A of $SL_2(\mathbb{Z})$, Cayley graphs $X(SL_2(\mathbb{Z}/p\mathbb{Z}), A_p)$ constitute a family of expanders as $p\to\infty$ if and only if the subgroup $\langle A\rangle$ of $SL_2(\mathbb{Z})$ generated by A does not contain a solvable subgroup of finite index. Here A_p is the image in $SL_2(\mathbb{Z}/p\mathbb{Z})$ of A (see Section 7 for the definition of Cayley graphs).

Thus far, we have confined ourselves to a special topic in discrete spectral geometry. There are, of course, other topics of interest which also fall under the same title. Let us list some of them.

(1) The notion of diameters is also important in the theory of communication networks since it is related to the transmission time of information. There are many articles handling spectral diameter estimates. For example, see N. Alon and V.D. Milman [**4**]. F. R. K. Chung, V. Faber and T. A. Manteuffel [**28**] and P. Sarnak [**105**] derived the following upper bound on the diameter,

$$\mathrm{diam}(X) \leq \frac{\operatorname{arccosh}(N-1)}{\operatorname{arccosh}\big((q+1)/\Lambda(X)\big)} + 1.$$

A partial improvement of this bound was given by G. Quenell [**103**].

(2) The graph version of Courant's celebrated Nodal Domain Theorem was established by E. B. Davies, G. M. Gladwell, J. Leydold and P. F. Stadler [**35**]. The problem is to count the number of nodal domains of an eigenfunction[21]. Remember that, in the continuous setting, a nodal domain of a continuous function is defined to be a connected component of the complement of $f^{-1}(0)$. A crucial point in the discrete setting is to find a reasonable formulation for nodal domains because a function on a graph can change the sign without having zero. For the recent development, see G. Berkolaiko [**12**] and R. Band, I. Oren, and U. Smilansky [**9**].

(3) Two finite graphs are said to be *isospectral* if they have the same eigenvalues with the same multiplicities. The first example of non-isomorphic isospectral graphs for the Laplacian $\mathcal{D}-\mathcal{A}$ was given by L. Collatz and U. Sinogowitz [**32**].

[19]The first example of Ramanujan graphs appeared implicitly in Y. Ihara [**69**].

[20]An explicit construction dates back to the pioneer work by G. A. Margulis [**92**].

[21]This question started with Sturm's oscillations theorem which states that a vibrating string is divided into exactly n nodal intervals by the zeros of its n-th mode of vibrations.

See [**34**] for more examples. The group theoretic method for constructing isospectral manifolds [**121**] can be exported to the discrete case (see R. Brooks [**19**] for interesting discussion).

5. Spectra of infinite graphs

In the case of an infinite weighted graph, we have, in general, the continuous spectrum which may make the spectral problem more complicated. As a matter of fact, much has not yet been explored in the general setting. However as far as the bounds of the spectrum are concerned, the ideas in the previous section work well.

W denote by $\sigma(T)$ the spectrum of a hermitian operator T acting in a Hilbert space. We shall make use of the following notations

$$\begin{aligned} \ell^2(V) &= \{f \in C(V);\ \|f\|^2 := \sum_{x\in V} f(x)^2 m_V(x) < \infty\}, \\ \ell^2(E) &= \{\omega \in C(E);\ \omega(\overline{e}) = -\omega(e),\ \|\omega\|^2 := \frac{1}{2}\sum_{e\in E}\omega(e)^2 m_E(e) < \infty\}, \end{aligned}$$

and we restrict ourselves to the class of weighted graphs satisfying the condition

$$k = \sup_{x\in V}\frac{1}{m_V(x)}\sum_{e\in E_x} m_E(e) < \infty,$$

under which Δ extends to a hermitian operator of $\ell^2(V)$. The *Cheeger constant* h for the infinite graph[22] is defined by

$$h = \inf_{A\subset V}\frac{m_E(\partial_E A)}{m_V(A)}, \tag{5.1}$$

where the infimum is taken over all finite subsets A of V. With this h, two inequalities (4.3), (4.4) hold for *every* function f on V with finite support. Indeed, if we look at that part in the proof of the inequality (4.4) where the constant h shows up, then we find that an arbitrary function f satisfies (4.4) because the condition $m_V(A) \le (1/2)m_V(V)$ is removed in the definition of h above. Therefore $k - (k^2 - h^2)^{1/2} \le \|df\|^2/\|f\|^2 \le k + (k^2 - h^2)^{1/2}$. This implies

$$\sigma(\Delta) \subset [k - (k^2 - h^2)^{1/2}, k + (k^2 - h^2)^{1/2}]. \tag{5.2}$$

We also have $\inf\sigma(\Delta) \le h$ since the defining function $f = \chi_A$ of a subset A satisfies $\|f\|^2 = m_V(A)$, and $\|df\|^2 = m_E(\partial_E A)$. In particular, $\inf\sigma(\Delta) = 0$ if and only if $h = 0$.

In the case of $\Delta = \mathcal{D} - \mathcal{A}$ on the regular tree of degree $q+1$, we have $k = q+1$ and $h = q-1$, and hence $\sigma(\Delta) \subset [q+1-2\sqrt{q}, q+1+2\sqrt{q}]$. This is equivalent to $\sigma(\mathcal{A}) \subset [-2\sqrt{q}, 2\sqrt{q}]$. Therefore the estimate above is sharp.

The *Cheeger constant at infinity* h_∞, which is related to the essential spectrum $\sigma_{ess}(\Delta)$, is defined as follows. For a finite subset K of V, we put

$$\begin{aligned} k(K) &= \sup_{x\in K^c}\frac{1}{m_V(x)}\sum_{e\in E_x} m_E(e), \\ h(K) &= \inf\left\{\frac{m_E(\partial_E A)}{m_V A}; A\subset V, |A| < \infty,\ \text{and } A\cap K = \emptyset\right\}. \end{aligned}$$

[22] This is a discrete analogue of the one for open manifolds (R. Brooks [**17**]).

We then put $h_\infty = \sup_{K\subset V} h(K)$, $k_\infty = \inf_{K\subset V} k(K)$ (note $h \le h_\infty \le k_\infty \le k$). It is shown (K. Fujiwara [**46**]) that $\sigma_{ess}(\Delta) \subset [k_\infty - (k_\infty^2 - h_\infty^2)^{1/2}, k_\infty + (k_\infty^2 - h_\infty^2)^{1/2}]$, and $\inf \sigma_{ess}(\Delta) \le h_\infty$. In particular, if $h_\infty = k_\infty$, then $\sigma_{ess}(\Delta) = \{k_\infty\}$ (the converse is also true). As a corollary, one can show that, if X is a *rapidly branching tree* in the sense that $\sup_{K\subset V} \inf_{x\in K^c} \deg x = \infty$, then $\sigma(\Delta)\backslash\{1\}$ consists of eigenvalues with finite multiplicity[23] for the canonical Laplacian $\Delta = I - \mathcal{D}^{-1}\mathcal{A}$.

We shall mention a few more results.

(1) For $\Delta = I - \mathcal{D}^{-1}\mathcal{A}$, $\inf \sigma(\Delta) + \sup \sigma(\Delta) \ge 2$ ([**64**]).

(2) For $\Delta = I - \mathcal{D}^{-1}\mathcal{A}$, $\sup \sigma_{ess}(\Delta) \ge 1 + 2d_\infty^{-1}$, where $d_\infty = \inf_K \sup_{x\in K^c} \deg x$ ([**61**]). In particular, if $\sup_{x\in V} \deg x = M(<\infty)$, then $\sup \sigma_{ess}(\Delta) \ge 1 + 2M^{-1}$.

(3) The line graph construction yields infinite graphs with many isolated eigenvalues ([**112**]).

(4) Let $B_n(x_0) = \{x \in V;\ d(x, x_0) \le n\}$ be the ball with radius n with respect to the graph distance d, and let $b_n = m_V(B_n(x_0))$. Define the *growth rate* of X by $b = \limsup_{n\to\infty} b_n^{1/n}$ $(\le \infty)$, which does not depend on the choice of x_0. One may easily prove the inequality $b \ge k^{-1}h + 1$. Under the condition $m_V(V) = \infty$, the following inequality was established by K. Fujiwara [**47**] for the canonical Laplacian

$$\inf \sigma_{ess}(\Delta) \le 1 - \frac{2}{b^{1/2} + b^{-1/2}}. \tag{5.3}$$

This inequality improves several previous results given by J. Dodziuk and L. Karp [**42**] and Y. Ohno and H. Urakawa [**100**]. Actually estimate (5.3) is sharp since the equality holds for the regular tree.

6. Covering graphs

The proof of inequality (4.6) due to Allon and Boppana relies on the fact that the universal covering graph over a regular graph is a regular tree, and that the spectrum of the adjacency operator on the regular tree of degree $q+1$ is $[-2\sqrt{q}, 2\sqrt{q}]$. This example leads us to a herd of problems about spectra of covering graphs over more general finite graphs. Another motivation stems from the spectral study of *Cayley graphs* which are nothing but regular covering graphs over bouquet graphs[24].

Consider a regular covering map $X \longrightarrow X_0 = (V_0, E_0)$ over a finite weighted graph, and let G be the covering transformation group. Lifting the weight functions on X_0 makes X a weighted graph. Our concern is the spectrum of Δ_X. It is observed that, if G is of infinite order, then there exist no eigenvalues with finite multiplicity so that $\sigma_{ess}(\Delta_X) = \sigma(\Delta_X)$.

A tool to treat this situation is *twisted Laplacians.* Given a unitary representation $\rho : G \longrightarrow U(\mathbb{H})$ on a Hilbert space $\mathbb{H}$, we set $\ell^2(\rho) = \{f; V \longrightarrow \mathbb{H};\ f(gx) = \rho(g)f(x)\}$, which has a natural Hilbert space structure. The twisted Laplacian Δ_ρ (resp. the twisted adjacency operator $\mathcal{A}_\rho$) is defined to be the restriction to $\ell^2(\rho)$ of the discrete Laplacian (resp. adjacency operator) acting on $\mathbb{H}$-valued functions. One can establish the following estimates for the bottom of the spectrum $\lambda_0(\rho) = \inf \sigma(\Delta_\rho)$ from below and above

$$c_1\delta(\rho, \mathbf{1})^2 \le \lambda_0(\rho) \le c_2\delta(\rho, \mathbf{1})^2. \tag{6.1}$$

[23]This is regarded as a discrete analogue of the result due to H. Donnelly and P. Li [**43**].

[24]A bouquet graph is a graph with a single vertex and several loop edges.

Here c_1, c_2 are positive constants independent of ρ (see [**125**], [**128**]). The quantity $\delta(\rho, \mathbf{1})$ is the *Kazhdan distance* between ρ and the trivial representation $\mathbf{1}$ defined by $\delta(\rho, \mathbf{1}) = \inf\{\sup_{g\in A} \|\rho(g)v - v\|;\ v \in \mathbb{H},\ \|v\| = 1\}$, where A is a finite set of generators of G. (6.1), in particular, says that $\lambda_0(\rho) = 0$ if and only if $\delta(\rho, \mathbf{1}) = 0$. Applying this to the regular representation[25] $\rho_H : G \longrightarrow U\big(\ell^2(H\backslash G)\big)$ associated with a subgroup H of G, we conclude that $\lambda_0(\Delta_{H\backslash X}) = 0$ if and only if $\delta(\rho_H, \mathbf{1}) = 0$ since Δ_{ρ_H} is unitarily equivalent to the discrete Laplacian $\Delta_{H\backslash X}$ on the quotient graph $H\backslash X$ [26]. In particular, since $\delta(\rho_G, \mathbf{1}) = 0$ if and only if G is *amenable*[27] (cf. R. J. Zimmer [**138**]), we find that $\lambda_0(\Delta_X) = 0$ if and only if G is amenable. This is a discrete analogue of a result due to R. Brooks [**16**].

Another effective use of (6.1) is seen in a construction of a family of expanders by means of a tower of covering graphs. For a fixed finite graph X_0, let $\cdots \to X_n \to \cdots \to X_1 \to X_0$ be a sequence of finite-fold covering maps. Suppose that every covering map $X_n \longrightarrow X_0$ is a subcovering map of a fixed regular covering map $X \longrightarrow X_0$. If the covering transformation group G satisfies the *(Kazhdan) property* (T), then there exists a positive constant c such that $\lambda_1(X_n) \geq c$ for every n. Here G is said to have the Kazhdan property (T) (or to be a Kazhdan group) if there exists a positive constant c such that $\delta(\rho, \mathbf{1}) \geq c$ for every non-trivial irreducible representation ρ of G ([**138**]).

As for the property (T), A. Zuk [**139**] gave an interesting criterion in terms of the smallest positive eigenvalue of a discrete Laplacian defined on a finite set of generators. A typical example of Kazhdan groups is $SL_n(\mathbb{Z})$ $(n \geq 3)$. The rotation group $SO(n)$ $(n \geq 5)$ has a finitely generated dense Kazhdan subgroup. This fact has been used to solve Ruziewicz's problem of the uniqueness of rotationally invariant finitely additive measures defined on Lebesgue sets on S^{n-1} (G. A. Margulis [**93**] and D. Sullivan [**118**]). The proof of uniqueness reduces to the existence of an ϵ-good set in $SO(n)$, where a finite set $A \subset SO(n)$ said to be an ϵ-good set if $\|L_a f - f\|_2 \geq \epsilon \|f\|_2$ for $a \in A$ and $f \in L^2(S^{n-1})$ with $\displaystyle\int_{S^{n-1}} g = 0$. Here $L_a f(x) = f(a^{-1}x)$. One easily infers that, if the group G generated by A is dense in $SO(n)$ and has the property (T), then A is an ϵ-good set for some $\epsilon > 0$. The uniqueness also holds for S^2 and S^3 (V. G. Drinfeld [**44**]). Although this problem is seemingly unrelated to an explicit construction of Ramanujan graphs, both problems can be solved using similar methods. See [**105**] and [**91**] for details[28].

Useful for the study of spectra (in principle at least) is the notion of (*integrated*) *densities of states*, which was originally introduced in quantum physics and quantifies how closely packed energy levels are in a quantum-mechanical system. To define it in our setting, we start with the notion of G-trace. For a G-equivariant bounded operator $T : \ell^2(V) \longrightarrow \ell^2(V)$ on a regular covering graph $X \xrightarrow{G} X_0$ over a finite graph, we define the G-*trace* of T (cf. Atiyah [**5**])[29] by

$$\mathrm{tr}_G T = \sum_{x\in\mathcal{F}} m_V(x)^{-1} \langle T(\delta_x), \delta_x\rangle = \sum_{x\in\mathcal{F}} t(x,x) m_V(x),$$

[25] ρ_H is the induced representation $\mathrm{Ind}_H^G(\mathbf{1})$ of the trivial representation.

[26] This implies that, if ρ_{H_1} and ρ_{H_2} are equivalent for two subgroups H_1, H_2, then $\Delta_{H_1\backslash X}$ and $\Delta_{H_1\backslash X}$ are unitary equivalent. This gives a method to construct isospectral graphs.

[27] This is equivalent to that G has an *invariant mean*.

[28] The uniqueness breaks down for S^1.

[29] The G-trace is a special case of von Neumann traces in operator algebras.

where $\mathcal{F}$ is a fundamental set in V for the G-action, and $t(x,y)$ stands for the kernel function of T; namely $Tf(x) = \sum_{y\in V} t(x,y)f(y)m_V(y)$. Suppose that T is hermitian, and let $T = \int_{-\infty}^{\infty} \lambda dE(\lambda)$ be the spectral resolution of T. The integrated density of states for T is defined by $\rho_T(\lambda) = \mathrm{tr}_G E(\lambda)$. Evidently the function ρ_T is non-decreasing, right-continuous and $\sigma(T) = \mathrm{supp}\ d\rho_T$.

Again let $X \to \cdots \to X_n \to \cdots \to X_1 \to X_0$ be a tower of finite-fold subcovering graphs over X_0. We denote by G_n the covering transformation group of the covering map $\pi : X \longrightarrow X_n$. Suppose that $t(x,y)$ has finite support as a function on $G\backslash(X \times X)$ (note that $t(\sigma x, \sigma y) = t(x,y)$ for every $\sigma \in G$). Then for each n, there exists a unique $T_n : C(V_n) \longrightarrow C(V_n)$ such that $(T_n f)\circ\pi = T(f\circ\pi)$ for $f \in C(V_n)$ (for instance, $T_n = \Delta_{X_n}$ for $T = \Delta_X$). Let $\{\lambda_k\}$ be the eigenvalue of T_n, and define the measure $d\rho_n$ on $\mathbb{R}$ by $\int_{\mathbb{R}} f\ d\rho_n = \sum_k f(\lambda_k)$. We then have

$$w\text{-}\lim_{n\to\infty} |G/G_n|^{-1} d\rho_n = d\rho_T, \tag{6.2}$$

provided that $\bigcap_n G_n = \{1\}$. This justifies the name "(integrated) density of states" for ρ_T. The proof relies on the following "pretrace formula"[30] for a G-equivariant operator S

$$\mathrm{tr}\ S_0 = \mathrm{tr}_G S + \sum_{g\neq 1\in G}\sum_{x\in\mathcal{F}} s(gx,x)m_V(x). \tag{6.3}$$

See [**126**] for the idea, and [**49**] for a generalization to nonregular covering graphs[31].

In case G is an abelian group of infinite order, the covering graph X is called a *crystal lattice*, an abstraction of a crystal. This being the case, the unitary equivalence between Δ_X and Δ_{ρ_G} and the irreducible decomposition of ρ_G yield the direct integral decomposition over the unitary character group $\widehat{G}$:

$$\Delta_X = \int_{\widehat{G}}^{\oplus} \Delta_\chi\ d\chi.$$

This is substantially the Bloch-Floquet theorem applied to discrete Laplacians. Let $0 \le \lambda_0(\chi) \le \lambda_1(\chi) \le \cdots \le \lambda_{N-1}(\chi)$ be the eigenvalues of Δ_χ. Then each λ_i is a continuous function on $\widehat{G}$, and $\sigma(\Delta_X) = \bigcup_{i=0}^{N-1}\{\lambda_i(\chi);\ \chi\in\widehat{G}\}$. In particular, the spectrum has a band structure in the sense that $\sigma(\Delta_X)$ is the union of N closed intervals, some of which possibly overlap. It is an interesting problem to find the number of gaps in the spectrum. An extreme situation is the case of the *maximal abelian covering graph* X over a finite graph. In this case, Yu. Higuchi and T. Shirai [**64**] proved that there is no gap for the canonical Laplacian $\Delta = I - \mathcal{D}^{-1}\mathcal{A}$ provided that the degree is *even*. Their proof relies on the famous solution to the puzzle of the Königsberg bridges due to Euler. Actually, one may construct a one-parameter family of unitary characters χ_t such that a branch of perturbed eigenvalues $\{\lambda_i(\chi_t)\}$ covers the whole interval $[0,2]$. Such χ_t is obtained by using a closed path (Euler path) in X_0 such that every unoriented edge appears in it once and only once. We conjecture that no-gap phenomenon holds for regular graphs of odd degree as well (see [**65**]).

[30](6.3) is formally transformed to the identity $\mathrm{tr}\ S_0 = \sum_{[g]\in[G]} \mathrm{tr}_{G_g}(gS)$, where $[G]$ is the set of conjugacy classes, and G_g denotes the centralizer of g in G.

[31]In [**49**], $d\rho_T$ is called the Kesten measure.

For a general abelian covering graph, one may introduce the notions of Bloch varieties and Fermi surfaces which have something to do with an integral representation of the Green kernel (cf. [**87**]). One can also consider twisted operators associated with real characters, which are not hermitian any more, but useful when we explore asymptotic behaviors of random walks on crystal lattices (see the last section, and [**89**] for nilpotent coverings).

7. Cayley graphs

It is worthwhile to treat the case of Cayley graphs separately as group structures are more directly reflected in the spectra. Here we restrict ourselves to the notion of *cogrowth* introduced by R. I. Grigorchuk [**48**][32].

First we shall fix our terminology. Let G be a group, and $i : A \longrightarrow G$ be a map of a finite set A into G such that $i(A)$ generates G. We put $q = 2|A| - 1$. Let $\overline{A} = \{\overline{a};\ a \in A\}$ be a disjoint copy of A. A *word* with letters in A means either void (denoted by $\emptyset$) or a finite sequence $w = (b_1, \ldots, b_n)$ with $b_i \in A \cup \overline{A}$. The *length* n of a word $w = (b_1, \ldots, b_n)$ is denoted by $|w|$ ($|\emptyset| = 0$). A word $w = (b_1, \ldots, b_n)$ is said to be *reduced* if $\overline{b_{i+1}} \neq b_i$ $(i = 1, \ldots, n-1)$, where $\overline{\overline{a}}$ is understood to be a. Denote by $g(w) \in G$ the product $i(b_1) \cdots i(b_n)$ $(g(\emptyset) = 1)$, where $i(\overline{a})$ is understood to be $i(a)^{-1}$.

The *Cayley graph* $X(G, A)$ is defined as follows[33]. The set V of vertices is just G. Oriented edges are the pairs (g, a), $g \in G$, $a \in A$. The origin and terminus of the edge (g, a) are defined to be g and $gi(a)$, respectively. Forgetting orientation, we get a connected regular graph $X(G, A)$ of degree $q+1$. The graph $X(G, A)$ has a natural free G-action. Thus given a subgroup H of G, one can consider the quotient graph $H\backslash X(G, A)$, called the *Schreier graph*, whose vertices are right cosets of H.

The adjacency operator on $X(G, A)$ is simply expressed as

$$\mathcal{A}f(g) = \sum_{a \in A} \big[f\big(gi(a)\big) + f\big(gi(a)^{-1}\big)\big].$$

$\mathcal{A} : \ell^2(G) \longrightarrow \ell^2(G)$ is G-equivariant. Note that the twisted adjacency operator $\mathcal{A}_\rho$ is identified with the operator $\sum_{a \in A} \big[\rho\big(i(a)\big) + \rho\big(i(a)^{-1}\big)\big]$ acting in $\mathbb{H}$, which is called the *Hecke operator* associated with ρ.

We define the cogrowth sequence $\{\ell_n\}_{n=0}^{\infty}$ of (G, A) by

$$\ell_n = \big|\{w;\ w \text{ is a reduced word over } A \text{ with } g(w) = 1 \text{ and } |w| \leq n\}\big|.$$

In terms of Cayley graphs, ℓ_n is the number of geodesic loops in $X(G, A)$ with the base point 1 and length $\leq n$. It is also described in the following way. Let F be the free group with the free basis A. Let H be the kernel of the canonical homomorphism of F onto G. Then $X(G, A)$ is identified with the quotient graph $H\backslash X(F, A)$. Since $X(F, A)$ is a regular tree, and is the universal covering graph over $X(G, A)$, applying the unique lifting property of covering maps, we have $\ell_n = \big|\{h \in H;\ d(1, h) \leq n\}\big|$, where d is the distance function on $X(F, H)$.

The cogrowth rate is defined to be $\ell = \lim\limits_{n \to \infty} \ell_n^{1/n}$. Grigorchuk showed that this limit exists, and

(i) $1 \leq \ell \leq q$;
(ii) $\ell = 1$ if and only if G is the free group with the basis A;

[32] See also J. M. Cohen [**30**].
[33] Our definiton of Cayley graphs is slightly different from the conventional one (cf. [**110**]).

(iii) $\ell = q$ if and only if G is amenable;
(iv) if G is not a free group, then $q^{1/2} < \ell \leq q$.

The cogrowth sequence is directly related to the adjacency operator $\mathcal{A}$ on the Cayley graph $X(G, A)$ by the formula

$$\sum_{n=0}^{\infty} \ell_n z^n = \mathrm{tr}_G \left(\frac{1+z}{1 - \mathcal{A}z + qz^2} \right). \tag{7.1}$$

Note that $\mathrm{tr}_G T = \langle T\delta_1, \delta_1 \rangle$. The identity (7.1) is a consequence of (2.2). Indeed, if we put

$$\begin{aligned} m_n &= \left| \{w;\ w \text{ is a reduced word over } A \text{ with } g(w) = 1 \text{ and } |w| = n\} \right| \\ &= \left| \{h \in H;\ d(1,h) = n\} \right|, \end{aligned}$$

then (2.2) and the identity $m_n = \mathrm{tr}_G \mathcal{A}_n$ lead to

$$\sum_{n=0}^{\infty} m_n z^n = \mathrm{tr}_G \left(\frac{1-z^2}{1 - \mathcal{A}z + qz^2} \right).$$

Since $\sum_{n=0}^{\infty} m_n z^n = (1-z) \sum_{n=0}^{\infty} \ell_n z^n$, we get (7.1).

In the case where G is of infinite order, we have, for $\alpha = \sup \sigma(\mathcal{A})$
(i) $2q^{1/2} \leq \alpha \leq q+1$;
(ii) $\alpha = 2q^{1/2}$ if and only if G is a free group with the basis A;
(iii) $\alpha = q+1$ if and only if G is amenable;
(iv) $\ell = \big(\alpha + (\alpha^2 - 4q)^{1/2}\big)/2$ provided that G is not free.

The cogrowth rate is a complementary concept of the growth rate which is more classical in theory of discrete groups. In group-theoretic terms, it is defined as $b = \lim_{n\to\infty} b_n^{1/n}$ where $b_n = \big|\{g \in G;\ \text{there exists a word } w \text{ with } g = g(w),\ |w| \leq n\}\big|$. The growth rate enjoys similar statements as the cogrowth rate.
(i) $1 \leq b \leq q$;
(ii) if $b = 1$, then G is amenable;
(iii) if G is the free group with the basis A, then $b = q$.

The relation between b and the spectrum is less exact. The following is a special case of inequality (5.3) (assume $|G| = \infty$).

$$\sup \sigma(\mathcal{A}) \ \geq \ \frac{2(q+1)}{b^{1/2} + b^{-1/2}}.$$

The spectrum of $\mathcal{A}$ seems to have a band structure for a broad class of finitely generated groups G. A sufficient condition is given in terms of group C^*-algebras. Let $C^*(G)$ be the C^*-algebra which is defined as the norm-completion of the algebra of equivariant bounded operators T on $\ell^2(V)$ such that $T\delta_x$ has finite support for every $x \in G$ (the algebra $C^*(G)$ is what we call the *reduced C^*-group algebra* of G). Obviously $\mathcal{A} \in C^*(G)$. Suppose that there exists a positive constant c such that $\mathrm{tr}_G P \geq c$ for every non-zero projection $P \in C^*(G)$. Then $E(a) - E(b) \in C^*(G)$ for $a > b$ with $a, b \notin \sigma(\mathcal{A})$, where $\{E(\lambda)\}$ is the spectral resolution for $\mathcal{A}$. Hence $\sigma(\mathcal{A})$ is a finite union of closed intervals. Free groups and surface groups are such examples satisfying this condition.

On the other hand, there is a class of Schreier graphs such that the spectra of the adjacency operators have peculiar structures; for instance, it may happen that the spectrum is the union of a set of isolated points S and a Cantor set C where C consists of accumulation points of S and is obtained as the Julia set of a certain polynomial. Examples belonging to such a class arise from the Hanoi Towers Game which is described in terms of a rooted tree and its automorphism group. See [**50**] for the detail[34]. The reference [**49**] provides many examples with interesting spectral properties.

Let us mention a useful formula which was skillfully applied to the construction of Ramanujan graphs in [**90**]. Suppose that G is a finite group with a finite set of generators A, and take the free group F and its subgroup H as before. If we put

$$t_n = \sum_{k=0}^{[n/2]} m_{n-2k} = \sum_{k=0}^{[n/2]} \big|\{h \in H;\ d(1,h) = n-2k\}\big|,$$

then, in veiw of (2.3), we find

$$\sum_{n=0}^{\infty} t_n z^n = \mathrm{tr}_G \frac{1}{1-\mathcal{A}z+qz^2} = \frac{1}{|G|}\mathrm{tr}\frac{1}{1-\mathcal{A}z+qz^2} = \frac{1}{N}\sum_{i=0}^{N-1}\frac{1}{1-\mu_i z+qz^2},$$

where $N = |G|$ and $q+1 = \mu_0 > \mu_1 \geq \cdots \geq \mu_{N-1} \geq -(q+1)$ are eigenvalues of $\mathcal{A}$. Recall that the *Chebychev polynomial* U_n of the second kind is defined by $U_n(\cos\theta) = \sin(n+1)\theta/\sin\theta$ and satisfies

$$\sum_{n=0}^{\infty} U_n(\mu) z^n = \frac{1}{1-2\mu z+z^2}. \tag{7.2}$$

Obviously $U_{2n-1}(-\mu) = -U_{2n-1}(\mu)$, $U_{2n}(-\mu) = U_{2n}(\mu)$, and

$$U_n(\mu) = \begin{cases} \frac{1}{\alpha-\beta}(\beta^{-n-1}-\alpha^{-n-1}) & (\mu^2 \neq 1) \\ n+1 & (\mu = 1) \\ (-1)^n(n+1) & (\mu = -1) \end{cases},$$

where α, β are the solutions of $z^2 - 2\mu z + 1 = 0$. Comparing (7.2) with (2.3) above, we have[35]

$$t_n = \frac{q^{n/2}}{N}\sum_{i=0}^{N-1} U_n\Big(\frac{\mu_i}{2\sqrt{q}}\Big). \tag{7.3}$$

We also find

$$\begin{cases} q^{n/2}U_n\Big(\frac{q+1}{2\sqrt{q}}\Big) = \frac{q^{n+1}-1}{q-1} = \sum_{d|q^n} d, \\ q^{n/2}U_n\Big(\frac{-(q+1)}{2\sqrt{q}}\Big) = (-1)^n\frac{q^{n+1}-1}{q-1} = (-1)^n\sum_{d|q^n} d, \\ q^{n/2}U_n\Big(\frac{\mu_i}{2\sqrt{q}}\Big) = o(q^n) \quad (|\mu_i| < q+1). \end{cases} \tag{7.4}$$

When $H = F$, we have $t_n = \sum_{d|q^n} d$, and $8t_n$ coincides with the number of representations of q^n as a sum of 4 squares provided that q is an odd prime (Jacobi). This observation gives rise to anticipation that a number theoretic meaning may be given to t_n for a class of normal sungroups H.

In [**90**], starting from the ring of integral quaternions $H(\mathbb{Z})$, Lubotzky, Phillips and Sarnak constructed a finite Cayley graph $X(G,A) = H\backslash X(F,A)$, with $G =$

[34]The group studied in [**50**] has many interesting properties. For instance, it is amenable and not elementary (see [**134**]).

[35]This formula can also be derived from the pre-trace formula (see [**90**]).

$PGL_2(\mathbb{Z}/p\mathbb{Z})$ or $PSL_2(\mathbb{Z}/p\mathbb{Z})$, of degree $q+1$ such that $2t_n$ is expressed as the number of representatives of q^n by the quadratic form ${x_1}^2+(2p)^2{x_2}^2+(2p)^2{x_3}^2+(2p)^2{x_4}^2$, where p, q are unequal primes both $\equiv 1 \pmod 4$. Thus $2t_n$ is the Fourier coefficient of a modular form of weight two for the congruence subgroup $\Gamma(16p^2)$, and hence expressed as the sum of the Fourier coefficient $a(q^n)$ of a cusp form and the coefficient $\delta(q^n)$ of an Eisenstein series. The *Ramanujan conjecture* (now a theorem) says $a(q^n) = O_\epsilon(q^{n(1/2+\epsilon)})$ for an arbitrary positive ϵ. On the other hand, δ is of the form $\delta(m) = \sum_{d|m} dS(d)$ with a periodic function S on $\mathbb{N}$. It is easily verified that, if $\sum_{d|q^n} dR(d) = o(q^n)$ for a periodic function R on $\mathbb{N}$, then $\sum_{d|q^n} dR(d) = 0$. Using this fact, (7.4), and the Ramanujan bound, we find that, in the equality (7.3) (twice both sides, to be exact, so that the left hand side is $a(p^n)+\delta(p^n)$), the terms $\delta(p^n)$ and $2N^{-1}q^{n/2}U_n\Big(\frac{q+1}{2\sqrt{q}}\Big)$ (and $2N^{-1}q^{n/2}U_n\Big(-\frac{q+1}{2\sqrt{q}}\Big)$ if this appears) are cancelled out. We therefore get, as $n \to \infty$,

$$\sum_{i=1}^{N-2} U_n\Big(\frac{\mu_i}{2\sqrt{q}}\Big) = O_\epsilon(q^{n\epsilon}) \quad \text{(bipartite case)}, \tag{7.5}$$

$$\sum_{i=1}^{N-1} U_n\Big(\frac{\mu_i}{2\sqrt{q}}\Big) = O_\epsilon(q^{n\epsilon}) \quad \text{(nonbipartite case)}. \tag{7.6}$$

This implies that $|\mu_i| \le 2\sqrt{q}$ for $1 \le i \le N-1$ in non-bipartite case, and $|\mu_i| \le 2\sqrt{q}$ for $1 \le i \le N-2$ in bipartite case. For, noting that $|\alpha| = |\beta| = 1$ if $|\mu| \le 1$, we find that, on the left hand sides of (7.5) and (7.6), the terms corresponding to μ_i with $|\mu_i| > 2\sqrt{q}$ grow faster than $q^{n\epsilon}$ if we take ϵ small enough, while the terms corresponding to μ_i with $|\mu_i| \le 2\sqrt{q}$ are bounded or has the bound $n+1$. Hence taking a look at (7.5) and (7.6) for even n (to avoid cancellations), we get the claim. Thus $\Lambda\big(X(G,A)\big) \le 2\sqrt{q}$, and the graph $X(G,A)$ is Ramanujan.

8. Ihara zeta functions

Zeta functions are the most favorite things for many mathematicians since the famous Riemann Hypothesis (RH) for the classical zeta function $\zeta(s) = \prod_p(1-p^{-s})^{-1}$ still remains to be solved as a challenging problem in the 21$^{\text{st}}$ century, whereas zeta functions some of which satisfy analogues of RH have been innovated in various fields. A typical zeta function satisfying the RH is the Weil zeta function associated with a projective algebraic variety defined over a finite field. A remarkable fact is that the Ramanujan conjecture for modular forms of even integral weight is reduced to the RH for the Weil zeta. Another classic zeta function satisfying the RH (nearly, to be exact) is the Selberg zeta function associated with a cocompact torsion-free discrete subgroup of $PSL_2(\mathbb{R})$, which may be expressed as an Euler product over prime closed geodesics in a Riemann surface of constant negative curvature.

In 1966, Y. Ihara [**70**] introduced an analogue of the Selberg zeta function for a cocompact torsion-free discrete subgroup of $PSL_2(\mathbb{Q}_p)$, and proved its rationality by establishing a determinant formula. In the background, there is a consensus that the p-adic version of the real hyperbolic plane $GL_2(\mathbb{R})/(\mathbb{R}^\times \times O(2))$ is $GL_2(\mathbb{Q}_p)/(\mathbb{Q}_p^\times \times GL_2(\mathbb{Z}_p))$[36]. The latter space is, as a special case of Bruhat-Tits

[36] $GL_2(\mathbb{Z}_p)$ and $O(2)$ are maximal compact subgroups of $GL_2(\mathbb{Q}_p)$ and $GL_2(\mathbb{R})$, respectively.

buildings, regarded as the set of vertices in the regular tree of degree $p+1$, and hence its compact quotient yields a finite regular graph. To explain this briefly, we use the fact that every element $g \in GL_2(\mathbb{Q}_p)$ can be written as the product $g = k_1 a k_2$, where $k_i \in GL_2(\mathbb{Z}_p)$ $(i = 1, 2)$ and $a = \begin{pmatrix} p^m & 0 \\ 0 & p^n \end{pmatrix}$. The unordered pair $\{m, n\}$ of integers depends only on g. Putting $\underline{d}(g) = |m - n|$, and $\underline{d}(g_1, g_2) = \underline{d}(g_1^{-1} g_2)$, we obtain the pseudo-distance $\underline{d}$ on $GL_2(\mathbb{Q}_p)$ satisfying $\underline{d}(gg_1, gg_2) = \underline{d}(g_1, g_2)$. Since $\underline{d}(g_1, g_2) = 0$ if and only if $g_1^{-1} g_2 \in \mathbb{Q}_p^\times \cdot GL_2(\mathbb{Z}_p)$, we get a $GL_2(\mathbb{Q}_p)$-invariant distance d on $GL_2(\mathbb{Q}_p)/(\mathbb{Q}_p^\times \times GL_2(\mathbb{Z}_p))$ with nonnegative integral values[37]. Defining the graph structure on $GL_2(\mathbb{Q}_p)/(\mathbb{Q}_p^\times \times GL_2(\mathbb{Z}_p))$ in such a way that x and y are adjacent if and only if $d(x, y) = 1$, we obtain the regular tree of degree $p + 1$.

Thus it is natural to interprete the Ihara zeta functions in terms of finite regular graphs. Let P be the set of all *prime cycles* in a finite regular graph X of degree $q + 1$. Here a prime cycle is an equivalence class of a closed geodesic which is not a power of another one. Two closed paths are said to be equivalent if one is obtained by a cyclic permutation of edges in another. In 1985, Sunada [**122**], following the suggestion stated in the preface of J.-P. Serre [**110**], expressed the Ihara zeta function as the Euler product

$$Z(u) = \prod_{\mathfrak{p} \in P} (1 - u^{|\mathfrak{p}|})^{-1},$$

and gave a graph theoretic proof for a determinant expression in terms of the adjacency operator. Sunada's observation was immediately generalized by K. Hashimoto and A. Hori [**59**] to the case of semi-regular graphs which correspond to p-adic semisimple group of rank one. It was H. Bass [**11**] who first noticed that "regularity" of graphs is not necessary for a determinant expression. More precisely, what he established for a general finite graph X is the equality

$$Z(u) = (1 - u^2)^{\chi(X)} \det \left(I - u\mathcal{A} + u^2(\mathcal{D} - I)\right)^{-1}, \tag{8.1}$$

where $\chi(X)$ denotes the Euler number of X. Moreover, if X is a non-circuit graph, then $u = 1$ is a pole of $Z(u)$ of order $b_1 = \dim H_1(X, \mathbb{R})$, and $\lim_{u \to 1}(1 - u)^{-b_1} Z(u)^{-1} = 2^{b_1} \chi(X) \kappa(X)$. In the case of a regular graph of degree $q + 1$, (8.1) reduces to the identity $Z(u) = (1 - u^2)^{(1-q)N/2} \det \left(I - u\mathcal{A} + qu^2 I\right)^{-1}$, from which it follows that *the Ihara zeta function of a regular graph X satisfies the RH if and only if X is Ramanujan*[38].

There have been many work on the Ihara zeta function including new proofs of (8.1) and various generalizations (L. Bartholdi [**10**], H. Mizuno and I. Sato [**94**], [**95**], C. K. Storm [**117**], I. Sato [**106**]). An instructive proof together with a generalization to weighted graphs is seen in M. D. Horton, H. M. Stark and A. A. Terras [**67**]. See also [**83**] in which the idea of discrete geodesic flows is employed. Actually, the Ihara zeta function is interpreted as the dynamical zeta function associated with a symbolic dynamical system.

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[37] It is interesting to compare this distance function with the Poincaré metric on $GL_2\mathbb{R})/(\mathbb{R}^\times \times O(n))$ induced from $d(g) = |\log \ ab^{-1}|$, where $g = k_1 \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} k_2$ with $k_1, \ k_2 \in O(2)$.

[38] The RH for the Ihara zeta means that every pole s of $\zeta(s) = Z(q^{-s})$ in the region $0 < \mathrm{Re}\ s < 1$ lies on the line $\mathrm{Re}\ s = 1/2$.

One may make a "miniature" of algebraic number theory in the graph setting which provides Ihara zeta functions with "moral support". Indeed, prime cycles behave like prime ideals in number fields under a finite-fold covering map. Hence the number theoretic notions such as ideal groups, Frobenius substitutions and ideal class groups have matching parts in graphs[39]. We may even formulate the class field theory[40] in the discrete setting. Note that the "ideal class group" of X is nothing but $H_1(X,\mathbb{Z})$. A big difference from number theory is that the ideal class group is of infinite order.

As a matter of course, we may imitate analytic number theory. For simplicity, we treat regular graphs of degree $q+1$ (for general graphs, a dynamical viewpoint is more convenient to handle this matter). The counting function $\pi(n) = \big|\{\mathfrak{p} \in P;\ |\mathfrak{p}| = n\}\big|$ has the following asymptotic behavior

(1) when X is non-bipartite; $\pi(n) \sim \dfrac{q^n}{n} \quad (n \to \infty)$.

(2) when X is bipartite, $\pi(n) \sim 2\dfrac{q^n}{n} \quad (n = \text{even} \to \infty)$.

These asymptotics[41], whose proof is almost trivial, are regarded as a discrete analogue of the prime number theorem. This observation brings us to the question "what is an analogue of the density theorem for primes in an arithmetic progression (and its generalization to number fields; that is, the Chebotarev density theorem) ?". The answer is an asymptotic formula, if any, for the counting function $\pi(n,[g]) = \big|\{\mathfrak{p};\ |\mathfrak{p}| = n,\ \varphi(\mathfrak{p}) \in [g]\}\big|$, where φ is a surjective homomorphism of the fundamental group $\pi_1(X)$ onto a finite group G. A tool is graph theoretic *L-functions*. The L-function associated with a finite dimensional unitary representation ρ of $\pi_1(X)$ is defined as $L_\rho(u) = \prod_{\mathfrak{p}\in P} \det(I - \rho(\mathfrak{p})u^{|\mathfrak{p}|})^{-1}$. It possesses a determinant formula involving the twisted adjacency operator $\mathcal{A}_\rho$. Mimicking the proof in number theory, we may establish a density theorem in this set-up ((see [**67**] for a generalization to weighted graphs))[42]. Much difficult is the case when G is of infinite order. So far abelian cases have been studied intensively. In the special case $G = H_1(X,\mathbb{Z})$, it is verified (cf. [**77**] and [**78**]) that, for $\alpha \in H_1(X,\mathbb{Z})$,

(1) (non-bipartite case) $\pi(n,\alpha) \sim \kappa(X)^{1/2}\Big(\dfrac{N(q-1)}{4\pi}\Big)^{b_1/2}\dfrac{q^n}{n^{b_1/2+1}} \quad (n \to \infty)$;

(2) (bipartite case) $\pi(n,\alpha) \sim 2\kappa(X)^{1/2}\Big(\dfrac{N(q-1)}{4\pi}\Big)^{b_1/2}\dfrac{q^n}{n^{b_1/2+1}} \quad (n = \text{even} \to \infty)$.

The zeta function for an infinite-fold covering graph over a finite regular graph was introduced by Grigorchuk and Zuk [**49**] as a suitable limit of Ihara zeta functions. In their discussion, the idea of densities of states is employed. They carried out explicit computation for a class of Schreier graphs associated with fractal groups. B. Clair and S. Mokhtari-Sharghi [**29**] handled the zeta function for an infinite graph with a group action which is defined as an Euler product over group orbits of prime cycles. As is expected, the notion of von Neumann trace comes up

[39] In the Riemannian geometric setting, a similar formulation has already been carried out in [**121**], [**120**]. See [**67**] and K. Hashimoto [**60**] for the case of graphs.

[40] This corresponds to the class field theory for unramified extensions, to be exact.

[41] Actually we have an exact formula for $\pi(n)$. We may obtain a similar asymptotic for a general graph.

[42] The idea for closed geodesics in manifolds is already seen in [**119**].

in their determinant expressions. See D. Guido, T. Isola, and M. L. Lapidus [**58**], [**57**] for the recent development in this direction. The zeta function for a regular covering graph is related to the counting function $\pi(n,[1])$. In this connection, it is interesting to establish the Chebotarev type theorem for non-abelian "Galois group" of infinite order.

A lucid meaning is given for the zeta functions of Cayley graphs as follows. A word $w=(b_1,\ldots,b_n)$ is said to be *cyclically reduced* if $\overline{b_{i+1}}\neq b_i$ $(i=1,2,\ldots,n-1)$ and $\overline{b_1}\neq b_n$. A cyclically reduced word w is said to be *prime* if it is not a power of another word. Two words w_1 and w_2 are *equivalent* if w_1 is obtained from w_2 by a cyclic permutation. Letting now P be the set of equivalence classes of cyclically reduced prime words w with $g(w)=1$, we define the zeta function $Z(u)$ in the same way as the Ihara zeta function. We then have, in a small neighborhood of $u=0$,

$$Z(u)=(1-u^2)^{-(q-1)/2}\mathrm{det}_G(1-\mathcal{A}u+qu^2)^{-1},$$

where $\det_G$ stands for the G-determinant defined by $\det_G(T)=\exp\mathrm{tr}_G(\log T)$[43].

We conclude this section with more about analogy between hyperbolic spaces and regular trees.

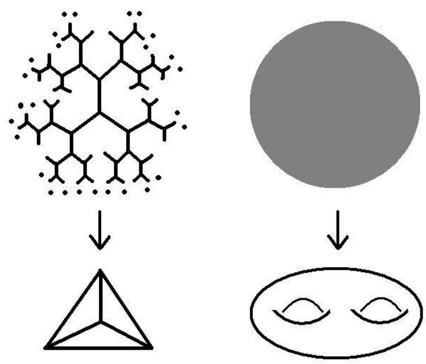

FIGURE 2

(1) As was carried out by P. Cartier [**22**], harmonic analysis on regular trees can be developed in a parallel way as that on hyperbolic spaces (see also A. Figa-Talamanca and C. Nebbia [**45**]). Indeed, the ideal boundary and horospheres are defined in a natural manner with which we may introduce the notion of spherical functions and the Fourier transformation. This allows us to perform the detailed spectral decomposition of the adjacency operator. For the reader's convenience, we shall give a brief account. Fix a vertex o in X, and put $S(n)=\{x\in V;\ d(x,o)=n\}$. For each $x\in S(n)$, there is a unique $y\in S(n-1)$ such that $d(x,y)=1$, so that one can define a map $\pi_n: S(n)\longrightarrow S(n-1)$ by setting $\pi_n(x)=y$. The *boundary* of X,

[43] This unpublished work was presented by the author at the memorial symposium for Hubert Pesce in 1998 at Grenoble. The definition of $\det_G$ here is rather formal and is justified by a functional-analytic argument. This determinat formula does not imply that the function $Z(u)$ is analytically continued to the whole complex plane $\mathbb{C}$.

which we denote by ∂X, is defined as the projective limit $\varprojlim(S(n), \pi_n)$. By Kolmogoroff's theorem, there exists a (unique) probability measure μ on ∂X satisfying $\mu\big(\widetilde{\omega}_n{}^{-1}(x)\big) = |S(n)|^{-1} = (q+1)^{-1}q^{1-n}$ for $x \in S(n)$, where $\widetilde{\omega}_n : \partial X \longrightarrow S(n)$ is the projection map. Let $b = (o, x_1, x_2, \ldots) \in \partial X$, and define the *horosphere* $H_n(b)$ through b and x_n by $H_n(b) = \{x \in V;\ d(x, x_k) = k - n$ for sufficiently large $k\}$. It is evident that $V = \coprod_n H_n(b)$. Define a function $\langle x; b\rangle$ on $X \times \partial X$ by putting $\langle x; b\rangle = n$ when $x \in H_n(b)$. We set $\lambda_\theta = q^{1/2}e^{\sqrt{-1}\theta}$, and define the Fourier transformation of a function $f \in C(V)$ with finite support by $(\mathfrak{F}f)(\theta, b) = \sum_{x\in V} f(x)\lambda_{-\theta}^{\langle x;b\rangle}$. We also define $a(\theta) = 2q(q+1)\sin^2\theta\big((q+1)^2 - 4q\cos^2\theta\big)^{-1}$. Then the map $f \longmapsto \mathfrak{F}f$ extends to a unitary isomorphism

$$\mathfrak{F} : \ell^2(V) \longrightarrow L^2([0,\pi] \times \partial X, \frac{1}{\pi}a(\theta)d\theta d\mu(b)),$$

and the following inversion formula holds

$$f(x) = \frac{1}{\pi}\int_0^\pi \int_{\partial X} \lambda_\theta^{\langle x;b\rangle}\hat{f}(\theta, b)a(\theta)d\mu(b)d\theta.$$

Furthermore we find that $\mathfrak{F}\mathcal{A}\mathfrak{F}^{-1}$ coincides with the multiplication operator by the function $2\sqrt{q}\cos\theta$.

(2) Quite a bit related to harmonic analysis is the Selberg trace formula, an indispensable tool to treat the Selberg zeta function. This also has a counterpart in regular trees (G. Ahumada [**2**]). Let $h(\theta)$ be a real analytic function on $\mathbb{R}$ satisfying (i) $h(\theta+2\pi) = h(\theta)$; (ii) $h(-\theta) = h(\theta)$; (iii) $h(\theta)$ is analytically continued to $|\mathrm{Im}\ \theta| < \frac{1}{2}\log q + \epsilon$ $(\epsilon > 0)$. Then, for a finite regular graph X_0 of degree $q+1$, we have

$$\sum_{i=0}^{N-1} h(\theta_i) = \frac{2N}{\pi}q(q+1)\int_0^\pi \frac{\sin^2\theta}{(q+1)^2 - 4q\cos^2\theta}h(\theta)\ d\theta + \sum_{\mathfrak{p}\in P}\sum_{n=1}^\infty |\mathfrak{p}|q^{-n|\mathfrak{p}|/2}\widehat{h}(n|\mathfrak{p}|).$$

Here $\mu_i = 2q^{1/2}\cos\theta_i$ is an eigenvalue of $\mathcal{A}$ and

$$\widehat{h}(k) = \frac{1}{2\pi}\int_0^{2\pi} h(\theta)e^{\sqrt{-1}k\theta}\ d\theta.$$

The trace formulae with some special test functions h have been applied to the study of Ramanujan graphs by many people (cf. [**18**], [**105**], [**133**], [**130**]). See also U. Smilansky [**115**] for trace formulae derived from a physical viewpoint.

In the proof of Ahumada's formula, the pre-trace formula (6.3) is employed for $G = \pi_1(X_0)$. In a sense, the determinant formula for the Ihara zeta function is considered a "product" version of the trace formula.

(3) One may apply the trace formulae to study the statistics of eigenvalues. Let $\{X_n\}_{n=1}^\infty$ be a family of regular graphs of degree $q+1$ such that $\lim_{n\to\infty}\mathrm{girth}(X_n) = \infty$[44]. Denote $\phi_n([a,b]) = \big|\{\text{eigenvalues } \mu \text{ of } \mathcal{A}_{X_n} \text{with } \mu \in [a,b]\}\big|$. Then we have

$$\lim_{n\to\infty}\frac{1}{|V_n|}\phi_n([a,b]) = \int_a^b \phi(\mu)\ d\mu, \tag{8.2}$$

[44]The *girth* is the number of edges in the shortest closed geodesic.

where

$$\phi(\mu) = \begin{cases} \dfrac{q+1}{2\pi}\dfrac{\sqrt{4q-\mu^2}}{(q+1)^2-\mu^2} & (|\mu| \le 2q^{1/2}) \\ 0 & (\text{otherwise}) \end{cases}. \tag{8.3}$$

This is an analogue of the result due to H. Huber [**68**] for eigenvalues of the Laplacians on hyperbolic surfaces.

We may also establish the *semi-circle law*[45]. Suppose a family of regular graphs $\{X_n\}_{n=1}^{\infty}$ satisfies $\mathrm{girth}(X_n) \to \infty$, $\deg X_n = q_n + 1 \to \infty$ and $\sqrt{\deg X_n} \times (\log \mathrm{girth}(X_n))^{-1} \to 0$. Let $\psi_n([a,b]) = \big|\{\text{eigenvalues } \mu \text{ of } \mathcal{A}_{X_n} \text{ such that } \mu \in [2q_n^{1/2}a, 2q_n^{1/2}b]\}\big|$. Define $\psi(x) = 2\sqrt{1-x^2}/\pi$ ($|x| \le 1$), $\psi(x) = 0$ otherwise. Then

$$\lim_{n\to\infty} \frac{1}{|V_n|}\psi_n([a,b]) = \int_a^b \psi(x)\ dx. \tag{8.4}$$

The function (8.3) is nothing but the density of states for $\mathcal{A}$. Actually (8.2) is a special case of (6.2) if $\{X_n\}$ is a tower of covering graphs.

(4) We may classify irreducible unitary representations of the automorphism group of a regular tree. This was put into practice by Ol'shanski [**101**],[**102**], and resembles the classification for $SL_2(\mathbb{R})$. See also [**45**]. Closely related is unitary representations of $G = PGL_2(\mathbb{Q}_p)$ which is a subgroup of the automorphism group of the regular tree of degree $p+1$. A sample of results in this theory asserts that, for a cocompact lattice Γ in G, no representation from the complementary series occurs in $L^2(\Gamma\backslash G)$ if and only if the graph $\Gamma\backslash G/PGL_2(\mathbb{Z}_p)$ is Ramanujan (cf. [**91**]).

9. Random walks

Another important source of discrete geometric analysis is the theory of random walks, a synonym in a loose sense for a time homogeneous Markov chain on a countable state space. In terms of a graph $X = (V,E)$, a random walk[46] is a stochastic process associated with a function p on E (called a transition probability) satisfying $p(e) \ge 0$ and $\sum_{e\in E_x} p(e) = 1$. The transition operator $P : C(V) \longrightarrow C(V)$ is defined by $Pf(x) = \sum_{e\in E_x} p(e)f\big(t(e)\big)$. The n-step transition probability $p(n,x,y)$ is defined to be the kernel function of P^n; namely $(P^n f)(x) = \sum_{y\in V} p(n,x,y)f(y)$. If we put $f_n = P^n f$, then $f_{n+1} - f_n = (P-I)f_n$, $f_0 = f$. Thus $\{f_n\}$ satisfies an analogue of the heat equation $\partial f/\partial t = -\Delta f$ if we think of $I-P$ as an analogue of Laplacians. In general, however, it is not a discrete Laplacian in our sense. It is the case only if the random walk is *symmetric*[47], namely there exists a positive-valued function m on V such that $p(e)m\big(o(e)\big) = p(\overline{e})m\big(t(e)\big)$. Indeed, this being the case, $I-P$ is the discrete Laplacian for the weight functions $m_V = m$, $m_E(e) = p(e)m\big(o(e)\big)$. Note that $p(n,x,y)m(y)^{-1} (= p(n,y,x)m(x)^{-1})$ is the kernel function of P^n with respect to the measure m. The most canonical one is the *simple* random walk with the transition probability defined by $p(e) = 1/\deg o(e)$. In this case, $I-P$ coincides with the canonical Laplacian $I - \mathcal{D}^{-1}\mathcal{A}$. In what follows, we only handle symmetric random walks.

[45] This was stated by the author in the Japanese magazine "Su-Semi" in 2001. See also [**96**].

[46] Strictly speaking, we consider the case of finite range random walks.

[47] The term "reversible" is also used.

A central theme for random walks is the properties of $p(n,x,y)$ as n goes to infinity (see [**137**] for an overview). One of the most classical problems is the recurrence-transience problem[48] which is concerned with the divergence and convergence of $\sum_{n=1}^{\infty} p(n,x,y)$. If the series diverges, then a walker tends to return again and again to finite regions in probability one; whereas, if it converges, then the walker goes to "infinity" in probability one. For instance, if the Cheeger constant is positive, then the random walk is transient. In the transient case, it is natural to consider the "ideal boundaries" (e.g., the Martin boundary or the Poisson boundary) which consist of "points" at infinity reached by the random walker when n goes to infinity. The ideal boundaries are related to the problem on the existence of positive or bounded harmonic functions and also to the Dirichlet problem at infinity (see V. A. Kaimanovich and W. Woess [**76**]).

The central limit theorem, a generalization of the Laplace-de Moivre theorem, has been studied in detail for random walks on crystal lattices with periodic transition probability. In F. Spitzer [**116**], a classical book on random walks on Cayley graphs $X(\mathbb{Z}^n, A)$, one can find various results including the central limit theorem. For general crystal lattices, see A. Y. Guivarc'h [**56**] and A. Krámli and D. Szász [**86**]. The perturbed eigenvalues $\lambda_0(\chi)$ for the twisted Laplacian (see Section 6) play an important role in the proof. Interesting is the fact that asymptotics for the simple random walk on a crystal lattice involve a generalization of discrete Albanese maps ([**80**], [**82**]). In fact, $p(n,x,y)$ for a d-dimensional crystal lattice has the following asymptotic expansion[49] at $n=\infty$.

$$p(n,x,y)\big(\deg y\big)^{-1} \sim (4\pi n)^{-d/2} C\big[1 + c_1(x,y)n^{-1} + c_2(x,y)n^{-2} + \cdots\big], \tag{9.1}$$

where the coefficient $c_1(x,y)$ is expressed as

$$c_1(x,y) = -\frac{C'}{4}\|\widetilde{\Phi}(x) - \widetilde{\Phi}(y)\|^2 + g(x) + g(y) + c \tag{9.2}$$

with a certain function $g(x)$ and a constant c. The map $\widetilde{\Phi} : X \longrightarrow \mathbb{R}^d$ is what we call the *standard realization*, which turns out to coincide with the lifting of the discrete Albanese map when X is the maximal abelian covering graph over a finite graph. In view of (9.1),(9.2), we have

$$C'\|\widetilde{\Phi}(x) - \widetilde{\Phi}(y)\|^2 = \lim_{n\to\infty} 2n\Big\{\frac{p(n,x,x)}{p(n,y,x)} + \frac{p(n,y,y)}{p(n,x,y)} - 2\Big\},$$

from which we may deduce that $\widetilde{\Phi}$ is a realization with maximal symmetry. This is rephrased as "the random walker detects somehow the most natural way for his crystal lattice to sit in space".

The theory of large deviations exhibits another aspect of asymptotics. This theory, in general, concerns the asymptotic behavior of remote tails of sequences of probability distributions[50]. In our setting, the problem is to find an asymptotic behavior of $p(n,x,y)$ while y is kept near the boundary $\partial B_n(x) = \{y \in V;\ d(x,y) =$

[48]Historically this question originates in Polya's observation in 1921 of the simple random walk on the hyper-cubic lattice $\mathbb{Z}^k$. What Polya showed is that the simple random walk on $\mathbb{Z}^2$ is recurrent, while it is transient on $\mathbb{Z}^k$ with $k \geq 3$.

[49]For a technical reason, we suppose that X is non-bipartite. In the bipartite case, a slight modification of the statement is required.

[50]We may trace back to Laplace and Cramér. It is Varadhan who introduced a unified formal definition (1966).

$n\}$. A satisfactory answer has been given for random walks on crystal lattices. Here we describe a weak version restricted to the case of the maximal abelian covering graph X over a finite graph X_0. It tells that *there exists a convex function H on $H_1(X_0,\mathbb{R})$ (possibly assuming $+\infty$) such that, for ξ in the interior of D_H, and $\{y_n\}_{n=1}^{\infty}$ in V such that $\{\widetilde{\Phi}(y_n)-n\xi\}$ is bounded, we have*

$$\lim_{n\to\infty}\frac{1}{n}\log p(n,x,y_n)=-H(\xi),$$

where $D_H=\{\xi;\ H(\xi)<\infty\}$. The convex function H is explicitly described in terms of the maximal positive eigenvalues $\mu_0(\omega)$ of the twisted transition operators[51] L_ω defined by $L_\omega f(x)=\sum_{e\in E_x}p(e)e^{\omega(e)}f\big(t(e)\big)$ $(\omega\in C^1(X_0,\mathbb{R}))$. More precisely, if we think of μ_0 as a function on $H^1(X_0,\mathbb{R})$ using the fact that $L_{\omega+df}=e^{-f}L_\omega e^f$ $(f\in C^0(X_0,\mathbb{R}))$, then

$$H(\xi)=\sup_{\mathbf{x}\in H^1(X_0,\mathbb{R})}\big[(\xi,\mathbf{x})-\log\mu_0(\mathbf{x})\big].$$

The convex domain D_H is identified with the scaling limit of "reachable" regions of the random walker starting from x, and expressed as $D_H=\{\xi\in H_1(X_0,\mathbb{R});\ \|\xi\|_1\le 1\}$, where the norm $\|\cdot\|_1$ on $C_1(X_0,\mathbb{R})$ is defined by $\Big\|\sum_{e\in E^o}a_e e\Big\|_1=\sum_{e\in E^o}|a_e|$ ($E^o\subset E$ is an orientation, i.e., $E^o\cap\overline{E^o}=\emptyset$ and $E^o\cup\overline{E^o}=E$). Thus D_H is a convex polyhedron. The metric space $H_1(X_0,\mathbb{R})$ with the distance $d_\infty(\xi_1,\xi_2)=\|\xi_1-\xi_2\|_1$ turns out to coincide with the Gromov-Hausdorff limit of $(X,n^{-1}d)$ as n goes to infinity ([**85**] and M. Gromov [**54**]).

Another extreme is the case of trees. S. Sawyer and T. Steger [**107**] obtained a central limit theorem for the radial part of random walks (possibly with infinite range) on trees. See Y. Derriennic [**39**] and Y. Guivarc'h [**55**] for a related matters. S. P. Lalley [**88**] established a local limit formula and a large deviation property for random walks with finite range on free groups. The local limit formula says that there are constants $R>1, C>0$ such that $p(n,x,x)\sim Cn^{-3/2}R^n$.

Gaussian estimates for $p(n,x,y)$ have been studied by many people. The business is to compare $p(n,x,y)$ with a geometrically defined function of the form $h(n,x,y)\exp\big(-g(x,y)\ n^{-\gamma}\big)$ (see [**132**]). Roughly speaking, the term $h(n,x,y)$ corresponds to the volume growth of the graph, and $\exp\big(-g(x,y)\ n^{-\gamma}\big)$ is related to a certain isoperimetric inequality. A typical result due to A. Grigor'yan [**51**] and L. Saloff-Coste [**104**] says that, under the assumption that $\inf p(e)>0$, the estimates

$$p(n,x,y)m(y)\le\frac{c_1}{m\big(B_{\sqrt{n}}(x)\big)}\exp\Big[-\frac{d(x,y)^2}{c_1 n}\Big], \tag{9.3}$$

$$p(n,x,y)m(y)+p(n+1,x,y)\ge\frac{c_2}{m\big(B_{\sqrt{n}}(x)\big)}\exp\Big[-\frac{d(x,y)^2}{c_2 n}\Big]$$

hold if and only if the following two conditions are satisfied:

(1) (Volume doubling condition) There exists a positive C such that $m\big(B_{2n}(x)\big)\le Cm\big(B_n(x)\big)$ for every n and $x\in V$.

[51] Actually, L_ω is linearly equivalent to the twisted operator associated with a real character of $H_1(X_0,\mathbb{Z})$.

(2) (Poincaré inequality) There exists a positive constant C such that

$$\sum_{y\in B_n(x)} \big(f(y)-\overline{f}_{B_n(x)}\big)^2 m(y) \le Cn^2 \sum_{\substack{e\in E\\ o(e),t(e)\in B_n(x)}} |df(e)|^2 p(e) m\big(o(e)\big)$$

for all $f \in C\big(B_n(x)\big)$, where $\overline{f}_{B_n(x)} = m\big(B_n(x)\big)^{-1}\sum_{y\in B_n(x)} f(y)m(y)$.

If we restrict ourselves to Gaussian upper bound, then the Faber-Krahn type inequality is involved. More precisely, under the volume doubling condition, (9.3) is equivalent to the following inequality

$$\lambda_1(A) \ge \frac{c}{n^2}\Big(\frac{m\big(B_n(x)\big)}{m(A)}\Big)^{\alpha} \quad (A \subset B_n(x))$$

where $\alpha, c > 0$ and $\lambda_1(A) = \inf\Big\{\|df\|^2/\|f\|^2;\ f \ne 0,\ \text{supp}\ f \subset A\Big\}$.

We see interesting effects on $p(n,x,y)$ when a graph has a fractal property. Indeed, the "sub-gaussian" function

$$\frac{c}{m\big(B_{n^\beta}(x)\big)} \exp\Big[-\Big(\frac{d(x,y)^\beta}{cn}\Big)^{\frac{1}{\beta-1}}\Big]$$

turns up as a bound. For more details, see the papers [**33**], [**49**], [**52**] by A. Grigor'yan, T. Coulhon, and A. Telcs.

At this point, it should not be surprising to be able to relate properties of random walks on the Cayley graph to the group structure, say group growth, amenability, hyperbolicity, etc. (see H. Kesten [**79**] as a pioneer work and [**75**] for recent results).

10. Concluding remarks

The present article dwells only on discrete analogue of Laplacians. Needless to say, there are a variety of related operators. For example, a discrete *elastic Laplacian* appears in the microscopic study of crystals wherein the idea of twisted operators proved to be very useful ([**114**]). This is actually a graph version of the differential operator showing up in the elastic wave equation, and has been implicitly introduced by physicists in the study of lattice vibrations[52]. We may also consider a discrete Schrödinger operator with a magnetic field. In particular, the notion of discrete Schrödinger operators with periodic magnetic fields on a covering graph $X \xrightarrow{G} X_0$ is formulated in a natural manner in terms of invariant cohomology classes. Magnetic fields, which are closed 2-forms in the continuous case, are identified with 2-cocycles in the group cohomology ([**124**], [**84**]). It is worthwhile to note the following exact sequence in connection with the set-up.

$$0 \to H^1\big(G, U(1)\big) \to H^1\big(X_0, U(1)\big) \to H^1\big(X, U(1)\big)^G \to H^2\big(G, U(1)\big) \to 1.$$

Even in the case of periodic magnetic fields on a crystal lattice (more specifically on $\mathbb{Z}^2$), non-commutative nature crops up in the study of spectra and causes very curious phenomena ([**71**]). Related to this is the almost Mathieu operator which acts in $\ell^2(\mathbb{Z})$ by the formula

$$H_{\alpha,\theta,\lambda} f(n) = f(n+1) + f(n-1) + 2\lambda\cos(2\pi\alpha n + \theta) f(n).$$

[52]It originated in the study of specific heat of crystalline solids by Einstein (1907) and Dybye (1912).

In their recent article [**6**], A. Avila and S. Jitomirskaya solved the Ten Martini Problem proposed by M. Kac and B. Simon[53] claiming that *the spectrum of the almost Mathieu operator is a Cantor set for all irrational* α *and for all* $\lambda \neq 0$ (see also A. Shubin [**113**]).

My choice of topics is admittedly biased. For instance, I did not make reference to the recent work of Green functions and random graphs. I also could not touch on the detail of self-similar graphs whose spectra display very interesting aspects. I believe, however, that the exploration into the references below would help the reader in gathering materials for various subjects which I could not take up in this article.

Acknowledgement. The author is grateful to Polly Wee Sy who has helped by commenting upon the first draft, and also to Stephen Hyde, Michael O'Keeffe, Davide M. Proserpio, Peter Kroll, Edwin Clark, Johannes Roth, and Alan Mackay for drawing his attention to relevant work in crystallography.

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[53] The reason of the naming is the following. In 1981, Mark Kac, during his talk in a session of American Mathematical Society, offered ten Martinis to anybody who could solve the problem. B. Simon who was then present at his talk formulated this problem in his review paper and called it the Ten Martini Problem.

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