Branching structure of uniform recursive trees

FENG Qunqiang, SU Chun & HU Zhishui

Department of Statistics and Finance, University of Science and Technology of China, Hefei 230026, China
Correspondence should be addressed to Feng Qunqiang (email: fengqq@mail.ustc.edu.cn)
Received October 17, 2003; revised January 28, 2005

Abstract The branching structure of uniform recursive trees is investigated in this paper. Using the method of sums for a sequence of independent random variables, the distribution law of η_n, the number of branches of the uniform recursive tree of size n are given first. It is shown that the strong law of large numbers, the central limit theorem and the law of iterated logarithm for η_n follow easily from this method. Next it is shown that η_n and ξ_n, the depth of vertex n, have the same distribution, and the distribution law of $\zeta_{n,m}$, the number of branches of size m, is also given, whose asymptotic distribution is the Poisson distribution with parameter $\lambda = \frac{1}{m}$. In addition, the joint distribution and the asymptotic joint distribution of the numbers of various branches are given. Finally, it is proved that the size of the biggest branch tends to infinity almost sure as $n \to \infty$.

Keywords: uniform recursive tree, branch, depth, distribution law, limit theorem.

DOI: 10.1360/03ys0299

1 Introduction

A tree is a connected simple graph without cycles [1]. The recursive tree of size n is a kind of random trees on n particles that attach to each other randomly. The process of generating a recursive tree is as follows (see ref. [2]): let the set of particles be $\{1, 2, \cdots, n\}$, and $\{p_{k,i},\ i = 1, 2, \cdots, k\}$, $k = 1, 2, \cdots, n-1$, be a sequence of probability mass functions, i.e.

$$p_{k,i} \geqslant 0, \quad \sum_{i=1}^{k} p_{k,i} = 1, \quad k = 1, 2, \cdots, n-1.$$

At step 1, put all particles in a plane; at step 2, particle 2 attaches to particle 1; at step 3, particle 3 attaches to particle 1 with probability p_{21} or to particle 2 with probability p_{22}. In general, at step $k+1$, particle $k+1$ attaches to one of the particles in the set $\{1, 2, \cdots, k\}$ with the probabilities $p_{k,i}, i = 1, 2, \cdots, k$, respectively. After n steps, the resulting tree with the root vertex 1 is called a recursive tree. If

$$p_{k,i} = \frac{1}{k}, \quad i = 1, 2, \cdots, k; \quad k = 1, 2, \cdots, n-1,$$

i.e. at each step the new particle attaches to a uniformly selected particle from the previous ones, independent of previous attachments, then we call it a uniform recursive tree, denoted by $\mathcal{T}_n$. At the kth$(k \geqslant 2)$ step we can make $k-1$ choices, so $(n-1)!$ different trees can be obtained, and each tree occurs with the same probability $\frac{1}{(n-1)!}$.

With many applications, recursive trees have been proposed as models for the spread of epidemics[3], the family trees of preserved copies of ancient or medieval texts[4], pyramid schemes[5], etc. Here we give an example of the model for the spread of epidemics.

Suppose there exist n persons infected by a specific infectious disease (e.g. SARS) in turn in some area, and only one of them is the original case. The second case must be infected by the original one. Unknowing the law of infection, we suppose that the third case was infected by one of the previous two with the probability $1/2$. In general, we suppose that the kth case was infected by one of the previous $k-1$ cases with respective probabilities $\frac{1}{k-1}$, $k = 2, 3, \cdots, n$. Let vertex k represent the kth case, and vertex i attaches to vertex j $(1 \leqslant i < j \leqslant n)$ if and only if the jth case was infected by the ith case. Then we obtain a uniform recursive tree. By this token, such a study of the uniform recursive trees can make the law of infection clear to a certain extent.

In $\mathcal{T}_n$, D_j denotes the set of vertices of the jth generation. A subtree with the root in D_1 is called a branch, which is also a uniform recursive tree [6]. Obviously, the number of branches is the total number of vertices in the set D_1, denoted by η_n. If the size of a branch is m $(1 \leqslant m \leqslant n-1)$, we call it an m-branch, and let $\zeta_{n,m}$ denote the number of the m-branches. In particular, if $m = 1$, the only vertex in the branch is called a child-leaf of the root 1. It is easy to see that $\eta_n = \sum_{i=1}^{n-1} \zeta_{n,i}$. Furthermore, if vertex $k \in D_j$, we say that the depth of vertex k is j, and let ξ_k denote the depth of vertex k.

Many authors have studied the depth of vertices. For example, Szymański has given the distribution of ξ_n, the depth of vertex n[7]; Devroye has proved the central limit theorem for ξ_n[8]; Mahmoud has done some further studies on the limiting behavior of ξ_n and $\sum_{k=1}^{n} \xi_k$ [9,10]; Meir and Moon have given the distribution of the number of vertices in each generation [6].

It is easy to see that the branching structure is one of the important properties of the uniform recursive trees, but as far as we know, no one has considered it. In this paper, our main purpose is to study it. In Section 2, taking advantage of the mutual independence of the events $(2 \in D_1), (3 \in D_1), \cdots, (n-1 \in D_1)$, we establish easily the strong law of large numbers, the central limit theorem and the law of iterated logarithm for η_n, and give the distribution law of η_n directly. In Section 3, we prove that η_n and ξ_n have the same distribution law. In Section 4, we give the distribution law and asymptotic distribution law of $\zeta_{n,m}$, but also give the joint distribution of the numbers of various branches and their expectations and covariance matrix, and simultaneously we prove the asymptotic independence of them. Finally, in Section 5, we show that ν_n, the size of the biggest branch of $\mathcal{T}_n$, tends to infinity almost sure as $n \to \infty$.

2 The number of branches of $\mathcal{T}_n$

In this section, we shall discuss the properties of η_n. Meir and Moon have given the distribution law of η_n, but their method is more complex and they have not discussed the properties of η_n ulteriorly [6].

Let $X_j = I(j+1 \in D_1)$. According to the process of generating a uniform recursive tree, it is just related to the jth step of the process that vertex j is in D_1 or not. Therefore, $X_1, X_2, \cdots, X_{n-1}$ are mutually independent Bernoulli random variables, and

$$\eta_n = \sum_{j=1}^{n-1} X_j = 1 + \sum_{j=2}^{n-1} X_j. \tag{1}$$

It is easy to see that

$$\mathrm{P}(X_j = 1) = \mathrm{P}(j+1 \in D_1) = \frac{1}{j}, \quad j = 1, \cdots, n-1,$$

then

$$\mathrm{E}X_j = \frac{1}{j}, \quad \mathrm{Var}\, X_j = \frac{j-1}{j^2}, \quad \mathrm{E}|X_j - \mathrm{E}X_j|^3 \leqslant \mathrm{E}X_j^3 = \frac{1}{j}, \quad j = 1, \cdots, n-1.$$

Let $\log x = \ln \max\{e, x\}$. Furthermore, as $n \to \infty$, we have

$$\mathrm{E}\eta_n = \sum_{j=1}^{n-1} \mathrm{E}X_j = \sum_{j=1}^{n-1} \frac{1}{j} = \log n + O(1); \tag{2}$$

$$B_n := \mathrm{Var}\, \eta_n = \sum_{j=1}^{n-1} \mathrm{Var}\, X_j = \sum_{j=1}^{n-1} \frac{j-1}{j^2} = \log n + O(1); \tag{3}$$

$$G_n := \sum_{j=1}^{n-1} \mathrm{E}|X_j - \mathrm{E}X_j|^3 \leqslant C \sum_{j=1}^{n-1} \mathrm{E}X_j^3 = C \sum_{j=1}^{n-1} \frac{1}{j} = C \log n + O(1). \tag{4}$$

Theorem 1 (Marcinkiewicz SLLN). For any $1 \leqslant p < 2$, as $n \to \infty$,

$$\frac{\eta_n - \log n}{\log^{1/p} n} \longrightarrow 0, \quad a.s. \tag{5}$$

In particular,

$$\frac{\eta_n}{\log n} \longrightarrow 1, \quad a.s. \tag{6}$$

Proof. By (1), η_n is a sum for mutually independent random variables, $X_1, X_2, \cdots, X_{n-1}$, and

$$\sum_{n=1}^{\infty} \frac{\mathrm{Var}\, X_n}{\log^{2/p} n} \leqslant \sum_{n=1}^{\infty} \frac{1}{n(\log n)^{2/p}} < \infty.$$

Therefore, according to Theorem 6.6 in ref. [11],

$$\frac{\eta_n - \mathrm{E}\eta_n}{\log^{1/p} n} = \frac{\sum\limits_{j=1}^{n-1} (X_j - \mathrm{E}X_j)}{\log^{1/p} n} \longrightarrow 0, \quad a.s.$$

And it is easy to see that

$$\frac{\mathrm{E}\eta_n - \log n}{\log^{1/p} n} \longrightarrow 0, \quad \frac{\mathrm{E}\eta_n}{\log n} \longrightarrow 1.$$

Hence, Theorem 1 holds. □

Theorem 2 (CLT).

$$\frac{\eta_n - \log n}{\sqrt{\log n}} \xrightarrow{d} N(0,1).$$

Proof. As $n \to \infty$,

$$\frac{1}{B_n^{3/2}} \sum_{j=1}^{n-1} \mathrm{E}|X_j - \mathrm{E}X_j|^3 \leqslant C(\log n)^{-1/2} \to 0.$$

Then $X_1, X_2, \cdots$ satisfy the Lyapunov's condition, i.e.,

$$\frac{\eta_n - \mathrm{E}\eta_n}{B_n} \xrightarrow{d} N(0,1).$$

Thus, by (2) and (3), Theorem 2 holds. □

Theorem 3 (LIL).

$$\limsup_{n\to\infty} \frac{\eta_n - \log n}{\sqrt{2\log n \log\log\log n}} = 1 \quad \text{a.s.};$$
$$\liminf_{n\to\infty} \frac{\eta_n - \log n}{\sqrt{2\log n \log\log\log n}} = -1 \quad \text{a.s.}$$

Proof. By (1), η_n is a sum for uniform bounded mutually independent random variables, and as $n \to \infty$,

$$\mathrm{E}\eta_n = \log n + O(1), \quad B_n = \log n + O(1), \quad \log\log B_n \sim \log\log\log n.$$

Therefore, Theorem 3 follows by the well-known Kolmogorov's law of iterated logarithm. □

Using (1), we can write out the distribution law of η_n easily. Assume that m_i, $i = 1, 2, \cdots$ only can take values on nature numbers, and let

$$\beta_{n,\,0} = 1, \quad \beta_{n,\,k} = \sum_{1\leqslant m_1<\cdots<m_k\leqslant n-2} m_1\cdots m_k, \qquad k = 1, 2, \cdots, n-2, \tag{7}$$

where the sum extends over all $m_1, \cdots, m_k \in \mathcal{N}$ satisfying $1 \leqslant m_1 < \cdots < m_k \leqslant n-2$, for a fixed $k \in \{1, 2, \cdots, n-2\}$.

Theorem 4. If $n \geqslant 2$, then

$$\mathrm{P}(\eta_n = k) = \frac{\beta_{n,n-1-k}}{(n-1)!}, \quad k = 1, \cdots, n-1. \tag{8}$$

Proof. Since η_n is the sum of $n-1$ mutually independent Bernoulli random variables, $X_1, X_2, \cdots, X_{n-1}$, the event $(\eta_n = k)$ occurs, if and only if k of them equal 1 and the rest equal 0. Thus,

$$\begin{aligned}\mathrm{P}(\eta_n = k) &= \sum_{1\leqslant j_1<\cdots<j_k\leqslant n-1} \left(\prod_{i=1}^{k} \frac{1}{j_i} \prod_{j\notin\{j_1,\cdots,j_k\}} \left(1 - \frac{1}{j}\right)\right) \\ &= \frac{1}{(n-1)!} \sum_{1\leqslant j_1<\cdots<j_k\leqslant n-1} \prod_{j\notin\{j_1,\cdots,j_k\}} (j-1)\end{aligned}$$

$$= \frac{1}{(n-1)!} \sum_{1 \leqslant m_1 < \cdots < m_{n-1-k} \leqslant n-2} m_1 \cdots m_{n-1-k} = \frac{\beta_{n,n-1-k}}{(n-1)!}. \qquad \square$$

3 The depth of vertex n

In this section, we shall prove that ξ_n, the depth of vertex n, and η_n have the same distribution law.

There exists only a shortest path between the root and vertex n in $\mathcal{T}_n$, denoted by Z_n. Except the root, the number of vertices in Z_n is just the depth of vertex n. Obviously, $n \in Z_n$, thus,

$$\xi_n = 1 + \sum_{j=2}^{n-1} I(j \in Z_n). \tag{9}$$

We shall discuss the distribution law of $I(j \in Z_n)$ first. Obviously, $\mathrm{P}(n-1 \in Z_n) = \frac{1}{n-1}$. The event $(n-2 \in Z_n)$ occurs, if and only if vertex $n-2$ is the parent or the grandparent of vertex n, therefore,

$$\mathrm{P}(n-2 \in Z_n) = \frac{1}{n-1} + \frac{1}{n-1} \cdot \frac{1}{n-2} = \frac{1}{n-2}.$$

Similarly, by induction,

$$\mathrm{P}(i \in Z_n) = \frac{1}{i}, \quad i = 2, \cdots, n-1. \tag{10}$$

In fact, we assume that for some $2 \leqslant i \leqslant n-2$, (10) holds for all $i < j \leqslant n-1$. We only need to prove that it also holds for i. Let $A_{i,\,j}$ be the event that vertex j is a child of vertex i. It is easy to see that

$$I(i \in Z_n) = \sum_{j=i+1}^{n} I(j \in Z_n,\ A_{i,\,j}).$$

$A_{i,\,j}$ is related to the jth step of the generating process only and the event $(j \in Z_n)$ is just related to step $j+1, \cdots, n$, therefore, the two events $A_{i,\,j}$ and $(j \in Z_n)$ are mutually independent. Hence,

$$\begin{aligned}\mathrm{P}(i \in Z_n) =& \mathrm{E}I(i \in Z_n) = \sum_{j=i+1}^{n} \mathrm{E}I(j \in Z_n,\ A_{i,\,j}) = \sum_{j=i+1}^{n} \mathrm{P}(j \in Z_n,\ A_{i,\,j}) \\ =& \sum_{j=i+1}^{n} \mathrm{P}(j \in Z_n)\mathrm{P}(A_{i,\,j}) = \sum_{j=i+1}^{n-1} \frac{1}{j(j-1)} + \frac{1}{n-1} = \frac{1}{i}.\end{aligned}$$

Secondly, the random variables $I(2 \in Z_n), \cdots, I(n-1 \in Z_n)$ are mutually independent. For any $2 \leqslant k \leqslant n-2$ and $2 \leqslant j_k < \cdots < j_2 < j_1 \leqslant n-1$, the event

$$(I(j_i \in Z_n) = 1,\ I(j \in Z_n) = 0,\ j \neq j_i;\ i = 1, 2, \cdots, k)$$

represents that vertex n is a child of j_1 and j_i is a child of j_{i+1}, $i = 1, \cdots k-1$. Let

$j_0 = n$. And by the rule of generating process,

$$\begin{aligned} &\mathrm{P}\left(I(j_i \in Z_n) = 1,\ I(j \in Z_n) = 0,\ j \neq j_i;\ i = 1, 2, \cdots, k\right) \\ &= \prod_{i=0}^{k} \frac{1}{j_i - 1} = \frac{1}{(n-1)!} \prod_{j \notin \{j_1, \cdots, j_k\}} (j-1) = \prod_{i=1}^{k} \frac{1}{j_i} \prod_{j \notin \{j_1, \cdots, j_k\}} \frac{j-1}{j} \\ &= \prod_{i=1}^{k} \mathrm{P}(I(j_i \in Z_n) = 1) \prod_{j \notin \{j_1, \cdots, j_k\}} \mathrm{P}(I(j_i \in Z_n) = 0), \end{aligned}$$

which yields that $I(2 \in Z_n), \cdots, I(n-1 \in Z_n)$ are mutually independent random variables.

Comparing (9) with (1), the expression of η_n (in (1), $X_1 = I(2 \in D_1) = 1$), we can see that they are the same, then ξ_n and η_n have the same distribution law. Therefore, Theorem 1, Theorem 2 and Theorem 3 still hold for ξ_n. The first two results can be found in Devroye[8] and Mahmoud[10], but their proof are much more complex.

4 The number of m-branches

In this section, we will give not only the distribution law and the asymptotic distribution of the numbers of various branches, but also the joint distribution law and the asymptotic joint distribution of all the different branches.

4.1 The distribution law and the asymptotic distribution of $\zeta_{n,1}$

In fact, the result in this subsection is a part of Theorem 6 in the next, but for the particularity of child-leaves, we give a different way here.

First we give a recursive formula for $\zeta_{n,1}$. According to the total probability formula,

$$\begin{aligned} \mathrm{P}(\zeta_{n+1,1} = k) =& \mathrm{P}(\zeta_{n,1} = k)\mathrm{P}(\zeta_{n+1,1} = k \mid \zeta_{n,1} = k) \\ &+ \mathrm{P}(\zeta_{n,1} = k-1)\mathrm{P}(\zeta_{n+1,1} = k \mid \zeta_{n,1} = k-1) \\ &+ \mathrm{P}(\zeta_{n,1} = k+1)\mathrm{P}(\zeta_{n+1,1} = k \mid \zeta_{n,1} = k+1) \\ =& \frac{n-k-1}{n}\mathrm{P}(\zeta_{n,1} = k) \\ &+ \frac{1}{n}\mathrm{P}(\zeta_{n,1} = k-1) + \frac{k+1}{n}\mathrm{P}(\zeta_{n,1} = k+1). \end{aligned}$$

That is,

$$\begin{aligned} &n\mathrm{P}(\zeta_{n+1,1} = k) - (n-1)\mathrm{P}(\zeta_{n,1} = k) \\ =& \mathrm{P}(\zeta_{n,1} = k-1) + (k+1)\mathrm{P}(\zeta_{n,1} = k+1) - k\mathrm{P}(\zeta_{n,1} = k). \end{aligned}$$

Replace n by j and sum up for $j = 1, 2, \cdots, n$, then

$$\begin{aligned} n\mathrm{P}(\zeta_{n+1,1} = k) =& \sum_{j=1}^{n} \mathrm{P}(\zeta_j = k-1) + \sum_{j=1}^{n} (k+1)\mathrm{P}(\zeta_j = k+1) \\ &- \sum_{j=1}^{n} k\mathrm{P}(\zeta_j = k). \end{aligned} \tag{11}$$

Using the recursive formula (11), we can prove the following theorem.

Theorem 5. For any $n \in \mathcal{N}$,

$$P(\zeta_{n,1} = k) = \frac{a_{n-k}}{k!}, \quad k = 0, 1, \cdots, n-1, \tag{12}$$

where

$$a_i = \sum_{j=0}^{i-1} \frac{(-1)^j}{j!}, \quad i \geqslant 1. \tag{13}$$

Furthermore, as $n \to \infty$, the asymptotic distribution of $\zeta_{n,1}$ is the Poisson distribution with parameter $\lambda = 1$.

Proof. It is easy to verify that the sequence $\{a_n\}$ satisfies the following recursive relation:

$$a_1 = 1, \quad a_2 = 0, \quad a_j = \frac{1}{j-1}\sum_{i=1}^{j-2} a_i, \quad j \geqslant 2, \tag{14}$$

and (12) holds for $n = 1, 2$.

Suppose that (12) holds for n $(n \geqslant 2)$. It suffices to prove that it still holds for $n+1$. Obviously,

$$P(\zeta_{n+1,1} = n-1) = 0, \quad P(\zeta_{n+1,1} = n) = \frac{1}{n!}.$$

By (11), (14) and inductive assumption, then for any $k \in \{0, 1, \cdots, n-2\}$,

$$\begin{aligned}
nP(\zeta_{n+1,1} = k) =& \sum_{j=1}^{n} P(\zeta_j = k-1) + \sum_{j=1}^{n}(k+1)P(\zeta_j = k+1) - \sum_{j=1}^{n} kP(\zeta_j = k) \\
=& \sum_{j=k}^{n} \frac{a_{j-(k-1)}}{(k-1)!} + \sum_{j=k+2}^{n} (k+1)\cdot\frac{a_{j-k-1}}{(k+1)!} - \sum_{j=k+1}^{n} k\cdot\frac{a_{j-k}}{k!} \\
=& \frac{a_{n-(k-1)}}{(k-1)!} + \frac{1}{k!}\sum_{j=1}^{n-k-1} a_j = \frac{(k+n-k)a_{n-(k-1)}}{k!} \\
=& \frac{na_{n+1-k}}{k!},
\end{aligned}$$

which yields that (12) holds for any $n \in \mathcal{N}$.

From (12), it is obvious that the asymptotic distribution of $\zeta_{n,1}$ is the Poisson distribution with parameter $\lambda = 1$. □

Moreover, by (12), a consequence of Theorem 5 is as follows.

Corollary. For any nature number $n \geqslant 3$,

$$E\zeta_{n,1} = 1, \quad \mathrm{Var}\zeta_{n,1} = 1.$$

4.2 The general cases

Obviously, for any $m \in \{1, \cdots, n-1\}$, we have

$$P(\zeta_{n,m} \geqslant 0) = 1, \quad P\left(\zeta_{n,m} > \left[\frac{n-1}{m}\right]\right) = 0, \tag{15}$$

where $[t]$ is the biggest integer not more than t.

Now we prove the following theorem.

Theorem 6. In uniform recursive trees of size n, the distribution law of $\zeta_{n,m}$, the number of the m-branches, is as follows:

$$\mathrm{P}(\zeta_{n,m}=k)=\frac{1}{m^k k!}\sum_{i=0}^{[\frac{n-1}{m}]-k}\frac{(-1)^i}{i!m^i},\quad k=0,1,\cdots,\left[\frac{n-1}{m}\right]. \tag{16}$$

Specially, if $m>\frac{n-1}{2}$, $\zeta_{n,m}$ is a Bernoulli random variable, i.e.

$$\mathrm{P}(\zeta_{n,m}=1)=1-\mathrm{P}(\zeta_{n,m}=0)=\frac{1}{m}.$$

Proof. From the set $\{2,3,\cdots,n\}$, i subsets of size m are chosen to make i m-branches (each may have $(m-1)!$ forms), and the rest of $n-mi-1$ vertices attach arbitrarily by the above rule . Therefore, the number of the ways of generating a recursive tree is

$$\begin{aligned}&\frac{\binom{n-1}{m}\binom{n-m-1}{m}\cdot\cdots\cdot\binom{n-m(i-1)-1}{m}((m-1)!)^i(n-mi-1)!}{i!}\\ =&\frac{(n-1)!}{m^i i!},\qquad 1\leqslant i\leqslant\left[\frac{n-1}{m}\right].\end{aligned} \tag{17}$$

On the other hand, by (15),

$$\sum_{j=0}^{[\frac{n-1}{m}]}\mathrm{P}(\zeta_{n,m}=j)=1.$$

Set $i=1$ in (17), then

$$\frac{\binom{n-1}{m}(m-1)!(n-m-1)!}{(n-1)!}=\frac{1}{m}.$$

In view of the fact that each recursive tree which has j m-branches exactly is counted j times and $|\mathcal{T}_n|=(n-1)!$, the left of the above formula is $\sum\limits_{j=1}^{[\frac{n-1}{m}]}\binom{j}{1}\mathrm{P}(\zeta_{n,m}=j)$. Hence,

$$\sum_{j=1}^{[\frac{n-1}{m}]}\binom{j}{1}\mathrm{P}(\zeta_{n,m}=j)=\frac{1}{m}. \tag{18}$$

Similarly,

$$\sum_{j=2}^{[\frac{n-1}{m}]}\binom{j}{2}\mathrm{P}(\zeta_{n,m}=j)=\frac{1}{2m^2}, \tag{19}$$

$$\cdots\cdots\cdots\cdots$$

$$\sum_{j=i}^{[\frac{n-1}{m}]}\binom{j}{i}\mathrm{P}(\zeta_{n,m}=j)=\frac{1}{m^i i!},$$

$$\cdots\cdots\cdots\cdots$$

$$\mathrm{P}\left(\zeta_{n,m} = \left[\frac{n-1}{m}\right] - 1\right) + \left(\left[\frac{n-1}{m}\right]\right) \mathrm{P}\left(\zeta_{n,m} = \left[\frac{n-1}{m}\right]\right)$$
$$= \frac{1}{m^{[\frac{n-1}{m}]-1}([\frac{n-1}{m}] - 1)!},$$
$$\mathrm{P}\left(\zeta_{n,m} = \left[\frac{n-1}{m}\right]\right) = \frac{1}{m^{[\frac{n-1}{m}]}([\frac{n-1}{m}])!}.$$

Consider the $[\frac{n-1}{m}] + 1$ formulae above from the bottom up, then it is easy to yield (16).

If $m > \frac{n-1}{2}$, $\zeta_{n,m}$ only can take values of 0 or 1. Since $[\frac{n-1}{2}] = 1$,

$$\mathrm{P}(\zeta_{n,m} = 1) = 1 - \mathrm{P}(\zeta_{n,m} = 0) = \frac{1}{m},$$

by (16). □

By (18) and (19), we can obtain the expectation and variance of $\zeta_{n,m}$:

Corollary. (1) For any $n \geqslant 2$,

$$\mathrm{E}(\zeta_{n,m}) = \frac{1}{m}, \quad m = 1, \cdots, n-1;$$

(2) For any $n \geqslant 3$,

$$\mathrm{Var}(\zeta_{n,m}) = \begin{cases} \frac{1}{m}, & 1 \leqslant m \leqslant \frac{n-1}{2}; \\ \frac{m-1}{m^2}, & \frac{n-1}{2} < m \leqslant n-1. \end{cases} \tag{20}$$

Proof. It follows from (18) that $\mathrm{E}\zeta_{n,m} = \frac{1}{m}$. And by (19), if $1 \leqslant m \leqslant \frac{n-1}{2}$,

$$\mathrm{E}(\zeta_{n,m}^2) - \mathrm{E}(\zeta_{n,m}) = \frac{1}{m^2},$$

thus,

$$\mathrm{Var}(\zeta_{n,m}) = \mathrm{E}(\zeta_{n,m}^2) - (\mathrm{E}(\zeta_{n,m}))^2 = \frac{1}{m};$$

if $\frac{n-1}{2} < m \leqslant n-1$, the result is obvious. □

From Theorem 6, it is easy to see

Theorem 7. For any $m \in \mathcal{N}$, the asymptotic distribution of $\zeta_{n,m}$ is the Poisson distribution with the parameter $\lambda = \frac{1}{m}$, as $n \to \infty$, i.e.

$$\lim_{n\to\infty} \mathrm{P}(\zeta_{n,m} = k) = e^{-1/m}\frac{1}{m^k k!}, \quad k = 0, 1, \cdots; \quad m = 1, 2, \cdots. \tag{21}$$

4.3 The joint distribution of $\zeta_{n,m}$

Next we give the joint distribution of random vector $(\zeta_{n,1}, \zeta_{n,2}, \cdots, \zeta_{n,n-1})$.

Theorem 8. In $\mathcal{T}_n$, the joint distribution of the numbers of various branches

$$(\zeta_{n,1},\ \zeta_{n,2},\ \cdots,\ \zeta_{n,n-1})$$

is as follows:

$$\mathrm{P}(\zeta_{n,1} = x_1,\ \zeta_{n,2} = x_2,\ \cdots,\ \zeta_{n,n-1} = x_{n-1}) = \prod_{m=1}^{n-1} \frac{1}{m^{x_m} x_m!}, \tag{22}$$

where $\{x_1, \cdots, x_{n-1}\}$ is any sequence of nonnegative integers satisfying the condition

$$\sum_{i=1}^{n-1} ix_i = n-1.$$

Proof. It suffices to compute the number of the elementary events (corresponding to a certain recursive tree) in the event $\{\zeta_{n,1} = x_1, \cdots, \zeta_{n,n-1} = x_{n-1}\}$. Consider the groups of $n-1$ vertices (the vertices of a group belong to the same branch). The number of the ways of grouping is

$$\frac{(n-1)!}{(1!)^{x_1}(2!)^{x_2}\cdots((n-1)!)^{x_{n-1}}} \cdot \frac{1}{x_1!x_2!\cdots x_{n-1}!}.$$

And m-branch has $(m-1)!$ different forms, so the number of the elementary events in $\{\zeta_{n,1} = x_1, \cdots, \zeta_{n,n-1} = x_{n-1}\}$ is

$$\frac{(n-1)!}{(1!)^{x_1}(2!)^{x_2}\cdots((n-1)!)^{x_{n-1}}} \cdot \frac{(0!)^{x_1}(1!)^{x_2}\cdots[(n-2)!]^{x_{n-1}}}{x_1!x_2!\cdots x_{n-1}!}$$
$$=\frac{(n-1)!}{1^{x_1}2^{x_2}\cdots(n-1)^{x_{n-1}}x_1!x_2!\cdots x_{n-1}!}.$$

Since the elementary events occur with the same probability $\frac{1}{(n-1)!}$,

$$\mathrm{P}(\zeta_{n,1} = x_1,\ \zeta_{n,2} = x_2,\ \cdots,\ \zeta_{n,n-1} = x_{n-1})$$
$$=\frac{1}{(n-1)!} \cdot \frac{(n-1)!}{1^{x_1}2^{x_2}\cdots(n-1)^{x_{n-1}}x_1!x_2!\cdots x_{n-1}!} = \prod_{m=1}^{n-1} \frac{1}{m^{x_m}x_m!}. \qquad \square$$

In the previous subsection, we have obtained the expectation of random vector $(\zeta_{n,1}, \zeta_{n,2}, \cdots, \zeta_{n,n-1})$, i.e.

$$\mathrm{E}(\zeta_{n,1},\ \zeta_{n,2},\ \cdots,\ \zeta_{n,n-1}) = \left(1,\ \frac{1}{2},\ \cdots,\ \frac{1}{n-1}\right). \tag{23}$$

Now we give its covariance matrix.

Theorem 9. For any $1 \leqslant k < l \leqslant n-1$, if $k+l \leqslant n-1$,

$$\mathrm{Cov}(\zeta_{n,k},\ \zeta_{n,l}) = 0; \tag{24}$$

and if $k+l > n-1$,

$$\mathrm{Cov}(\zeta_{n,k},\ \zeta_{n,l}) = -\frac{1}{kl}. \tag{25}$$

Proof. If $1 \leqslant k < l \leqslant n-1$ and $k+l > n-1$, it is obvious that

$$\mathrm{P}(\zeta_{n,k} = i,\ \zeta_{n,l} = j) = 0, \quad i, j > 0,$$

thus,

$$\mathrm{E}\zeta_{n,k}\zeta_{n,l} = 0, \quad \mathrm{Cov}(\zeta_{n,k},\ \zeta_{n,l}) = -\mathrm{E}\zeta_{n,k}\mathrm{E}\zeta_{n,l} = -\frac{1}{kl}.$$

For $1 \leqslant k < l \leqslant n-1$ and $k+l \leqslant n-1$, if $i,j > 0$, $ik+jl \leqslant n-1$, the number of the uniform recursive trees which exactly have i k-branches and j l-branches is $(n-1)! \cdot \mathrm{P}(\zeta_{n,k} = i, \zeta_{n,l} = j)$. Let A and B be two disjoint subsets of the set $\{2,3,\cdots,n\}$, whose sizes are k and l, respectively. Then the number of uniform recursive trees, which have a k-branch and a l-branch consisting of the vertices in A and B, is $(k-1)!(l-1)!(n-k-l-1)!$. Noting that A and B can be chosen arbitrarily, the number multiplied by $\binom{n-1}{k}\cdot\binom{n-k-1}{l}$ is

$$M := \binom{n-1}{k}\cdot\binom{n-k-1}{l}(k-1)!(l-1)!(n-k-l-1)! = \frac{(n-1)!}{kl}.$$

It is easy to see that in M, each recursive tree which exactly has i k-branches and j l-branches is counted ij times. Then

$$\sum_{(i,j):\ i,j>0,\ ik+jl\leqslant n-1} ij(n-1)!\mathrm{P}(\zeta_{n,k}=i,\zeta_{n,l}=j) = M = \frac{(n-1)!}{kl}.$$

That is

$$\sum_{(i,j):\ ij>0,\ ik+jl\leqslant n-1} ij\mathrm{P}(\zeta_{n,k}=i,\zeta_{n,l}=j) = \frac{1}{kl},$$

hence,

$$\mathrm{E}\zeta_{n,k}\zeta_{n,l} = \frac{1}{k\cdot l} = \mathrm{E}\zeta_{n,k}\mathrm{E}\zeta_{n,l},\quad 1\leqslant k<l\leqslant n-1,\ k+l\leqslant n-1,$$

by (23). And $\mathrm{Cov}(\zeta_{n,k},\zeta_{n,l})=0$ follows. □

From this theorem, the following consequence is obvious.

Corollary. The covariance matrix of random vector $(\zeta_{n,1},\zeta_{n,2},\cdots,\zeta_{n,n-1})$ is $B_n=(b_{ij})_{(n-1)\times(n-1)}$, where

$$b_{ii} = \begin{cases} \dfrac{1}{i}, & 1\leqslant i\leqslant \frac{n-1}{2} \\ \dfrac{i-1}{i^2}, & \frac{n-1}{2}<i\leqslant n-1 \end{cases};\quad b_{ij} = \begin{cases} 0, & i\neq j,\ i+j\leqslant n-1 \\ -\dfrac{1}{ij}, & i\neq j,\ i+j>n-1 \end{cases}.$$

4.4 The asymptotic joint distribution of $\zeta_{n,m}$

To study the asymptotic joint distribution of $\zeta_{n,m}$, we prove a lemma first.

Lemma 1. For any $m\in\mathcal{N}$, as $n\to\infty$, the limit of $\mathrm{P}(\zeta_{n,1}=0,\ \cdots,\ \zeta_{n,m}=0)$ exists.

Proof. It holds for $m=1$ by Theorem 7. Consider the case $m=2$. Let

$$\mathrm{P}(\zeta_{n,1}=0,\ \zeta_{n,2}=0)=a_n,\quad \mathrm{P}(\zeta_{n,1}=0)=b_n.$$

By Theorem 8,

$$b_n=\mathrm{P}(\zeta_{n,1}=0)=\sum_{2x_2+3x_3+\cdots+(n-1)x_{n-1}=n-1}\prod_{m=2}^{n-1-i}\frac{1}{m^{x_m}x_m!};$$

$$a_n=\mathrm{P}(\zeta_{n,1}=0,\ \zeta_{n,2}=0)=\sum_{3x_3+\cdots+(n-1)x_{n-1}=n-1}\prod_{m=2}^{n-1-i}\frac{1}{m^{x_m}x_m!}.$$

Therefore,

$$b_n = \sum_{j=0}^{[(n-1)/2]} \frac{1}{2^j j!} \sum_{3x_3+\cdots+(n-1)x_{n-1}=n-2j-1} \prod_{m=3}^{n-2j-1} \frac{1}{m^{x_m} x_m!}$$
$$= \sum_{j=0}^{[(n-1)/2]} \frac{1}{2^j j!} a_{n-2j}. \tag{26}$$

Suppose that the limit of a_n does not exist as $n \to \infty$, then there exist $0 \leqslant \alpha < \beta \leqslant 1$, satisfying

$$\liminf_{n\to\infty} a_n = \alpha, \qquad \limsup_{n\to\infty} a_n = \beta.$$

Noting that $16 - 9\sqrt{e} > \sqrt{256} - \sqrt{81 \times 3} > 0$, let

$$\delta = \frac{\beta - \alpha}{4} > 0, \quad \delta_1 = \left(4 - \frac{9\sqrt{e}}{4}\right)\delta > 0.$$

For any fixed $0 < \varepsilon < (11\sqrt{e} - 18)\delta$, since $\lim\limits_{n\to\infty} b_n = e^{-1}$, there exists an n_0, such that for any $n_2 > n_1 > n_0$,

$$|b_{n_1} - b_{n_2}| < \varepsilon. \tag{27}$$

And when n_0 is sufficiently large,

$$\alpha - \delta < a_n < \beta + \delta, \quad n \geqslant n_0; \tag{28}$$

$$\sum_{j=n+1}^{\infty} \frac{1}{2^j j!} < \delta_1, \quad n \geqslant n_0. \tag{29}$$

Then fix an n_0, which satisfies the above three formulae (27)—(29).

When n is sufficiently large, rewrite (26) as follows:

$$b_n = a_n + \sum_{j=1}^{n_0} \frac{1}{2^j j!} a_{n-2j} + \sum_{j=n_0+1}^{[(n-1)/2]} \frac{1}{2^j j!} a_{n-2j}. \tag{30}$$

Since $\limsup\limits_{n\to\infty} a_n = \beta$, $\liminf\limits_{n\to\infty} a_n = \alpha$, there exist $n_2 > n_1 > 3n_0$, satisfying

$$a_{n_1} > \beta - \delta_1, \quad a_{n_2} < \alpha + \delta_1.$$

Noting that $n_2 - 2n_0 > n_1 - 2n_0 > n_0$, (28) holds for all $n = n_1 - 2j$, $n = n_2 - 2j$, $j \in \{1, 2, \cdots, n_0\}$. Hence, combining (29) and (30), we have that

$$b_{n_1} = a_{n_1} + \sum_{j=1}^{n_0} \frac{1}{2^j j!} a_{n_1-2j} + \sum_{j=n_0+1}^{[(n-1)/2]} \frac{1}{2^j j!} a_{n_1-2j}$$
$$> (\beta - \delta_1) + \sum_{j=1}^{n_0} \frac{1}{2^j j!} (\alpha - \delta) + \sum_{j=n_0+1}^{[(n-1)/2]} \frac{1}{2^j j!} a_{n_1-2j}$$
$$> (\beta - \delta_1) + (\sqrt{e} - 1)(\alpha - \delta) - \sum_{j=n_0+1}^{\infty} \frac{1}{2^j j!}$$
$$> (\beta - \delta_1) + (\sqrt{e} - 1)(\alpha - \delta) - \delta_1$$

and

$$\begin{aligned} b_{n_2} &= a_{n_2} + \sum_{j=1}^{n_0} \frac{1}{2^j j!} a_{n_2-2j} + \sum_{j=n_0+1}^{[(n-1)/2]} \frac{1}{2^j j!} a_{n_2-2j} \\ &< (\alpha+\delta_1) + \sum_{j=1}^{n_0} \frac{1}{2^j j!}(\beta+\delta) + \sum_{j=n_0+1}^{[(n-1)/2]} \frac{1}{2^j j!} a_{n_1-2j} \\ &< (\alpha+\delta_1) + (\sqrt{e}-1)(\beta+\delta) + \sum_{j=n_0+1}^{\infty} \frac{1}{2^j j!} \\ &< (\alpha+\delta_1) + (\sqrt{e}-1)(\beta+\delta) + \delta_1. \end{aligned}$$

Therefore,

$$\begin{aligned} & b_{n_1} - b_{n_2} > (2-\sqrt{e})(\beta-\alpha) - 2(\sqrt{e}-1)\delta - 4\delta_1 \\ =& 4\delta - 2(\sqrt{e}-1)\delta - (16-9\sqrt{e})\delta = (11\sqrt{e}-18)\delta, \end{aligned}$$

which is in contradiction with (27). Thus, as $n \to \infty$, the limit of $\mathrm{P}(\zeta_{n,1}=0,\ \zeta_{n,2}=0)$ exists.

It is not hard to prove that Lemma 1 is holds for all $m \in \mathcal{N}$ by induction, whose process is similar as $m=1 \Rightarrow m=2$. □

The main result in this subsection is the following theorem.

Theorem 10. In uniform recursive trees, the numbers of various branches are asymptotical independent. Furthermore, for any $m \in \mathcal{N}$ and any sequence of nonnegative integers $\{x_1, \cdots, x_m\}$,

$$\lim_{n\to\infty} \mathrm{P}(\zeta_{n,1}=x_1,\ \cdots,\ \zeta_{n,m}=x_m) = \prod_{j=1}^{m} \lim_{n\to\infty} \mathrm{P}(\zeta_{n,j}=x_j) = \prod_{j=1}^{m} e^{-1/j} \frac{1}{j^{x_j} \cdot x_j!}.$$

Proof. The proof is divided into two parts: (1) the limit $\lim\limits_{n\to\infty} \mathrm{P}(\zeta_{n,1}=x_1, \cdots, \zeta_{n,m}=x_m)$ exists; (2) for any $m \in \mathcal{N}$ and sequence of nonnegative integers $\{x_1, \cdots, x_m\}$,

$$\lim_{n\to\infty} \mathrm{P}(\zeta_{n,1}=x_1,\ \cdots,\ \zeta_{n,m}=x_m) = \prod_{j=1}^{m} e^{-1/j} \frac{1}{j^{x_j} \cdot x_j!}. \tag{31}$$

By Theorem 7, it is shown that for any $j \in \mathcal{N}$ and nonnegative integer i,

$$\lim_{n\to\infty} \mathrm{P}(\zeta_{n,j}=i) = e^{-1/j} \frac{1}{j^i \cdot i!},$$

which yields that the theorem holds for $m=1$. In particular,

$$\lim_{n\to\infty} \mathrm{P}(\zeta_{n,1}=0) = e^{-1}. \tag{32}$$

Consider the case $k=2$. By (22),

$$\begin{aligned} \mathrm{P}(\zeta_{n,1}=i,\ \zeta_{n,2}=j) =& \frac{1}{i! \cdot 2^j j!} \sum_{3x_3+\cdots+(n-1-i)x_{n-1-i}=n-1-i-2j} \prod_{m=3}^{n-1-i} \frac{1}{m^{x_m} x_m!} \\ =& \frac{1}{i! \cdot 2^j j!} \mathrm{P}(\zeta_{n-i-2j,1}=0,\ \zeta_{n-i-2j,2}=0). \end{aligned}$$

Hence, by Lemma 1, the limit

$$\lim_{n\to\infty} \mathrm{P}(\zeta_{n,1}=i,\ \zeta_{n,2}=j)=\frac{1}{i!\cdot 2^j j!}\lim_{n\to\infty}\mathrm{P}(\zeta_{n-i-2j,\,1}=0,\ \zeta_{n-i-2j,\,2}=0)$$

exists. We only need to prove that

$$\lim_{n\to\infty}\mathrm{P}(\zeta_{n,1}=0,\ \zeta_{n,2}=0)=e^{-1-1/2}. \tag{33}$$

Let

$$\lim_{n\to\infty}\mathrm{P}(\zeta_{n,1}=0,\ \zeta_{n,2}=0)=\mathrm{P}(\zeta_1=0,\ \zeta_2=0):=p,\quad 0\leqslant p\leqslant 1. \tag{34}$$

Note that

$$\mathrm{P}(\zeta_{n,1}=0,\ \zeta_{n,2}=0)=\sum_{3x_3+\cdots+(n-1-i)x_{n-1}=n-1}\ \prod_{m=2}^{n-1}\frac{1}{m^{x_m}x_m!}.$$

On the other hand, (32) can be rewrite as follows:

$$\begin{aligned}
&\lim_{n\to\infty}\sum_{2x_2+3x_3+\cdots+(n-1-i)x_{n-1}=n-1}\ \prod_{m=2}^{n-1}\frac{1}{m^{x_m}x_m!}\\
&=\lim_{n\to\infty}\sum_{j=0}^{[(n-1)/2]}\frac{1}{2^j j!}\sum_{3x_3+\cdots+(n-1-2j)x_{n-1-2j}=n-1-2j}\ \prod_{m=3}^{n-1-2j}\frac{1}{m^{x_m}x_m!}\\
&=\lim_{n\to\infty}\sum_{j=0}^{[(n-1)/2]}\frac{1}{2^j j!}\mathrm{P}(\zeta_{n-2j,1}=0,\ \zeta_{n-2j,2}=0)=e^{-1}.
\end{aligned}$$

That is,

$$\lim_{n\to\infty} e\sum_{j=0}^{[(n-1)/2]}\frac{1}{2^j j!}\mathrm{P}(\zeta_{n-2j,\,1}=0,\ \zeta_{n-2j,\,2}=0)=1. \tag{35}$$

And (35) shows that for a sufficiently large nature number n_0 and if $n>3n_0$, we have

$$\begin{aligned}
&e\sum_{j=0}^{n_0}\frac{1}{2^j j!}\mathrm{P}(\zeta_{n-2j,\,1}=0,\ \zeta_{n-2j,\,2}=0)\\
&\leqslant e\sum_{j=0}^{[(n-1)/2]}\frac{1}{2^j j!}\mathrm{P}(\zeta_{n-2j,\,1}=0,\ \zeta_{n-2j,\,2}=0)\\
&\leqslant e\sum_{j=0}^{n_0}\frac{1}{2^j j!}\mathrm{P}(\zeta_{n-2j,\,1}=0,\ \zeta_{n-2j,\,2}=0)+e\sum_{j=n_0+1}^{\infty}\frac{1}{2^j j!}.
\end{aligned}$$

By (34) and (35), let $n\to\infty$, then

$$ep\sum_{j=0}^{n_0}\frac{1}{2^j j!}\leqslant 1\leqslant ep\sum_{j=0}^{n_0}\frac{1}{2^j j!}+e\sum_{j=n_0+1}^{\infty}\frac{1}{2^j j!}.$$

And let $n_0\to\infty$ in the above formula, too. Then

$$e^{1+\frac{1}{2}}p=1,$$

i.e. (33) follows and the theorem holds for $m=2$.

For the case $m \geqslant 3$, by induction, it is not hard to be proved, whose process is similar as $m = 1 \Rightarrow m = 2$. Hence the proof is completed. □

5 The biggest branch of $\mathcal{T}_n$

As described in the introduction, in $\mathcal{T}_n$, ν_n denotes the size of the biggest branch, i.e.

$$\nu_n = \max\{m : \zeta_{n,m} \geqslant 1\}.$$

In the model for spread of epidemics, ν_n represents the largest number of the sufferers, who were infected directly or indirectly by someone infected by the origin case directly.

Proposition 1.

$$\lim_{n\to\infty} \mathrm{P}\left(\nu_n > \frac{n-1}{2}\right) = \ln 2.$$

Proof. The event

$$\left(\nu_n > \frac{n-1}{2}\right) = \bigcup_{m=[(n-1)/2]+1}^{n-1} (\nu_n = m)$$

and two of the events $(\nu_n = [(n-1)/2]+1)$, $(\nu_n = [(n-1)/2]+2)$, $\cdots$, $(\nu_n = n-1)$ cannot occur at the same time, therefore, as $n \to \infty$,

$$\mathrm{P}\left(\nu_n > \frac{n-1}{2}\right) = \sum_{m=[(n-1)/2]+1}^{n-1} \mathrm{P}(\nu_n = m) = \sum_{m=[(n-1)/2]+1}^{n-1} \frac{1}{m} \to \ln 2. \quad \square$$

Remark. The theorem shows that the probability of an existing branch, whose size is more than $[(n-1)/2]$, is very large. To a certain extent, it shows that there existed some super-infectors in the spread of SARS.

Moreover, it can be proved that ν_n tends to infinity almost sure, as $n \to \infty$. It is easy to see that

$$\nu_n \leqslant \nu_{n+1} \leqslant \nu_n + 1,$$

then ν_n is increasing in n and the limit of ν_n exists.

Theorem 11.

$$\lim_{n\to\infty} \nu_n = \infty, \quad a.s.$$

Proof. Recall that η_n denotes the number of all branches, and ν_n denotes the size of the biggest branch, then $\eta_n \nu_n \geqslant n-1$, from this and (6),

$$\liminf_{n\to\infty} \frac{\log n \cdot \nu_n}{n} \geqslant \lim_{n\to\infty} \frac{\log n}{\eta_n} = 1, \quad \text{a.s.}$$

Since $\lim_{n\to\infty} \frac{\log n}{n} = 0$, we have $\lim_{n\to\infty} \nu_n = \infty$, a.s. □

Acknowledgements This work was supported by the National Natural Science Foundation (Grant No. 10371117) and Special Foundation of USTC. The authors are grateful to Prof. Mahmoud for providing his paper[10].

References

1. Xu, J. M., Graph Theory and its Applications, Hefei: Press of University of Science and Technology of China, 1998.
2. Bergeron, F., Flajolet, P., Salvy, B., Varieties of increasing trees, in Proc.17th Coll. Trees in Algebra and Programming (Lecture Notes Comput. Sci. 581), Berlin: Springer, 1992, 24–48.
3. Moon, J. W., The distance between nodes in recursive trees, in London Mathematics Society Lecture Notes Series, London: Cambridge University Press, 1974, 13: 125–132.
4. Najock, D., Heyde, C., On the number of the terminal vertices in certain random trees with an applications to stemma construction in philology, J. Appl. Prob., 1982, 19: 675–680.
5. Gastwirth, J., A probability model of a pyramid scheme, Amer. Statist., 1977, 31: 79–82.
6. Meir, A., Moon, J. W., On the altitude of nodes in random trees, Canad. J. Math., 1978, 30: 997–1015.
7. Szymański, J., On the maximum degree and the height of a random recursive tree, Random Graphs'87 (eds. Karoński M. Ruciński A.), North-Holland: John Wiley & Sons, 1990, 313–324.
8. Devroye, L., Applications of the theory of records in the study of random trees, Acta Informatica, 1988, 26: 123–130.
9. Mahmoud, H. M., Distances in random plane-orients recursive trees, J. Comput. Appl. Math., 1992, 41: 237–245.
10. Mahmoud, H. M., Limiting distributions for path lengths in recursive trees, Probab. Engrg. Inform. Sci., 1991, 5: 53–59.
11. Petrov, V. V., Limit Theorems of Probability Theory, Oxford: Clarendon Press, 1995.