HEAVY-TAILED DISTRIBUTIONS AND THEIR APPLICATIONS *

CHUN SU and QIHE TANG

Department of Statistics and Finance,
University of Science and Technology of China,
Hefei 230026, Anhui, China
E-mail: suchun@ustc.edu.cn

Department of Quantitative Economics,
University of Amsterdam,
Roetersstraat 11, 1018 WB Amsterdam, The Netherlands
E-mail: q.tang@uva.nl

Based on some recent investigations by the authors, this paper gives a short review on the concept of heavy-tailedness. Some commonly used and newly introduced classes of heavy-tailed distributions are examined. Applications to precise large deviations and to finite time ruin probabilities are proposed.

1. Heavy-tailed Distributions

Let X be a random variable (r.v.) with a distribution function (d.f.) $F(x) = \mathbb{P}(X \leq x)$. Throughout, we denote by $\overline{F}(x) = 1 - F(x)$ the tail probability of the r.v. X and write by

$$F_e(x) = (\mu_F^+)^{-1} \int_0^x \overline{F}(z)\mathrm{d}z \tag{1.1}$$

the equilibrium distribution of F provided

$$0 < \mu_F^+ = \int_0^\infty \overline{F}(z)\mathrm{d}z < \infty.$$

Hereafter, we will be using the notation μ_F^+ silently. We say that the r.v. X or its d.f. F is heavy tailed to the right if its moment generating function

$$\mathbb{E}\exp\{tX\} = \infty \qquad \forall\, t > 0; \tag{1.2}$$

*This work was supported by the national science foundation of china (no.10371117) and the dutch organization for scientific research (no. nwo 42511013).

otherwise, we say that it is light tailed to the right.

The most important class of heavy-tailed distributions is the subexponential class, denoted by $\mathcal{S}$. This class was first introduced by Chistyakov (1964) in a more general case. Let $\{X_k, k \geq 1\}$ be a sequence of independent and identically distributed (i.i.d.) r.v.'s with common d.f. F supported on $(0, \infty)$. By definition, the d.f. F belongs to the class $\mathcal{S}$ if and only if for each $n \geq 2$, the tail probabilities of the sum S_n and the maximum $X_{(n)}$ of the first n r.v.'s of $\{X_k, k \geq 1\}$ are asymptotically of the same order, i.e.,

$$\mathbb{P}(S_n > x) \sim \mathbb{P}\left(X_{(n)} > x\right) \quad \text{as } x \to \infty. \tag{1.3}$$

In the recent literature, the d.f. F supported on $(-\infty, \infty)$ is still said to belong to $\mathcal{S}$ if the d.f. of the r.v. $X^+ = X\mathbb{I}_{(X>0)}$ belongs to $\mathcal{S}$.

Some other important classes of heavy-tailed distributions are listed as follows, where the d.f. F involved is always assumed to be supported on $(-\infty, \infty)$:

1. the class $\mathcal{L}$ (long-tailed): F is said to belong to the class $\mathcal{L}$ if
$$\lim_{x\to\infty} \frac{\overline{F}(x+z)}{\overline{F}(x)} = 1$$
holds for any $z > 0$ (or equivalently for some $z > 0$).
2. the class $\mathcal{D}$ (of dominated variation): F is said to belong to the class $\mathcal{D}$ if
$$\limsup_{x\to\infty} \frac{\overline{F}(xz)}{\overline{F}(x)} < \infty$$
holds for any $0 < z < 1$ (or equivalently for some $0 < z < 1$).
3. the class $\mathcal{S}^*$: F is said to belong to the class $\mathcal{S}^*$ if $0 < \mu_F^+ < \infty$ and
$$\lim_{x\to\infty} \int_0^x \frac{\overline{F}(x-z)}{\overline{F}(x)} \overline{F}(z)\mathrm{d}z = 2\mu_F^+.$$
4. the class $\mathcal{C}$ (of consistent variation): F is said to belong to the class $\mathcal{C}$ if
$$\lim_{l\uparrow 1} \limsup_{x\to\infty} \frac{\overline{F}(lx)}{\overline{F}(x)} = 1. \tag{1.4}$$
5. the class ERV (with extended regular variation): F is said to belong to the class $ERV(-\alpha, -\beta)$ for some $0 < \alpha \leq \beta < \infty$ if
$$v^{-\beta} \leq \liminf_{x\to\infty} \frac{\overline{F}(vx)}{\overline{F}(x)} \leq \limsup_{x\to\infty} \frac{\overline{F}(vx)}{\overline{F}(x)} \leq v^{-\alpha}, \qquad \forall v > 1. \tag{1.5}$$

Especially, if $\alpha = \beta > 0$ then relation (1.5) describes the class $\mathcal{R}_{-\alpha}$, which is a well-known subclass of subexponential distributions.

The regularity property in (1.4) was first introduced and called "intermediate regular varying" property by Cline (1994). This class has been used in different studies of applied probability such as queueing system and ruin theory; see, for example, Schlegel (1998), Jelenković and Lazar (1999), Ng *et al.* (2004), and references therein. Specifically, the class $\mathcal{C}$ covers the class ERV. Most recently Cai and Tang (2004) constructed a simple, but non-trivial, example to illustrate that the inclusion $ERV \subset \mathcal{C}$ is strict; another similar example can be found in Cline and Samorodnitsky (1994).

For the classes above, we have, for any $0 < \alpha \leq \gamma \leq \beta < \infty$,

$$\mathcal{R}_{-\gamma} \subset ERV(-\alpha, -\beta) \subset \mathcal{C} \subset \mathcal{D} \cap \mathcal{L} \subset \mathcal{S} \subset \mathcal{L}, \quad \mathcal{S} \not\subset \mathcal{D}, \quad \mathcal{D} \not\subset \mathcal{S}. \tag{1.6}$$

Furthermore, if $0 < \mu_F^+ < \infty$, then

$$F \in \mathcal{D} \cap \mathcal{L} \Longrightarrow F \in \mathcal{S}^* \subset \mathcal{S}. \tag{1.7}$$

For more details and related discussions, we refer to Bingham *et al.* (1987), Embrechts *et al.* (1997), and Goldie and Klüppelberg (1998), among others.

Recently, we have introduced some other classes of heavy-tailed distributions.

6. the class $\mathcal{A}^*$ (Konstantinides *et al.* (2002)): F is said to belong to the class $\mathcal{A}^*$ if $0 < \mu_F^+ < \infty$, $F_e \in \mathcal{S}$ and

$$\liminf_{x\to\infty} \frac{x\overline{F}(x)}{\int_x^\infty \overline{F}(u)\mathrm{d}u} > 0. \tag{1.8}$$

7. the class $\mathcal{M}$ (Tang (2001), Su *et al.* (2002) and Su and Tang (2003)): F is said to belong to the class $\mathcal{M}$ if $0 < \mu_F^+ < \infty$ and

$$\lim_{x\to\infty} \frac{\overline{F}(x)}{\int_x^\infty \overline{F}(u)\mathrm{d}u} = 0.$$

8. the class $\mathcal{M}^*$ (Tang (2001), Su *et al.* (2002) and Su and Tang (2003)): F is said to belong to the class $\mathcal{M}^*$ if $0 < \mu_F^+ < \infty$ and

$$\limsup_{x\to\infty} \frac{x\overline{F}(x)}{\int_x^\infty \overline{F}(u)\mathrm{d}u} < \infty.$$

2. Some asymptotics and bounds

2.1. *Applications of the classes $\mathcal{L}$ and $\mathcal{S}$ to asymptotics*

Let $\{X_k, k \geq 1\}$ be a sequence of independent r.v.'s, each X_k with a d.f. F_k supported on $(-\infty, \infty)$, $k \geq 1$. With $X_0 = 0$, we write

$$X_{(n)} = \max_{0\leq k\leq n} X_k, \quad S_n = \sum_{k=0}^n X_k, \quad S_{(n)} = \max_{0\leq k\leq n} S_k, \qquad n \geq 1.$$

The following result shows that, for a large class of heavy-tailed distributions, the tail probability of the maxima of the partial sum is asymptotically equal to the tail probability of the sum of the individual random variables:

Theorem 2.1. *(Ng et al. (2002)) If $F_k \in \mathcal{L}$, $k \geq 1$, then for any $n \in \mathbb{N}$,*

$$\mathbb{P}\left(S_{(n)} > x\right) \sim \mathbb{P}(S_n > x). \tag{2.1}$$

Now we reduce the class $\mathcal{L}$ to the class $\mathcal{S}$ and obtain

Theorem 2.2. *(Ng et al. (2002)) Assume that $\lim_{x\to\infty} \overline{F_k}(x)/\overline{F}(x) = c_k$ holds for some constants $c_k \geq 0$ and some d.f. $F \in \mathcal{S}$, $k \geq 1$. Then for any $n \in \mathbb{N}$, we have*

$$\lim_{x\to\infty} \frac{\mathbb{P}\left(S_{(n)} > x\right)}{\overline{F}(x)} = \lim_{x\to\infty} \frac{\mathbb{P}(S_n > x)}{\overline{F}(x)} = \lim_{x\to\infty} \frac{\mathbb{P}\left(X_{(n)} > x\right)}{\overline{F}(x)} = \sum_{k=1}^n c_k. \tag{2.2}$$

Let F be a d.f. with $0 < \mu_F^+ < \infty$. Recall its equilibrium distribution function defined by (1.1). Clearly, F_e is absolutely continuous and supported on $(0, \infty)$ with a hazard rate function

$$r_e(x) = -\frac{\mathrm{d}}{\mathrm{d}x} \ln \overline{F_e}(x) = \frac{\overline{F}(x)}{\int_x^\infty \overline{F}(t)\mathrm{d}t}, \qquad x > 0, \tag{2.3}$$

where the latter equality holds almost everywhere with respect to the Lebesgue measure. The following result extends (2.2) to the random case:

Theorem 2.3. *(Ng and Tang (2004)) Let $\{X_k, k \geq 1\}$ be a sequence of i.i.d. r.v.'s with common d.f. F and finite expectation μ_F. Assume that τ is a nonnegative and integer-valued r.v. with finite expectation and independent of $\{X_k, k \geq 1\}$. If one of the following three conditions holds:*

(1) $F \in \mathcal{S}$ and for some $\delta > 0$ we have $\mathbb{E}\exp\{\delta\tau\} < \infty$
(2) $F \in \mathcal{D} \cap \mathcal{L}$, $\mu < 0$ and $\mathbb{P}(\tau > x) = o(r_e(x))$
(3) $F \in ERV(-\alpha, -\beta)$, $1 < \alpha \leq \beta < \infty$, and $\mu_F < 0$

then

$$\mathbb{P}\left(S_{(\tau)} > x\right) \sim \mathbb{P}\left(S_\tau > x\right) \sim \mathbb{P}\left(X_{(\tau)} > x\right) \sim \mathbb{E}\tau\overline{F}(x). \tag{2.4}$$

Compared with (1.3), the asymptotic relations (2.2) and (2.4) describe some very natural properties of subexponential distributions. Theorem 2.2 partly improves the related result in Sgibnev (1996). The second sufficient condition of Theorem 2.3 weakens the conditions of Theorem 2.3 in Ng *et al.* (2002). We refer the reader to Kaas and Tang (2003) and Foss and Zachary (2003) for closely related discussions in more general probabilistic models. The asymptotic relation $\mathbb{P}\left(S_\tau > x\right) \sim \mathbb{E}\tau\overline{F}(x)$ in (2.4) describes the tail asymptotic property of the compound sum

$$\mathbb{P}(S_\tau \le x) = \sum_{n=0}^{\infty} \mathbb{P}(\tau = n)F^{(n)}(x),$$

where and throughout $F^{(n)}(x)$ denotes the n-fold convolution of the d.f. F whereas $F^{(0)}(x)$ denotes a d.f. degenerate at 0. Related discussions can be found in Chover *et al.* (1973), Embrechts *et al.* (1979), Embrechts and Goldie (1982), Cline (1987), among others.

2.2. *Sums of random variables with dominated variation*

A classical inequality in the literature says that if $F \in \mathcal{S}$, then for any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that

$$\mathbb{P}\left(S_n > x\right) \le C_\varepsilon(1+\varepsilon)^n\overline{F}(x) \tag{2.5}$$

holds for all $n \in \mathbb{N}$ and $x \ge 0$. In the original work by Chistyakov (1964) and Athreya and Ney (1972), the d.f. F is assumed to be supported on $(0,\infty)$, but the extension of this result to the general case of F supported on $(-\infty,\infty)$ is immediate. We remark that the bound given in (2.5) is too rough since it grows exponentially fast with respect to n.

Now we present some more precise bounds as follows:

Theorem 2.4. *(Tang and Yan (2002)) Assume that d.f. $F \in \mathcal{D}$ has a finite expectation μ. Then*

(1) for any $\gamma > 0$, there exists $C(\gamma) > 0$ such that

$$\overline{F^{(n)}}(x) \ge C(\gamma)n\overline{F}(x) \tag{2.6}$$

holds for any $n \ge 1$ and $x \ge \gamma n$;

(2) for any $\gamma > \max\{\mu, 0\}$, there exists $D(\gamma) > 0$ such that

$$\overline{F^{(n)}}(x) \le D(\gamma)n\overline{F}(x) \tag{2.7}$$

holds for any $n \ge 1$ and $x \ge \gamma n$.

A closely related reference in this field is Cline and Hsing (1991), who considered more general x-regions T_n. We think that the most important x-region for applications is $T_n = [\gamma n, \infty)$, $\gamma > 0$.

By Theorem 2.4, for any $\gamma > \max\{\mu, 0\}$, we have

$$0 < \liminf_{n\to\infty} \inf_{x \ge \gamma n} \frac{\overline{F^{(n)}}(x)}{n\overline{F}(x)} \le \limsup_{n\to\infty} \sup_{x\ge\gamma n} \frac{\overline{F^{(n)}}(x)}{n\overline{F}(x)} < \infty. \tag{2.8}$$

i.e. the relation $\overline{F^{(n)}}(x) \asymp n\overline{F}(x)$ holds uniformly for $x \ge \gamma n$ as $n \to \infty$. The following corollary is immediate:

Corollary 2.1. *(Tang and Yan (2002)) Assume that d.f. $F \in \mathcal{D}$ has a finite expectation μ. Then for any $\gamma > \mu$, there exist two positive constants $C(\gamma)$ and $D(\gamma)$ such that*

$$C(\gamma)n\overline{F}(x - n\mu) \le \overline{F^{(n)}}(x) \le D(\gamma)n\overline{F}(x - n\mu) \tag{2.9}$$

holds for any $n \ge 1$ and $x \ge \gamma n$.

2.3. *Applications of the class $\mathcal{S}^*$ to local asymptotics*

For notational convenience we introduce

$$\mathbb{Q}_h(X;x) = \mathbb{Q}_h(F;x) = \frac{1}{h}\left(\overline{F}(x) - \overline{F}(x+h)\right) = \frac{1}{h}F(x, x+h], \quad h > 0.$$

This section is devoted to describing the local asymptotic behavior of the sums S_n and S_τ and the maxima $S_{(n)}$ and $S_{(\tau)}$. Recent investigations in this field can be found in Klüppelberg (1989), Bertoin and Doney (1994), Sgibnev (1996), Tang (2002), and Asmussen *et al.* (2002_b, 2003). Hereafter, the d.f. F is always assumed to satisfy that

$$\mathbb{Q}_h(F;x) \sim Cf(x) \quad \text{for some } C > 0 \text{ and any } h > 0, \tag{2.10}$$

where the function $f(x) : \mathbb{R}^+ \to \mathbb{R}^+$ is proportional to the tail of some d.f. from the class $\mathcal{S}^*$. Without loss of generality, we assume that $f(x)$ is non-increasing in $x \in (0,\infty)$.

Asmussen *et al.* (2002_b, Lemma 1) obtained that if F_1 and F_2 are supported on $(0,\infty)$ and satisfy condition (2.10) with the coefficients C_1, $C_2 > 0$, then it holds for any $h > 0$ that

$$\mathbb{Q}_h(F_1 * F_2; x) \sim (C_1 + C_2) f(x). \tag{2.11}$$

This indicates that the operator $\mathbb{Q}_h(\cdot; x)$ is additive in the asymptotic sense. The extension of (2.11) to the general case where F_i are supported on $(-\infty,\infty)$ is immediate. By induction, we can derive from (2.11) that $\mathbb{Q}_h(S_n; x) \sim n\mathbb{Q}_h(X; x)$, $n \geq 1$. On the other hand, based on (2.11), going along the same line as the proof of Theorem in Sgibnev (1996), we can also obtain that $\mathbb{Q}_h\left(S_{(n)}; x\right) \sim n\mathbb{Q}_h(X; x)$, $n \geq 1$. Furthermore, the proof for $\mathbb{Q}_h\left(X_{(n)}; x\right) \sim n\mathbb{Q}_h(X; x)$, $n \geq 1$, is straightforward. Hence, we summarize:

Theorem 2.5. *(Ng and Tang (2004)) If the d.f. F satisfies (2.10), then it holds for any $n \in \mathbb{N}$ and $h > 0$ that*

$$\mathbb{Q}_h(S_{(n)}; x) \sim \mathbb{Q}_h(S_n; x) \sim \mathbb{Q}_h(X_{(n)}; x) \sim n\mathbb{Q}_h(X; x). \tag{2.12}$$

Now we aim to extend Theorem 2.5 to the randomized case. By copying the proof of Lemma 2 in Asmussen *et al.* (2002_b) with some adjustments we can obtain that if F supported on $(-\infty,\infty)$ satisfies condition (2.10), then for any $\varepsilon > 0$, there exists some $C(\varepsilon, h) > 0$ such that

$$\mathbb{Q}_h\left(F^{(n)}; x\right) \leq C(\varepsilon, h)(1+\varepsilon)^n f(x) \tag{2.13}$$

holds for all $n \in \mathbb{N}$ and $x \geq 0$. Since $\mathbb{Q}_h\left(S_{(n)}; x\right) \leq \sum_{k=1}^{n} \mathbb{Q}_h(S_k; x)$, from (2.13) we immediately obtain that for any $\varepsilon > 0$, there exists some $D(\varepsilon, h) > 0$ such that

$$\mathbb{Q}_h\left(S_{(n)}; x\right) \leq D(\varepsilon, h)(1+\varepsilon)^n f(x) \tag{2.14}$$

holds for all $n \in \mathbb{N}$ and all $x \geq 0$. Hence, if we assume $\mathbb{E}\exp\{\delta\tau\} < \infty$ for some $\delta > 0$, then by the dominated convergence theorem, it follows from (2.12) that $\mathbb{Q}_h(S_\tau; x) \sim \mathbb{E}\tau\mathbb{Q}_h(X; x)$ holds. Similarly, we can obtain $\mathbb{Q}_h\left(S_{(\tau)}; x\right) \sim \mathbb{E}\tau\mathbb{Q}_h(X; x)$. Finally, the proof for the relationship $\mathbb{Q}_h\left(X_{(\tau)}; x\right) \sim \mathbb{E}\tau\mathbb{Q}_h(X; x)$ is also immediate. Hence, we summarize again:

Theorem 2.6. *(Ng and Tang (2004)) If the d.f. F satisfies (2.10) and $\mathbb{E}\exp\{\delta\tau\} < \infty$ for some $\delta > 0$, then it holds for any $h > 0$ that*

$$\mathbb{Q}_h(S_{(\tau)}; x) \sim \mathbb{Q}_h(S_\tau; x) \sim \mathbb{Q}_h(X_{(\tau)}; x) \sim \mathbb{E}\tau\mathbb{Q}_h(X; x). \tag{2.15}$$

3. The classes $\mathcal{M}$ and $\mathcal{M}^*$

3.1. *Introduction*

In the literature, heavy-tailed distributions are often assumed to be long tailed. The following example, however, indicates that even the union $\mathcal{D} \cup \mathcal{L}$ still fails to contain some heavy-tailed distributions of interest:

Example 3.1. (Su and Tang (2003)) Assume that the r.v. X is geometrically distributed with a parameter $p \in (0,1)$. Then for any $r > 1$, the d.f. F_r, say, of the r.v. X^r is heavy tailed but $F_r \notin \mathcal{D} \cup \mathcal{L}$.

In our recent investigation on heavy-tailed distributions we introduced two other classes, $\mathcal{M}$ and $\mathcal{M}^*$; see Tang (2001), Su *et al.* (2002), Su and Tang (2003). Let F be a d.f. with $0 < \mu_F^+ < \infty$. Recall the equilibrium hazard rate $r_e(x)$ of F defined by (2.3). We see that the d.f. F belongs to the class $\mathcal{M}$ if and only if

$$\lim_{x\to\infty} r_e(x) = 0 \tag{3.1}$$

and that the d.f. F belongs to the class $\mathcal{M}^*$ if and only if

$$\limsup_{x\to\infty} x r_e(x) < \infty. \tag{3.2}$$

It is easy to check that the d.f. F_r in Example 3.1 belongs to the class $\mathcal{M}$ and that any d.f. $F \in \mathcal{D} \cup \mathcal{L}$ with $0 < \mu_F^+ < \infty$ should be an element of the class $\mathcal{M}$..

Recently, we extensively investigated the classes $\mathcal{M}$ and $\mathcal{M}^*$ and discovered the following implications and inclusions; see Tang (2001), Su *et al.* (2002), Su and Tang (2003) and Su and Hu (2003):

(1) $F \in \mathcal{M} \Longleftrightarrow F_e \in \mathcal{L}$;
(2) $F \in \mathcal{M}^* \Longleftrightarrow F_e \in \mathcal{C} \Longleftrightarrow F_e \in \mathcal{D}$;
(3) $F \in \mathcal{A}^* \cap \mathcal{M}^* \Longleftrightarrow F_e \in ERV(-\alpha_F, -\beta_F)$, where

$$0 < \alpha_F =: \liminf_{x\to\infty} \frac{x\overline{F}(x)}{\int_x^\infty \overline{F}(u)\mathrm{d}u} \leq \limsup_{x\to\infty} \frac{x\overline{F}(x)}{\int_x^\infty \overline{F}(u)\mathrm{d}u} =: \beta_F < \infty. \tag{3.3}$$

(4) $\mathcal{A}^* \cap \mathcal{M}^* \subset \mathcal{D}$.

3.2. *Tails of distributions from the classes $\mathcal{M}$ and $\mathcal{M}^*$*

A mistake often encountered in the literature is the assertion that relation (1.2) implies

$$\lim_{x\to\infty} e^{tx}\overline{F}(x) = \infty, \quad \forall t > 0. \tag{3.4}$$

n fact counterexamples to this assertion can easily be provided. For any .v. X with a d.f. F supported on $(-\infty, \infty)$, we define its moment index by

$$\mathbb{I}_F = \mathbb{I}(X) = \sup\{v : \mathbb{E}X_+^v < \infty\}. \tag{3.5}$$

This index indeed describes a characteristic of the right tail of the r.v. X. We refer to Daley (2001) and Tang and Tsitsiashvili (2003) for some nteresting discussions. Our further investigation discovers the following fact:

Theorem 3.1. *(Su and Hu (2003)) If $F \in \mathcal{M}$ has a moment index $\mathbb{I}_F > 1$, then (3.4) holds. But there exist d.f.'s in the class $\mathcal{M}$ with only finite 1st moment such that (3.4) doesn't hold.*

Hence, those distributions with only finite 1st moment possesses some abnormal properties, which may have not been discovered so far. Moreover, we have the following result:

Theorem 3.2. *(Su and Hu (2003)) If $F \in \mathcal{M}^*$ has a moment index $\mathbb{I}_F > 1$, then we have*

$$\lim_{x\to\infty} x^\gamma \overline{F}(x) = \infty, \quad \forall \gamma > \frac{\mathbb{I}_F \beta_F}{\mathbb{I}_F - 1}, \tag{3.6}$$

where the quantity β_F is defined by (3.3). But there exist d.f.'s in the class $\mathcal{M}^*$ with only finite 1st moment such that (3.6) doesn't hold.

The theorem shows that, as an extension of $\mathcal{D}$, the class $\mathcal{M}^*$ inherits some properties of the class $\mathcal{D}$. For the equilibrium distribution of a d.f. F in the class $\mathcal{M}^*$, a more interesting property is discovered:

Theorem 3.3. *(Su and Hu (2003)) For $F \in \mathcal{M}^*$, we have*

$$\lim_{x\to\infty} x^\gamma \overline{F_e}(x) = \infty, \quad \forall \gamma > \beta_F. \tag{3.7}$$

3.3. *Closure of the classes $\mathcal{M}$ and $\mathcal{M}^*$*

The following theorem indicates the closure of the classes $\mathcal{M}$ and $\mathcal{M}^*$ under convolution:

Theorem 3.4. *(Su et al. (2002)) Let $F = F_1 * F_2$. If $\overline{F_2}(x) = O(\overline{F_1}(x))$, then*

$$F \in \mathcal{M}^* \Longleftrightarrow F_1 \in \mathcal{M}^*. \tag{3.8}$$

If $\overline{F_2}(x) = o\left(\int_x^\infty \overline{F_1}(t)\mathrm{d}t\right)$, then

$$F_1 \in \mathcal{M} \Longleftrightarrow F \in \mathcal{M}. \tag{3.9}$$

By Theorem 3.4 we know that, for any $n \in \mathbb{N}$,

$$F \in \mathcal{M}^* \Longleftrightarrow F^{(n)} \in \mathcal{M}^*, \qquad F \in \mathcal{M} \Leftrightarrow F^{(n)} \in \mathcal{M}.$$

Now we consider the difference

$$X = Z - Y, \tag{3.10}$$

where Z and Y are two independent and nonnegative r.v.'s which are not degenerate at 0, and Z is distributed by G. Then the d.f. F, say, of the difference X is therefore concentrated on $(-\infty, \infty)$. The following result describes the intimate relationship between the right tails of the r.v.'s X and Z.

Theorem 3.5. *(Su et al. (2002)) Consider the independent difference (3.10). If $\mathbb{E}Z \neq \mathbb{E}Y$, then*

$$G \in \mathcal{M}^* \Longleftrightarrow F \in \mathcal{M}^*, \qquad G \in \mathcal{M} \Longleftrightarrow F \in \mathcal{M}.$$

4. Precise large deviations

Throughout this section, $\{X, X_k, k \geq 1\}$ denotes a sequence of i.i.d. and nonnegative r.v.'s with common d.f. F and finite expectation μ. As before, we denote by S_n the nth partial sum, $n \geq 1$, of the sequence $\{X_k, k \geq 1\}$. All limit relationships are for $n \to \infty$ or $t \to \infty$ unless stated otherwise.

The main stream of research on precise large deviation probabilities has been concentrated on the study of the asymptotics

$$\mathbb{P}(S_n - n\mu > x) \sim n\overline{F}(x) \tag{4.1}$$

uniformly for some x-region T_n. The uniformity in (4.1) is understood in the following sense:

$$\lim_{n\to\infty} \sup_{x \in T_n} \left| \frac{\mathbb{P}(S_n - n\mu > x)}{n\overline{F}(x)} - 1 \right| = 0.$$

Some classical work on precise large deviations can be found in Nagaev ($1969_{a,\ b}$), Heyde ($1967_{a,\ b}$, 1968). See also Nagaev (1973, 1979), who studied the asymptotic relation (4.1) for r.v.'s with d.f. $F \in \mathcal{R}_{-\alpha}$ for some $\alpha > 1$. A nice review of recent research on precise large deviations is Mikosch and Nagaev (1998). Other reviews can be found in Nagaev

(1979) and Rozovski (1993). See also the monographs Vinogradov (1994), Gnedenko and Korolev (1996) and Meerschaert and Scheffler (2001).

Recently, the precise large deviations for the randomly indexed sum (random sum),

$$S_{N(t)} = \sum_{k=1}^{N(t)} X_k, \qquad t \geq 0,$$

have been investigated by many researchers such as Cline and Hsing (1991), Klüppelberg and Mikosch (1997), Mikosch and Nagaev (1998) and Tang *et al.* (2001), among others. Here the random index $N(t)$ is a nonnegative and integer-valued process, independent of the sequence $\{X, X_k, k \geq 1\}$.. We always suppose that the mean function $\lambda(t) \to \infty$ as $t \to \infty$. Most recently, Ng *et al.* (2003) established the large deviations result for a so-called prospective-loss process under a very mild condition on the counting process $N(t)$.

We cite here the following result of Theorem 3.1 in Klüppelberg and Mikosch (1997); see also Proposition 7.1 in Mikosch and Nagaev (1998) for a further extension:

Proposition 4.1. *(Klüppelberg and Mikosch (1997)) Let the common d.f. $F \in ERV(-\alpha, -\beta)$ for some $1 < \alpha \leq \beta < \infty$. If the process $N(t)$ satisfies that*

$$N(t)/\lambda(t) \to_p 1 \qquad \textit{Assumption A}$$

and that for some $\varepsilon > 0$ and any $\delta > 0$,

$$\sum_{k>(1+\delta)\lambda(t)} (1+\varepsilon)^k \mathbb{P}(N(t) > k) = o(1), \qquad \textit{Assumption B}$$

then for any $\gamma > 0$, we have

$$\mathbb{P}\left(S_{N(t)} - \mu\lambda(t) > x\right) \sim \lambda(t)\overline{F}(x) \quad \text{uniformly for } x \geq \gamma\lambda(t). \tag{4.2}$$

As verified in Klüppelberg and Mikosch (1997), homogenous Poisson process satisfies both assumptions A and B above. Some applications of result (4.2) to insurance and finance can be found in Chapter 8 of Embrechts *et al.* (1997) and some of the above-mentioned references. Lately, Su *et al.* (2001) and Tang *et al.* (2001) weakened Assumptions A and B into one as

$$\mathbb{E}\left((N(t))^{\beta+\varepsilon}\, \mathbb{I}_{(N(t)>(1+\delta)\lambda(t))}\right) = O\left(\lambda(t)\right). \qquad \text{Assumption C}$$

for some $\varepsilon > 0$ and any $\delta > 0$. This is a much weaker condition, which can be satisfied, for example, by the general renewal process and the compound renewal process.

Recently, we have made some progress on precise large deviations.

Theorem 4.1. *(Ng et al. (2004)) If $F \in \mathcal{C}$ has a finite expectation μ, then for any fixed $\gamma > 0$, it holds that*

$$\mathbb{P}\left(S_n - n\mu > x\right) \sim n\overline{F}(x) \quad \text{uniformly for } x \geq \gamma n. \tag{4.3}$$

Using the method of Klüppelberg and Mikosch (1997), we can extend Theorem 4.1 to the random case. To this end, we introduce the following notation. Let X be a r.v. with a d.f. F supported on $(-\infty, \infty)$. For any $y > 0$ we set

$$\overline{F}_*(y) = \liminf_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)}, \qquad \overline{F}^*(y) = \limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)}, \tag{4.4}$$

and then define

$$\mathbb{J}_F^+ = \inf\left\{-\frac{\log \overline{F}_*(y)}{\log y} : y > 1\right\} = -\lim_{y\to\infty} \frac{\log \overline{F}_*(y)}{\log y}, \tag{4.5}$$

$$\mathbb{J}_F^- = \sup\left\{-\frac{\log \overline{F}^*(y)}{\log y} : y > 1\right\} = -\lim_{y\to\infty} \frac{\log \overline{F}^*(y)}{\log y}. \tag{4.6}$$

In the terminology of Bingham *et al.* (1987), here the quantities $\mathbb{J}_F^+$ and $\mathbb{J}_F^-$ are the upper and lower Matuszewska indices of the function $f(x) = \left(\overline{F}(x)\right)^{-1}$, $x \geq 0$. Without any confusion, we simply call $\mathbb{J}_F^\pm$ the upper/lower Matuszewska index of the d.f. F. The latter equalities in (4.5) and (4.6) are due to Theorem 2.1.5 in Bingham *et al.* (1987). Trivially, for any d.f. F it holds that $0 \leq \mathbb{J}_F^- \leq \mathbb{J}_F^+ \leq \infty$. But from inequality (2.1.9) in Theorem 2.1.8 in Bingham *et al.* (1987) we see that $F \in \mathcal{D}$ if and only if $\mathbb{J}_F^+ < \infty$. Moreover, if $F \in \mathcal{R}_{-\alpha}$ for some $\alpha \geq 0$, then $\mathbb{J}_F^\pm = \alpha$. For more details of the Matuszewska indices, see Chapter 2.1 of Bingham *et al.* (1987), Cline and Samorodnitsky (1994) and Tang and Tsitsiashvili (2003).

Theorem 4.2. *(Ng et al. (2004)) If $F \in \mathcal{C}$ has a finite expectation μ and $\{N(t), t \geq 0\}$ satisfies*

$$\mathbb{E}N^p(t)\mathbb{I}_{(N(t)>(1+\delta)\lambda(t))} = O\left(\lambda(t)\right) \qquad \textit{Assumption D}$$

for some $p > \mathbb{J}_F^+$ and any $\delta > 0$, then, the large deviations result (4.2) holds.

Now we propose a new application for Theorem 4.2. In the context of insurance risk theory, the claim sizes X_k, $k \geq 1$, are often assumed to be i.i.d. nonnegative r.v.'s with common d.f. F. Their occurrence times σ_k, $k \geq 1$, constitute an ordinary renewal counting process

$$N(t) = \sup\{n : \sigma_n \leq t\}, \qquad t \geq 0, \tag{4.7}$$

with $\lambda(t) = \mathbb{E}N(t) < \infty$ for any $t \geq 0$. With $\sigma_0 = 0$, we can write the inter-arrival times by $Y_k = \sigma_k - \sigma_{k-1}$, $k \geq 1$, which therefore form a sequence of i.i.d. nonnegative r.v.'s. These are standard assumptions in the ordinary renewal model. Now we consider the model in Denuit *et al.* (2002). In their model, the kth insurance policy, $k \geq 1$, is associated with a Bernoulli variable I_k, and the variable I_k has a common expectation $0 < q \leq 1$, where q is the claim occurrence probability of the kth policy, $k \geq 1$. Then, the total claim amount up to time t is

$$S_1(t) = \sum_{k=1}^{N(t)} X_k I_k, \qquad t \geq 0. \tag{4.8}$$

We assume that the three sequences $\{X_k, k \geq 1\}$, $\{Y_k, k \geq 1\}$, and $\{I_k, k \geq 1\}$ are mutually independent and that the sequence $\{I_k, k \geq 1\}$ is negatively associated. Therefore, $S_1(t)$ is a non-standard random sum unless the parameter $q = 1$. Generally speaking, a sequence $\{I_k, k \geq 1\}$ is said to be negatively associated (NA) if, for any disjoint finite subsets A and B of $\{1, 2, \cdots\}$ and any coordinatewise monotonically increasing functions f and g, the inequality

$$\mathrm{Cov}[f(I_i; i \in A),\ g(I_j; j \in B)] \leq 0$$

holds whenever the moment involved exists. For details about the notation of NA, please refer to Alam and Saxena (1981), Joag-Dev and Proschan (1983), and Shao (2000), among others. We proved the following result:

Theorem 4.3. *(Ng et al. (2004)) If $F \in \mathcal{C}$ has a finite expectations μ, then it holds for any $\gamma > 0$ and all $x \geq \gamma\lambda(t)$ that*

$$\mathbb{P}\left(S_1(t) - \mu q \lambda(t) > x\right) \sim q\lambda(t)\overline{F}(x).$$

5. The finite time ruin probability

The following renewal risk model has been extensively investigated in the literature since it was introduced by Sparre Anderson (1957). As described in Section 4.3, the costs of claims X_i, $i \geq 1$, form a sequence of i.i.d., nonnegative r.v.'s with common d.f. F, and the inter-occurrence times $Y_i, i \geq 1$, form another sequence of i.i.d. nonnegative r.v.'s which are not degenerate at 0. We assume that the sequences $\{X_i, i \geq 1\}$ and $\{Y_i, i \geq 1\}$ are mutually independent. The locations of the successive claims constitute a renewal process defined by (4.7). The surplus process of the insurance company is then expressed as

$$R(t) = x + ct - \sum_{i=1}^{N(t)} X_i, \qquad t \geq 0,$$

where $x \geq 0$ denotes the initial surplus, $c > 0$ denotes the constant premium rate, and $\sum_{i=1}^{0} X_i = 0$ by convention. If Y_1 is exponentially distributed, hence $N(t)$ is a homogenous Poisson process, the model above is just the classical risk model, which is also called the compound Poisson risk model.

We define, as usual, the time to ruin of this model with initial surplus $x \geq 0$ as

$$\tau(x) = \inf\{t \geq 0 : R(t) < 0 \mid R(0) = x\}.$$

Hence, the probability of ruin within finite time $t \geq 0$ can be defined as a bivariate function

$$\psi(x; t) = \mathbb{P}\left(\tau(x) \leq t \mid R(0) = x\right), \tag{5.1}$$

and the ultimate ruin probability can be defined as

$$\psi(x; \infty) = \lim_{t \to \infty} \psi(x; t) = \mathbb{P}\left(\tau(x) < \infty \mid R(0) = x\right).$$

In order for the ultimate ruin not to be certain, it is natural to assume that the safety loading condition $\mu = c\mathbb{E}Y_1 - \mathbb{E}X_1 > 0$ holds.

Under the assumption that the equilibrium d.f. F_e defined by (1.1) is subexponential, Veraverbeke (1979) and Embrechts and Veraverbeke (1982) established a celebrated asymptotic relation for the ultimate ruin probability that

$$\psi(x; \infty) \sim \frac{1}{\mu} \int_x^\infty \overline{F}(u)\mathrm{d}u. \tag{5.2}$$

As universally admitted, the study of the finite time ruin probability is more practical, but much harder, than that of the infinite time ruin probability. The problem of finding accurate approximations to the finite time ruin probability for the renewal model has a long history and during the period many methods have been developed.

In this section we aim at a simple asymptotic relation for the finite time ruin probability $\psi(x; t)$ as $x \to \infty$ with special request that this asymptotic relation should be uniform for t in a relevant infinite time interval. Under some mild assumptions on the distribution functions of the heavy-tailed claim size X_1 and of the inter-occurrence time Y_1, applying a recent result by Tang (2004) on the tail probability of the maxima over a finite horizon inspired by Korshunov (2002), we obtained the following result:

Theorem 5.1. *(Tang (2003)) Consider the renewal risk model with the safety loading condition. If $F \in \mathcal{C}$ with its upper Matuszewska index $\mathbb{J}_F^+$ and $\mathbb{E}Y_1^p < \infty$ for some $p > \mathbb{J}_F^+ + 1$, then for arbitrarily given $t_0 \geq 0$ such that $\lambda(t_0) > 0$, it holds uniformly for $t \in [t_0, \infty]$ that*

$$\psi(x;t) \sim \frac{1}{\mu}\int_x^{x+\lambda(t)\mu} \overline{F}(u)\mathrm{d}u. \tag{5.3}$$

Moreover, relation (5.3) can be rewritten as

$$\psi(x;t) \sim \frac{1}{\mu}\int_x^{x+\lambda\mu t} \overline{F}(u)\mathrm{d}u$$

if the horizon t is restricted to a smaller region $[t(x), \infty]$ for arbitrarily given function $t(x) \to \infty$, where $\lambda = 1/\mathbb{E}Y_1$.

The uniformity of the asymptotic relation (5.3) allows us to flexibly vary the horizon t as the initial surplus x increases. For example, under the conditions of Theorem 5.1, from (5.3) we can obtain that the relation

$$\psi(x;t(x)) \sim \frac{1}{\mu}\int_x^{x+\lambda\mu t(x)} \overline{F}(u)\mathrm{d}u \tag{5.4}$$

holds for any real function $t(x)$ such that $t_0 \leq t(x) \leq \infty$. If we specify the d.f. $F \in \mathcal{R}_{-\alpha}$ for some $\alpha > 1$, a more explicit formula can be derived immediately, which extends Corollary 1.6(a) of Asmussen and Klüppelberg (1996) to the renewal model.

The uniformity of relation (5.3) also makes it possible to change the horizon t into a random variable as long as it is independent of the risk system, as done by Asmussen *et al.* (2002) and Avram and Usabel (2003). In fact, we have the following consequence of Theorem 5.1:

Corollary 5.1. *(Tang (2003)) Let the conditions of Theorem 5.1 be valid and let T be an independent r.v. with a d.f. H satisfying $\mathbb{E}T < \infty$ and $\mathbb{P}(T \geq t_0) = 1$ for some $t_0 \geq 0$ such that $\lambda(t_0) > 0$. Then it holds that*

$$\psi(x;T) \sim \mathbb{E}\lambda(T) \cdot \overline{B}(x). \tag{5.5}$$

It is clear that

$$\psi(x;T) = \mathbb{P}\left(\max_{1\leq k\leq N(T)} \sum_{i=1}^{k} (Z_i - c\theta_i) > x\right). \tag{5.6}$$

A similar result as (3.3) has recently been established by Foss and Zachary (2003). However, we remark that our Corollary 5.1 is never a special case of theirs since the r.v. $N(T)$ in (5.6) cannot be explained as a stopping time in their sense.

References

1. Alam, K., Saxena, K.M.L., 1981. Positive dependence in multivariate distributions. *Comm. Statist. A–Theory Methods* **10**, no. **12**, 1183–1196.
2. Asmussen, S., Avram, F., Usabel, M., 2002$_a$. Erlangian approximations for finite-horizon ruin probabilities. *Astin Bull.* **32**, no. **2**, 267–281.
3. Asmussen, S., Foss, S., Korshunov, D.A., 2003. Asymptotics for sums of random variables with local subexponential behaviour. *J. Theor. Probab.*, **16**, no. **2**, 489–518.
4. Asmussen, S., Kalashnikov, V., Konstantinides, D., Klüppelberg, C., Tsitsiashvili, G., 2002$_b$. A local limit theorem for random walk maxima with heavy tails. *Statist. Probab. Lett.* **56**, no. **4**, 399–404.
5. Asmussen, S., Klüppelberg, C., 1996. Large deviations results for subexponential tails, with applications to insurance risk. *Stochastic Process. Appl.* **64**, no. **1**, 103–125.
6. Athreya, K.B., Ney, P.E., 1972. *Branching Processes.* Springer-Verlag, New York-Heidelberg.
7. Avram, F., Usabel, M., 2003. Finite time ruin probabilities with one Laplace inversion. *Insurance Math. Econom.* **32**, no. **3**, 371–377.
8. Bertoin, J., Doney, R.A., 1994. On the local behaviour of ladder height distributions. *J. Appl. Probab.* **31**, no. **3**, 816–821.
9. Bingham, N.H., Goldie, C.M., Teugels, J.L., 1987. *Regular Variation.* Cambridge University Press, Cambridge.
10. Cai, J., Tang, Q., 2004. On max-sum-equivalence and convolution closure of heavy-tailed distributions and their applications. *J. Appl. Probab.* **41**. no. **1**, to appear.
11. Chistyakov, V.P., 1964. A theorem on the sums of independent positive random variables and its applications to branching random processes. Theory Prob. Appl. 9, 640–648.
12. Chover, J., Ney, P., Wainger, S., 1973. Functions of probability measures. *J. Analyse Math.* **26**, 255-302.
13. Cline, D.B.H., 1987. Convolutions of distributions with exponential and subexponential tails. *J. Austral. Math. Soc. Ser. A* **43**, 347–365.
14. Cline, D.B.H., 1994. Intermediate regular and Π variation. *Proc. London Math. Soc. (3)* **68**, no. **3**, 594–616.

15. Cline, D.B.H., Hsing, T., 1991. Large deviation probabilities for sums and maxima of random variables with heavy or subexponential tails. *Preprint.* Texas A&M University.
16. Cline, D.B.H., Samorodnitsky, G., 1994. Subexponentiality of the product of independent random variables. *Stochastic Process. Appl.* 49, no. 1, 75–98.
17. Daley, D.J., 2001. The moment index of minima. *J. Appl. Probab.* **38A**, 33–36.
18. Denuit, M., Lefèvre, C., Utev, S., 2002. Measuring the impact of dependence between claims occurrences. *Insurance Math. Econom.* **30**, no. **1**, 1–19.
19. Embrechts, P., Goldie, C.M., 1982. On convolution tails. *Stoch. Proc. Appl.* **13**, 263–278.
20. Embrechts, P., Goldie, C.M., Veraverbeke, N., 1979. Subexponentiality and infinite divisibility. *Z. Wahrsch. Verw. Gebiete* **49**, no. **3**, 335–347.
21. Embrechts, P., Klüppelberg, C., Mikosch, T., 1997. *Modelling Extremal Events for Insurance and Finance.* Berlin: Springer.
22. Embrechts, P., Veraverbeke, N., 1982. Estimates for the probability of ruin with special emphasis on the possibility of large claims. *Insurance Math. Econom.* **1**, no. **1**, 55–72.
23. Foss, S., Zachary, S., 2003. The maximum on a random time interval of a random walk with long-tailed increments and negative drift. *Ann. Appl. Probab.* **13**, no. **1**, 37–53.
24. Gnedenko, B.V., Korolev, V.Y., 1996. *Random Summation. Limit Theorems and Applications.* CRC Press.
25. Goldie, C.M., Klüppelberg, C., 1998. Subexponential distributions. A practical Guide to Heavy-Tails: Statistical Techniques and Applications. Eds. Adler, R., Feldman, R., Taqqu, M.S. Birkhäuser, Boston.
26. Heyde, C.C., 1967_a. A contribution to the theory of large deviations for sums of independent random variables. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **7**, 303–308.
27. Heyde, C.C., 1967_b. On large deviation problems for sums of random variables which are not attracted to the normal law. *Ann. Math. Statist.* **38**, 1575–1578.
28. Heyde, C.C., 1968. On large deviation probabilities in the case of attraction to a nonnormal stable law. *Sankhyā* **30**, 253–258.
29. Jelenković, P.R., Lazar, A.A., 1999. Asymptotic results for multiplexing subexponential on-off processes. *Adv. Appl. Prob.* **31**, 394–421.
30. Joag-Dev, K., Proschan, F., 1983. Negative association of random variables with applications. *Ann. Statist.* **11**, 286–295.
31. Kaas, R., Tang, Q., 2003. Note on the tail behavior of random walk maxima with heavy tails and negative drift. *North Amer. Actuar. J.* **7**, no. **3**, 57–61.
32. Kalashnikov, V., 1997. *Geometric sums: bounds for rare events with applications. Risk analysis, reliability, queueing.* Kluwer Academic Publishers Group, Dordrecht.
33. Klüppelberg, C., 1989. Subexponential distributions and characterization of related classes. *Probab. Th. Rel. Fields* **82**, no. **2**, 259–269.
34. Klüppelberg, C., Mikosch, T., 1997. Large deviations of heavy-tailed random sums with applications in insurance and finance. *J. Appl. Prob.* **34**, 293–308.
35. Konstantinides, D., Tang, Q., Tsitsiashvili, G., 2002. Estimates for the ruin probability in the classical risk model with constant interest force in the presence of heavy tails. *Insurance Math. Econom.* **31**, no. **3**, 447–460.
36. Korshunov, D.A., 2002. Large-deviation probabilities for maxima of sums of independent random variables with negative mean and subexponential distribution. *Theory Prob. Appl.* **46**, no. **2**, 355–366.
37. Meerschaert, M.M., Scheffler, H.P., 2001. *Limit distributions for sums of independent random vectors. Heavy tails in theory and practice. Wiley Series in Probability and Statistics: Probability and Statistics.* John Wiley & Sons, Inc., New York.
38. Mikosch, T., Nagaev, A.V., 1998. Large deviations of heavy-tailed sums with applications in insurance. *Extremes* **1**, no. **1**, 81–110.
39. Nagaev, A.V., 1969_a. Integral limit theorems for large deviations when Cramér's condition is not fulfilled I, II. *Theory Prob. Appl.* **14**, 51-64, 193–208.
40. Nagaev, A.V., 1969_b. Limit theorems for large deviations when Cramér's conditions are violated (In Russian). *Fiz-Mat. Nauk.* **7**, 17–22.
41. Nagaev, S.V., 1973. Large deviations for sums of independent random variables. In Trans. *Sixth prague conf. on Information Theory, Random Processes and Statistical Decision Functions.* Academic, Prague. 657–674.
42. Nagaev, S.V., 1979. Large deviations of sums of independent random variables. *Ann. Prob.* **7**, 745-789.
43. Ng, K., Tang, Q., 2004. Asymptotic behavior of sums of subexponential random variables on the real line. *J. Appl. Prob.* **41**. no. **1**, to appear.
44. Ng, K., Tang, Q., Yang, H., 2002. Maxima of sums of heavy-tailed random variables. *Astin Bull.* **32**, no. **1**, 43–55.
45. Ng, K., Tang, Q., Yan, J., Yang, H., 2003. Precise large deviations for the prospective-loss process. *J. Appl. Probab.* **40**, no. **2**, 391–400.
46. Ng, K., Tang, Q., Yan, J., Yang, H., 2004. Precise large deviations for sums of random variables with consistently varying tails. *J. in Appl. Probab.* **41**. no. **1**, to appear.
47. Rozovski, L.V., 1993. Probabilities of large deviations on the whole axis. *Theory. Probab. Appl.* **38**, 53–79.
48. Schlegel, S., 1998. Ruin probabilities in perturbed risk models. The interplay between insurance, finance and control (Aarhus, 1997). *Insurance Math. Econom.* **22**, no. **1**, 93–104.
49. Sgibnev, M.S., 1996. On the distribution of the maxima of partial sums. *Stat. Prob. Letters* **28**, 235–238.
50. Shao, Q.M., 2000. A comparison theorem on moment inequalities between negatively associated and independent random variables. *J. Theoret. Probab.* **13**, no. **2**, 343–356.
51. Sparre Anderson, E., 1957. On the collective theory of risk in the case of contagious between the claims. *In: Transactions of the XV International Congress of Actuaries.*
52. Su, C., Hu, Z., 2003. On Two Broad Classes of Heavy-Tailed Distributions.

Submited to Statistica Sinica.
53. Su, C., Tang, Q., 2003. Characterizations on heavy-tailed distributions by means of hazard rate. *Acta Math. Appl. Sinica (English Ser.)* **19**, no. **1**, 135–142.
54. Su, C., Tang, Q., Chen, Y., Liang, H., 2002. Two broad classes of heavy-tailed distributions and their applications to insurance. *Obozrenie Prikladnoi i Promyshlenni Matematiki*, to appear.
55. Su, C., Tang, Q., Jiang, T., 2001. A contribution to large deviations for heavy-tailed random sums. *Sci. China Ser. A* **44**, no. **4**, 438–444.
56. Tang, Q., 2001. *Extremal values of risk processes for insurance and finance: with special emphasis on the possibility of large claims.* Doctoral Thesis, University of Science and Technology of China.
57. Tang, Q., 2002. An asymptotic relationship for ruin probabilities under heavy-tailed claims. *Science in China (Series A)* **45**, no. **5**, 632–639.
58. Tang, Q., 2004. Uniform estimates for the tail probability of maxima over finite horizons with subexponential tails. *Probab. Engrg. Inform. Sci.* **18** (2004), no. **1**, 71–86.
59. Tang, Q., 2003. Precise Large deviations for the finite time ruin probability in the renewal risk model with consistently varying tails. Submitted.
60. Tang, Q., Su, C., Jiang, T., Zhang, J., 2001. Large deviations for heavy-tailed random sums in compound renewal model. *Stat. Prob. Letters* **52**, no. **1**, 91–100.
61. Tang, Q., Tsitsiashvili, G., 2003. Precise estimates for the ruin probability in finite horizon in a discrete-time model with heavy-tailed insurance and financial risks. *Stochastic Process. Appl.* **108** (2003), no. **2**, 299–325.
62. Tang, Q., Yan, J., 2002. A sharp inequality for the tail probabilities of sums of i.i.d. r.v.'s with dominatedly varying tails. *Science in China (Series A)* **45**, no. **8**, 1006–1011.
63. Veraverbeke, N., 1977. Asymptotic behaviour of Wiener-Hopf factors of a random walk. *Stochastic Processes Appl.* **5**, no. **1**, 27–37.
64. Vinogradov, V., 1994. *Refined large deviation Limit theorems.* Longman, Harlow.

THE INSURANCE REGULATORY REGIME IN HONG KONG (WITH AN EMPHASIS ON THE ACTUARIAL ASPECT)

AUGUST CHOW

Office of the Commissioner of Insurance
21/F Queensway Government Offices
66 Queensway, Hong Kong
E-mail: augustchow@oci.gov.hk

1. The Insurance Regulatory Regime in Hong Kong

In this article, I would like to share with you the insurance regulatory regime in Hong Kong, and in particular some of the actuarial applications for purposes of monitoring the life insurance industry in Hong Kong today.

2. Outline

This article will cover 3 areas:

First, to enable you to have a better understanding as to what we do, I would like to give you an overview of the life insurance market and our insurance regulatory philosophy in Hong Kong.

Then, I would like to cover the Appointed Actuary System that we have adopted in Hong Kong and the roles played by the actuaries in the market place.

And finally, I would like to provide you with an example of an application on Reserving for Investment Guarantees with respect to the retirement scheme business sold in Hong Kong.

3. An Overview of the Hong Kong Life Insurance Market

Compared with other territories in the region, Hong Kong is a free and open city with many insurers competing in the market, and has the highest number of insurers per capita.

However, the penetration ratio, which is defined as 100% times the number of policies per capita, is only 70% in Hong Kong which is relatively