The Reference Reading Report

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Outline

Chapter One Mathematical Fundamentals

1.1 Introduction

This chapter deals with mathematical fundamentals of curves and surfaces, and more generally manifolds and varieties.

By defining mappings between manifolds such as Euclidean spaces, we are able to uncover the local properties of their subspaces. In geometric modelling, we are particularly interested in properties such as, for example, local smoothness, i.e. to know whether the neighbourhood of a point in a submanifold is (visually) smooth, or the point is a singularity.

In other words, we intend to study the relationship between smoothness of mappings and smoothness of manifolds. The idea is to show that a mathematical theory exists to describe manifolds and varieties (e.g. curves and surfaces), regardless of whether they are defined explicitly, implicitly, or parametrically.

1.2 Functions and Mappings

In more formal terms, a function is a particular type of binary relation between two sets, say X and Y . The set X of input values is said to be the domain of f, while the set Y of output values is known as the codomain of f. The range of f is the set $\{f(x) : x \in X\}$, i.e. the subset of Y which contains all output values of f.

There are three major types of functions, namely, injections, surjections and bijections. An injection (or one-to-one function) has the property that if $f(a) = f(b)$, then a and b must be identical. A surjection (or onto function) has the property that for every y in the codomain there is an x in the domain such that $f(x) = y$. Finally, a bijection is both one-to-one and onto.

Functions can be even further extended in order to have several outputs. In this case, we have a component function for each output.

Functions with several outputs or component functions are here called mappings.

It is also useful to review how functions are classified in respect to the properties of their derivatives. Let $f : X \mapsto Y$ be a mapping of X into Y , where X, Y are open subsets of R^m, R^n respectively. If $n = 1$, we say that the function f is C^r (orC^r differentiable or differentiable of class C^r, or C^r smooth or smooth of class C^r) on X, for $r \in N$, if the partial derivatives of f exist and are continuous on X, that is, at each point $x \in X$.

If $n > 1$, the mapping f is C^r if each of the component functions $f_i (1 \leqslant i \leqslant n)$ of f is C^r. We say that f is C^∞ (or just differentiable or smooth) if it is C^r for all $r \geqslant 0$.

Moreover, f is called a $C^r diffeomorphism$ if: (i) f is a homeomorphism and (ii) both f and f^{-1} are C^r differentiable, $r \geqslant 1$ (when $r = \infty$ we simply say diffeomorphism).

1.3 Differential of a Smooth Mapping

Let U, V be open sets in R^m, R^n, respectively. Let $f : U \to V$ be a mapping with component functions $f_1, \ldots, f_n$.

a mapping $f : U \to V$ is smooth (or differentiable) if f has continuous partial derivatives of all orders. And we call f a diffeomorphism of U onto V when it is a bijection, and both f, f^{-1} are smooth.

Let $f : U \to V$ be a smooth (or differentiable or C^∞) and let $p \in U$. The linear mapping $Df(p) : R^m \to R^n$ whose matrix is the Jacobian is called the derivative or differential of f at p; the Jacobian $Jf(p)$ is also denoted by $[Df(p)]$.

Theorem 1.1. Let U, V be open sets in R_m, R_n respectively. If $f : U \to V$ is a diffeomorphism, at each point $p \in U$ the differential $Df(p)$ is invertible, so that necessarily $m = n$.

The justification for $m = n$ is that it is not possible to have a diffeomorphism between open subspaces of Euclidean spaces of different dimensions

In fact, a famous theorem of algebraic topology (Brouwer's invariance of dimension) asserts that even a homeomorphism between open subsets of R^m andR^n ,$m \neq n$, is impossible.

1.4 Invertibility and Smoothness

The smoothness of a submanifold that is the image of a mapping depends not only on smoothness but also the invertibility of its associated mapping.

Before proceeding, let us then brie y review the invertibility of mappings in the linear case.

Definition 1.2. Let X, Y be Euclidean spaces, and $f : X \to Y$ a continuous linear mapping. One says that f is invertible if there exists a continuous linear mapping $g : Y \to X$ such that $g \circ f = id_X$ and $f \circ g = id_y$ where id_X and id_Y denote the identity mappings of X and Y , respectively. Thus, by definition, we have:

$$g(f(x)) = x \ \ and \ \ f(g(y)) = y$$

for every $x \in X$ and $y \in Y$. We write f^{-1} for the inverse of f.

But, unless we have an algorithm to evaluate whether or not a mapping is invertible, smoothness analysis of a point set is useless from the geometric modelling point of view. Fortunately, linear algebra can help us at this point.

Consider the particular case $f : R^n \to R^n$. The linear mapping f is represented by a matrix $A = (a_{ij})$. It is known that f is invertible iff A is invertible.

Definition 1.3. Let U be an open subset of X and $f : U \to Y$ be a C^1 mapping, where X, Y are Euclidean spaces. We say that f is C^1-invertible on U if the image of f is an open set V in Y , and if there is a C^1 mapping $g : V \to U$ such that f and g are inverse to each other, i.e.

$$g(f(x)) = x \ \ and \ \ f(g(y)) = y$$

for every $x \in U$ and $y \in V$. We write f^{-1} for the inverse of f.

From the theorem that states that a C^r mapping that is a C^1 diffeomorphism is also a C^r diffeomorphism (see Hirsch [190]), it turns out that if f is a C^1 -invertible, and if f happens to be C^r , then its inverse mapping is also C^r . This is the reason why we emphasise C^1 at this point.

Let us now see the behaviour of invertibility under composition. Let $f : U \to V$ and $g : V \to W$ be invertible C^r mappings, where V is the image of f and W is the image of g. It follows that $g \circ f$ and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$are C^r-invertible, because we know that a composite of C^r mappings is also C^r.

Definition 1.5. Let $f : X \to Y$ be a C^r mapping, and let $p \in X$. One says that f is locally C^r-invertible at p if there exists an open subset U of X containing f is C^r-invertible on U.

Theorem 1.6. (Inverse Mapping Theorem) Let U be an open subset of R^m, let $p \in U$, and let $f : U \to R^n$ be a C^1 mapping. If the derivative Df is invertible, f is locally C^1-invertible at p. If f^{-1} is its local inverse, and $y = f(x)$, then $Jf^{-1}(y) = [Jf(x)]^{-1}$.

This is equivalent to saying that there exists open neighbourhoods U, V of p, $f(p)$, respectively, such that f maps U diffeomorphically onto V . Note that, by Theorem 1.1, R^m has the same dimension as the Euclidean space R^n, that is, $m = n$.

Corollary 1.8. Let U be an open subset of Rn and $f : U \to R^n$. A necessary and sufficient condition for the C^r mapping f to be a C^r diffeomorphism from U to $f(U)$ is that it be one-to-one and Jf be nonsingular at every point of U.

Thus, diffeomorphisms have nonsingular Jacobians.

1.5 Level Set, Image, and Graph of a Mapping

1.5.1 Mapping as a Parametrisation of Its Image

Definition 1.9. (Baxandall and Liebeck [35, p. 26]) Let U be open in R^m. The image of a mapping $f : U \subset Rm \to Rn$ is the subset of R^n given by

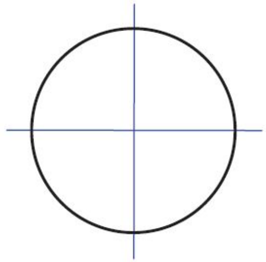
$$Imagef = \{y \in R^n | y = f(x), \forall x \in X\}$$

being f a parametrisation of its image with parameters $(x_1, \dots, x_m)$.

This definition suggests that practically any mapping is a parametrisation" of something [197, p. 263].

Example 1.10. The mapping $f : R \to R^2$ defined by $f(t) = (cost, sint), t \in R$, has an image that is the unit circle $x^2 + y^2 = 1$ in R^2 (Figure 1.1(a)). A distinct function with the same image as f is the mapping $g(t) = (cos2t, sin2t)$.

Example 1.10 suggests that two or more distinct mappings can have the same image. In fact, it can be proven that there is an infinity of different parametrisations of any nonempty subset of R^n [35, p. 29].



(a)

Fig. 1.1. (a) Image and

1.5.2 Level Set of a Mapping

Definition 1.13.(Dineen [112, p. 6]) Let U be open in R^m. Let $f : U \subset Rm \to Rn$ and $c = (c_1, \ldots, c_n)$ a point in R^n. A level set of f, denoted by $f^{-1}(c)$, is defined by the formula

$$f_{-1}(c) = \{x \in U | f(x) = c\}$$

In terms of coordinate functions $f_1, \ldots, f_n$ of f, we write

$$f(x) = c \Leftrightarrow f_i(x) = c_i \quad for \quad i = 1, \ldots, n$$

and thus

$$f^{-1}(c) = \bigcap_{i=1}^{n} \{x \in U | f_i(x) = c_i\} = \bigcap_{i=1}^{n} f_i^{-1}(c_i)$$

Theorem 1.14. (Implicit Function Theorem, Baxandall [35, p. 145]) A set $X \in R^m$ is a smooth variety if it is a level set of a C^1 function $f : R^m \to R$ such that $Jf(x) \neq 0$ for all $x \in X$.

Example 1.15. The circle $x^2 + y^2 = 4$ is a variety in R^2 that is a level set corresponding to the value 4 (i.e. point 4 in R) of a function f : R2 ! R given by $f(x, y) = x^2 + y^2$. Its Jacobian is given by $Jf(x, y) = [2x, 2y]$ which is null at $(0, 0)$. However, the point $(0, 0)$ is not on the circle $x^2 + y^2 = 4$; hence the circle is a smooth curve.

1.5.3 Graph of a Mapping

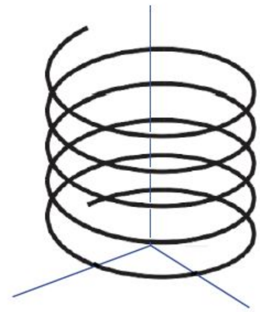
Definition 1.22. (Dineen [112, p. 6]) Let U be open in R^m. The graph of a mapping $f : U \subset R^m \to R^n$ is the subset of the product space $R^{m+n} = R^m \times R^n$ defined by

$$graph f = \{(x, y)|x \in U \; and \; y = f(x)\}$$

or

$$graph f = \{(x, f(x))|x \in U\}$$

Example 1.23. Let us consider both mappings $f(t) = (cost, sint)$ and $g(t) = (cos2t, sin2t)$ of Example 1.10. They have the same image in R^2, say a unit circle. However, their graphs are distinct point sets in R^3. The graph of f is a circular helix $(t, cost, sint)$ in R^3, Figure 1.1(b). But, although the graph of g is a circular helix with windings being around the same circular cylinder, those windings have half the pitch.

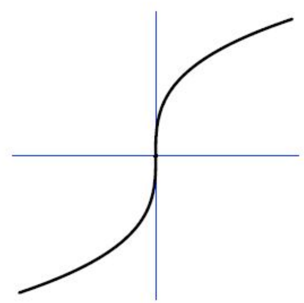


(b)

This suggests that there is a one-to-one correspondence between a mapping and its graph, that different mappings have distinct graphs. This leads us to think of a possible relationship between the smoothness of a mapping and the smoothness of its graph. In other words, the smoothness of a mapping determines the smoothness of its graph. This is corroborated by the following theorem.

Theorem 1.24. (Baxandall [35, p. 147]) The graph of a C_1 mapping $f : U \subset R^m \to R^n$ is a smooth variety in $R^m \times R^n$.

The graph of the function $f(x) = x^{1/3}$, depicted in Figure 1.6(c), is a smooth curve. Note that the curve is smooth despite the function being not differentiable at $x = 0$. This happens because the curve is the graph of the function $x = f(y) = y^3$ that is differentiable.



(c)

The relationship between graphs and level sets plays an important role in the study of varieties.

□ It is easy to see that every graph is a level set.
Let us consider a mapping $f : U \subset R^m \to R^n$. We define $F : U \times R^m \to R^n$ by $F(x, y) = f(x) - y$. If 0 is the origin in R^n, we have

$$(x, y) \in F^{-1}(0) \Leftrightarrow F(x, y) = 0$$

$$\Leftrightarrow f(x) - y = 0$$

$$(x, y) \in grahpf.$$

Thus, $F^{-1}(0) = graphf$ and every graph is a level set.

□ From the implicit mapping theorem,every smooth level set is locally a graph.

Level sets correspond to implicit representations, say functions, on some Euclidean space, while graphs correspond to explicit representations.

Definition 1.27. (Baxandall and Liebeck [35, p. 226]) Let $f : X \subset R^m \to R$ be a function, where $m \geqslant 2$. If there exists a function $g : Y \subset R^{m-1} \to R$ such that for all $(x_1, \ldots, x^{m-1}) \in Y$

$$f(x_1, \ldots, x_{m-1}, g(x_1, \ldots, x_{m-1})) = 0,$$

then the function g is said to be defined implicitly on Y by the equation

$$f(x_1, \ldots, x_m) = 0.$$

Likewise, the graph of $g : Y \subset R^{m-1} \to R$ is the subset of R^m given by

$$\{(x_1, \ldots, x_{m-1}, x_m) \in R^m \mid x_m = g(x_1, \ldots, x_{m-1})\}.$$

The expression $x_m = g(X)$ is called the equation of the graph [35, p.100]. Hence, g is said to be explicitly defined on Y by the equation $x_m = g(x_1, \ldots, x_{m-1})$.

Example 1.28. The graph of the function $f(x, y) = -x^2 - y^2$ has equation $-z = x^2 + y^2$. This graph is a 2-manifold in R^3 called a paraboloid (Figure 1.7). The equation $-z = x^2 + y^2$ explicitly defines the paraboloid in R^3.

For $c < 0$ the plane $z = c$ intersects the graph in a circle lying below the level set $x^2 + y^2 = -c$ in the (x, y)-plane. The equation $x^2 + y^2 = -c$ of a circle (i.e. a 1-manifold) in R^2 is said to define y implicitly in terms of x. This circle is said to be an implicit 1-manifold.

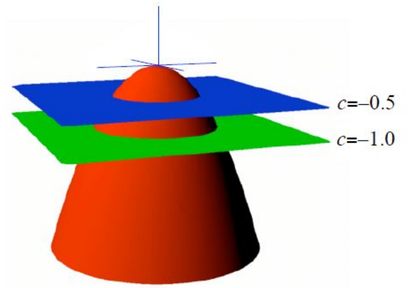


Fig. 1.7. The paraboloid $-z = x^2 + y^2$ in $\mathbb{R}^3$.

1.6 Rank-based Smoothness

Now, we are in position to show that the rank of a mapping gives us a general approach to check the C^r invertibility or C^r smoothness of a mapping, and whether or not a variety is smooth.

This smoothness test is carried out independently of how a variety is defined, implicitly, explicitly or parametrically, i.e. no matter whether a variety is considered a level set, a graph, or an image of a mapping, respectively.

Definition 1.29. (Olver [313, p. 11]) The rank of a mapping $f : R^m \to R^n$ at a point $p \in R^m$ is defined to be the rank of the $n \times m$ Jacobian matrix Jf of any local coordinate expression for f at the point p. The mapping f is called regular if its rank is constant.

It is proved in differential geometry that the set of points where the rank of f is maximal is an open submanifold of the manifold R^m (which is dense if f is analytic), and the restriction of f to this subset is regular.

The subsets where the rank of a mapping decreases are singularities [313, p. 11].

From linear algebra we have

$$rank Jf = k \Leftrightarrow k rows of Jf are linearly independent$$

$$\Leftrightarrow k columns of Jf are linearly independent$$

$$\Leftrightarrow Jf has a k \times k submatrix that has nonzero determinant.$$

Rank Theorem tells us that a mapping of constant rank k behaves locally as a projection of $R^m = R^k \times R^{m-k}$ to R^k

1.6.1 Rank-based Smoothness for Parametrisations

Theorem 1.31. (Rank Theorem for Parametrisations) Let U be an open set in R^m and $f : U \to R^n$. A necessary and sufficient condition for the C^1 mapping f to be a diffeomorphism from U to $f(U)$ is that it be one-to-one and the Jacobian Jf have rank m at every point of U.

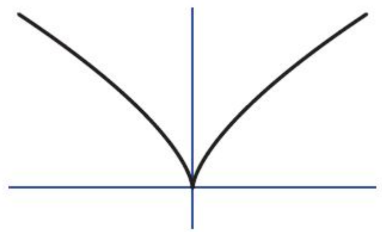
Three points should be noticed for this thereom.

◇ f should be smooth enough,i.e. f passes the differentiability test

Example 1.32. We know that the bent curve in R^2 defined by the parametrisation $f(t) = (t, |t|)$ is not differentiable at $t = 0$, even though its rank is 1 everywhere.

◇ Jf have rank m at every point of U. (rank test)

Example 1.33. A parametrised curve that passes the differentiability test, but not the rank test, is the cuspidal cubic in R^2 given by $f(t) = (t^3, t^2)$ (Figure 1.2(a)). The component functions are polynomials and therefore differentiable. However, the rank $Jf(t) = [3t^2, 2t]$ is not 1 (i.e. its maximal value) at $t = 0$; in fact it is zero. This means that the parametrised cuspidal cubic is not smooth at $t = 0$, that is, it possesses a singularity at $t = 0$.



(a)

Fig. 1.2. (a) Cuspidal cubic $x^3 = y^2$

◇ f should be one-to-one

Example 1.35. The curve parametrised by the differentiable mapping $f(t) = (t^3 - 3t - 2, t^2 - t - 2)$ is not smooth at (00),despite the differentiability of f and its maximal rank. In fact, we get the same point $(0, 0)$ on the curve for two distinct points $t = -1$ and $t = 2$ of the domain, that is, $f(1) = f(2) = (0, 0)$, and thus f is not one-to-one. These singularities are known as self-intersections in geometry or topological singularities in topology.

1.6.2 Rank-based Smoothness for Implicitations

The implicit function theorem is particularly useful for geometric modelling because it provides us with a computational tool to test whether an implicit manifold, and more generally a variety, is smooth in the neighbourhood of a point. Specifically, it gives us a local parametrisation for which it is possible to check the local C^r-invertibility by means of its Jacobian.

Theorem 1.38. (Rank Theorem for Implicitations) Let U be open in R^m and let $f : U \to R$ be a C^r function on U. Let $(p, q) = (p_1, \ldots, p_{m-1}, q) \in U$ and assume that $f(p, q) = 0$ but $\frac{\partial f}{\partial x_m}(p, q) \neq 0$ Then the mapping

$$F : U \to R^{m-1} \times R = R^m$$

given by

$$(X, y) \longrightarrow (X, f(X, y))$$

is locally C^r-invertible at (p, q).

Theorem 1.39. (Multivariate Implicit Function Theorem) Let U be open in R^m and let $f : U \to R$ be a C^r function on U. Let $(p, q) = (p_1, \ldots, p_{m-1}, q) \in U$ and assume that $f(p, q) = 0$ but $\frac{\partial f}{\partial x_m}(p, q) \neq 0$. Then there exists an open ball V in R^{m-1} centred at p and a C^r function

$$g : V \to R$$

such that g(p) = q and

$$f(X, g(X)) = 0$$

for all $x \in V$.

1.7 Submanifolds

Definition 1.41. Let $f : M \to N$ be a smooth mapping with constant rank. Then, for all $p \in M$, f is called:

$$animmersion\ \ if\ \ rankf = dimM,$$

$$asubmersion\ \ if\ \ rankf = dimN:$$

Definition 1.43. A smooth (analytic) m-dimensional immersed submanifold of a manifold N is a subset $M' \subset N$ parametrised by a smooth (analytic), one-to-one mapping $f : M \to M' \subset N$, whose domain M, the parameter space, is a smooth (analytic) m-dimensional manifold, and such that f is everywhere regular, of maximal rank m.

Thus, an m-dimensional immersed submanifold M0 is the image of an immersion $f : M \to M' = f(M)$. To verify that f is an immersion it is necessary to check that the Jacobian has rank m at every point.

Observe that an immersed submanifold is defined by a parametrisation. Thus, an immersed submanifold is nothing more than a parametrically defined submanifold, or simply a parametric submanifold.

Despite its smoothness, an immersed or parametric submanifold may include self-intersections.

Parametric submanifolds without self-intersections, are called parametric embedded submanifolds.

Definition 1.44. An embedding is a one-to-one immersion $f : M \to N$ such that the mapping $f : M \to f(M)$ is a homeomorphism (where the topology on $f(M)$ is the subspace topology inherited from N). The image of an embedding is called an embedded submanifold.

Definition 1.49. An m-dimensional smooth submanifold $M \subset N$ is regular if, in addition to the regularity of the parametrising mapping, there is a cov- ering $\{U_i\}$ of M by open sets of N such that, for each i, $U_i \bigcap M$ is a single open connected subset of M.

Let us see a counterexample of regular submanifolds.

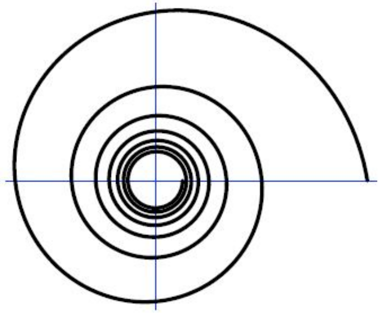
Example 1.50. Let $f : [1, \infty] \to R^2$ be a mapping given by

$$f(t) = (\frac{1}{t}cos(2\pi t), \frac{1}{t}sin(2\pi t)).$$

Its image (Figure 1.10(a)) in R^2 is an embedded curve because the image of every point $t \in [1, \infty]$ is a point in R^2; hence, f is a homeomorphism. Note that even near $t = \infty$, f is still a homeomorphism because its image is a point, the origin $(0, 0)$.

However, the image of $[1, \infty]$ is not a regular curve because it spirals to $(0, 0)$ as $t \to \infty$ and tends to $(1, 0)$ as $t \to 1$, Figure 1.10(a).

near (in a neighbourhood of) $t = \infty$ the relative neighbourhood in the image curve has several (possibly an infinite number of) components.



(a)

Fig. 1.10. Counterexamples

1.7.2 Implicit Submanifolds and Varieties

An alternative to the parametric approach for submanifolds is to define them implicitly as a common or intersecting level set of a collection of functions

Theorem 1.53. (Olver [313, p. 14]) A n-dimensional submanifold $N \subset R^m$ is regular if and only if for each point $p \in N$ there exist local coordinates $x = (x_1, \ldots, x_m)$ defined on a neighbourhood U of p such that $U \bigcap N = \{X : x_1 = \ldots = x_{m-n} = 0\}$.

Therefore, every regular n-dimensional submanifold of an m-dimensional manifold locally looks like a n-dimensional subspace of R^m.

Definition 1.54. Let F be a family of smooth real-valued functions $f_i : M \to R$, with M, R smooth manifolds. The rank of F at a point $p \in M$ is the dimension of the space spanned by their differentials. The family is regular if its rank is constant on M.

Definition 1.58. A variety (or system of equations) V_F is regular if it is not empty and the rank of F is constant.

Theorem 1.59. Let F be a family of functions defined on an m-dimensional manifold M. If the associated variety $VF \subset M$ is regular, it defines a regular submanifold of dimension $m - r$.

Definition 1.62. Let $f : U \subset R^m \to R^r$ be a smooth mapping. A point $p \in R^m$ is a regular point of f, and f is called a submersion at p, if the differential $Df(p)$ is surjective. This is the same as saying that the Jacobian matrix of f at p has rank r (which is only possible if $r \leqslant m$). A point $q \in R^r$ is a regular value of f if every point of $f^{-1}(q)$ is regular.

Definition 1.63. Let $f : U \subset R^m \to R^r$ be a smooth mapping. A point $p \in R^m$ is a singular point of f if the rank of its Jacobian matrix falls below its largest possible value $min(m, r)$. Likewise, a singular value is any $f(p) \in R^r$ where p is a singular point.

Chapter Six Continuation Methods

Continuation methods are based on piecewise linear approximation of a variety (e.g. curve or surface) by means of numerical solution of an initial value problem [7]. In other words, they compute solution varieties of nonlinear systems usually expressed in terms of an equation

$$f(p) = 0$$

with $f : R^{n+d} \rightarrow R^n$ a real function. The solution of this equation is called zero set (i.e. a particular level set).

As studied in Chapter 1, a zero set is a variety that consists of regular pieces called manifolds, which are joined at singular solutions (which are also solution manifolds, but of a system with lower d). The regular pieces are manifold curves when $d = 1$, manifold surfaces when $d = 2$, and d-manifolds in general.

6.1 Introduction

The essential idea behind a continuation method is very simple: first, compute a piece of the solution manifold near one solution, then select another solution from this set and repeat the process. As long as the new piece covers some new part of the solution manifold the computation progresses. So the basic issues are:

1. How to compute the solution manifold near some point p_i at which $f(p_i) = 0$.
2. How to select a new point.
3. How to avoid recomputing the same part of the manifold.

There are two ways to perform the first task, which lead us to two types of continuation methods: simplicial continuation methods and predictor- corrector methods.

6.2 Piecewise Linear Continuation

Piecewise linear continuation (or PL continuation), also called simplicial continuation, operates on a triangulation (simplicial complex) of a given domain $\Omega \subset R^{n+d}$ to approximate and to sample a a manifold (e.g. a curve or a surface).

This is illustrated in Figure 6.1 for a curve that lies in $\Omega \subset R^2$. The triangulation (a) splits Ω into equilateral triangles, where the red ones are those triangles which the curve passes through. The triangulation (b) partitions into isosceles triangles, where the green ones are those which better approximate the curve.

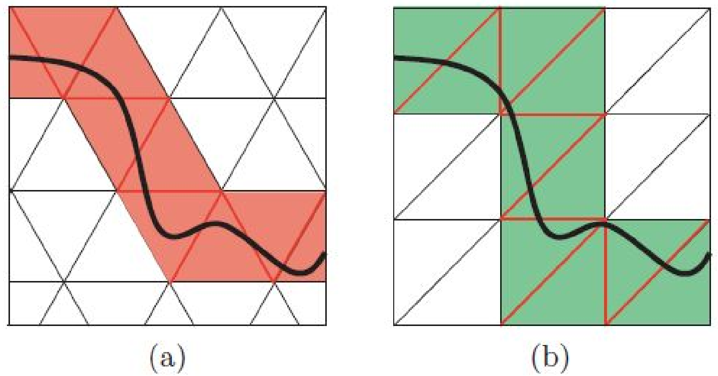


Fig. 6.1. Two simplicial approximations of a curve in $\Omega \subset \mathbb{R}^2$.

6.2.1 Preliminary Concepts

For any positive integer N, and for any set $\{p_0, \ldots, p_N\}$ of points in some linear space which are affinely independent (or, equivalently, $\{p_1 - p_0, \ldots, p_N - p_0\}$ are linearly independent), the convex hull $[p_0, \ldots, p_N]$ is called the N-simplex with vertices $p_0, \ldots, p_N$.

As known, the possible N-simplices in R^3 are: vertices (N = 0), edges (N = 1), triangles (N = 2), and tetrahedra (N = 3). Also, for each subset of $K + 1$ vertices $\{q_0, \ldots, q_k\} \subset \{p_0, \ldots, p_N\}$, the K-simplex $[q_0, \ldots, q_K]$ is called a K-face of $[p_0, \ldots, p_N]$. In particular, 0-faces are vertices, 1-faces are edges, 2-faces are triangles, and $(K - 1)$-faces are facets. Simplices are the building bricks" that allow us to construct different sorts of triangulations in R^N.

Definition 6.1. Let T be a non-empty collection of N-simplices in R^N. We call T a triangulation of R^N if the following properties are satisfied:
(1) $\bigcup_{\sigma\in T} = R^N$;
(2) the intersection $\sigma_1 \bigcap \sigma_2$ of two simplices $\sigma_1, \sigma_2 \in T$ is empty or a common facet of both simplices;
(3) the collection T is locally finite, i.e. any compact subset of R^d meets only a finite number of simplices of $\sigma \in T$.

This definition applies not only to triangulations of R^N but also to its subspaces, as needed in computer graphics.

6.2.2 Types of Triangulations

We would like to have triangulations with the following properties:
(1) It should be easy to find the simplex that shares a facet with a given simplex.
(2) It should be possible to label the vertices of all the simplexes at the same time with indexes $0, \ldots, N$, such that each of the $N+1$ vertices of an N-simplex has a different label.
(3) It should be desirable for all the simplexes to have almost the same size.
(4) It should be desirable for all the simplexes to have roughly the same dimensions in all directions.

6.2.3 Construction of Triangulations

(1)Freudenthal's Triangulation

The fact that Coxeter's triangulations are monohedral means that a simplex must have dihedral angles that are each a submultiple of 2π. That is, for each pair of facets of the simplex, there is an integer $i > 1$ such that the dihedral angle between the two facets is $\frac{2\pi}{i}$.

Let us now formalise the Freudenthal rule. Let $\sigma = [p_0, \ldots, p_N]$ be an N-simplex.For each $k \in \{0, \ldots, N\}$ let us define $k-1$ and $k+1$ as the "left" and "right" neighbours of k in the cyclic ordering of $(0, 1, \ldots, N)$. Analogously, for vertices, p_{k-1} $(k \neq 0)$ and p_{k+1} $(k \neq N)$ are defined to be the "left" and "right" neighbours of the vertex p_k.It is clear that p_N is the left neighbour of p_0 and, conversely, p_0 is the right neighbour of p_N.

Definition: The vertex obtained as follows

$$\hat{p}_k = p_{k-1} - p_k + p_{k+1}$$

is called the "reflection" of p_k across the centre of the "neighbouring edge" $[p_{k-1}, p_{k+1}]$.

Freudenthal's triangulations are got by the pivot operation $\Phi_k([p_0, \ldots, p_k, \ldots, p_N]) = [p_0, \ldots, \hat{p}_k, \ldots, p_N]$, where $\hat{p}_k = p_{k-1} - p_k + p_{k+1}$, $p_{-1} = p_N, p_{N+1} = p_0$.

This is illustrated in Figure 6.4:

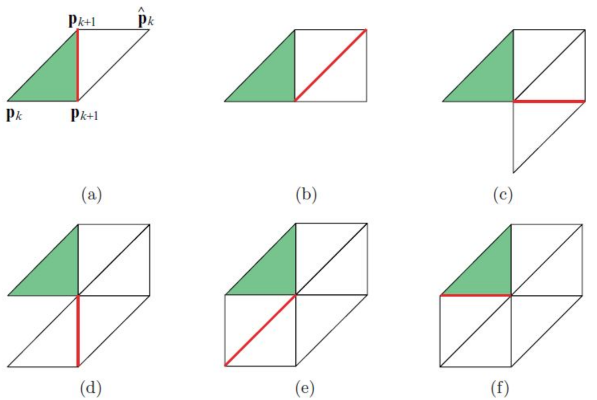


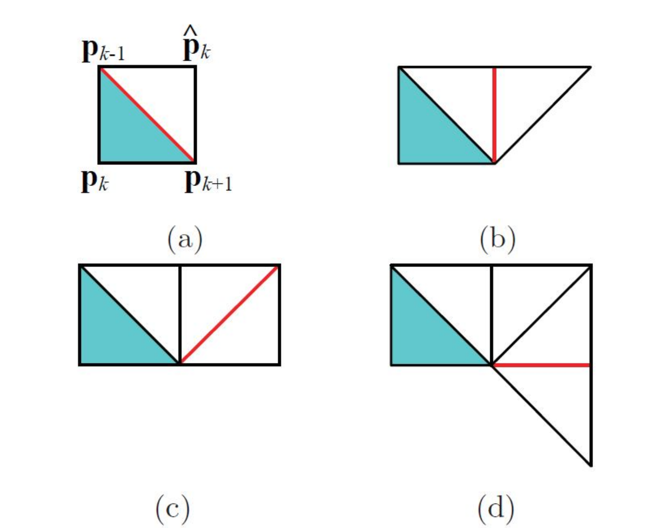
Fig. 6.4. Generation of isosceles triangles by reflection in $\mathbb{R}^2$.

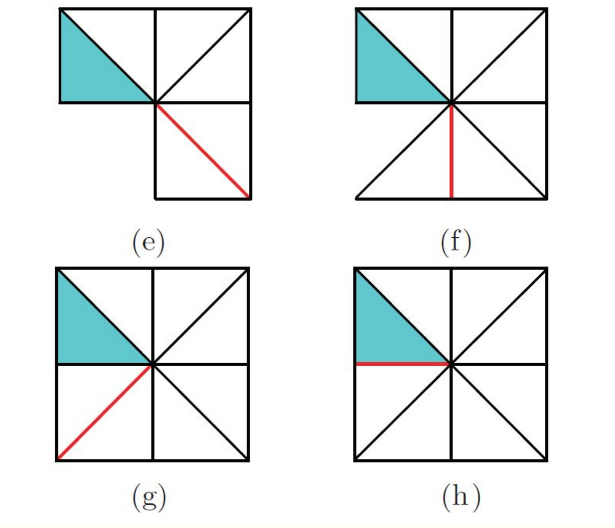
6.2.3 Construction of Triangulations

(2)Todd's Triangulation J_1

The J_1 triangulations are got by the pivot operation $\Theta_k([p_0, \ldots, p_k, \ldots, p_N]) = [p_0, \ldots, \hat{p}_k, \ldots, p_N]$, where

$$\hat{p}_k = p_{k-1} - p_k + p_{k+1}, \ \ if \ \ 0 < k < N, \hat{p}_k = 2p_{k+1} - p_k, \ \ if \ \ k = 0,$$
$$\hat{p}_k = 2p_{k-1} - p_k, \ \ if \ \ k = N,$$




(e)
(f)
(g)
(h)

6.3 Integer-Labelling PL Algorithms

In addition to pivoting rules to generate triangulations, PL algorithms use labellings with the following purposes:

◇ To keep track the PL approximation of a manifold. That is, by labelling the simplexes that intersect a manifold, also called transverse simplexes, we are able to follow such a manifold.

◇ To prevent the cycling phenomenon.

A simplicial algorithm provides a piecewise linear zero set of f via an auxiliary map L_f called labelling induced by f. The values of L_f at vertices are then used in determining whether a given n-simplex is ”completely labelled” or not.

Definition 6.2. For $p \in R^{n+d}$, the labelling of p is defined by $L_f(p) = j$, where $j \in \{0, \ldots, n\}$ is the number of leading nonnegative components of $f(p) \in R^n$.

Definition 6.3. An $(n+d)$-simplex $[p_0, \ldots, p^{n+d}]$ is said to be completely labelled if $L_f\{p_0, \ldots, p^{n+d}\} = \{0, \ldots, n\}$.

Definition A 1-simplex is called "completely labelled" when the labels of the vertices are different.

Definition6.6 An $(n+m)$-simplex $(m = 0, \ldots, d)$ is said to be transverse if it contains a completely labelled n-face.

In our case, f has only one component function $(n = 1)$, which means that there are only two possible labels for any vertex v of the triangulation: either 0 when $f(v) < 0$ or 1 when $f(v) \geqslant 0$.

Example 6.4. Let $\Omega = [-2,2]\times[-2,2] \in R^2$, $f(x,y) = y - x^2 + \frac{1}{2}$. Looking again at Figure 6.6, we note that the vertices $p_0 = (0,0), p_1 = (1,0), p_2 = (1,1)$ define a 2-simplex or triangle in Ω. The labels of these three vertices are:

- $L_f(0,0) = 1$ *because* $f(0,0) = \frac{1}{2} > 0$;
- $L_f(1,0) = 0$ *because* $f(1,0) = -\frac{1}{2} < 0$;
- $L_f(1,1) = 1$ *because* $f(1,1) = \frac{1}{2} > 0$;

Thus, the 1-simplices or edges $[p_0, p_1]$ and $[p_1, p_2]$ are completely labelled, but not the edge $[p_0, p_2]$.

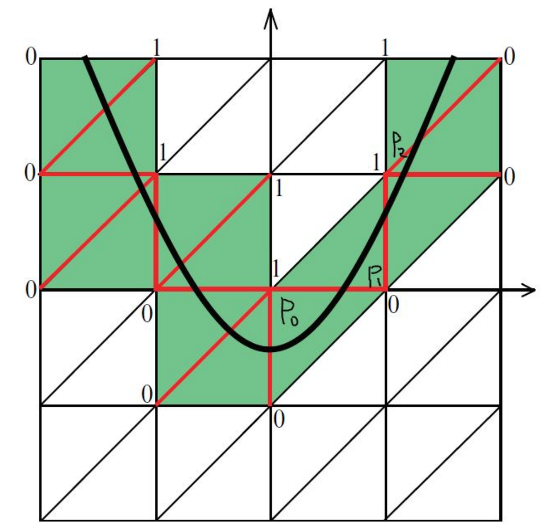


Fig. 6.6. Integer labelling of a triangulation in $\mathbb{R}^2$.

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Algorithm 12 Integer-Labelling PL Algorithm for Manifolds

procedure ALLGOWER-GEORG-SCHMIDT(f,Ω,$\mathcal{T}$)
    $\mathcal{T} \leftarrow \varnothing$ ▷ set of pivoted, transverse $(n+d)$-simplices of $\mathcal{T}$
    $S \leftarrow \varnothing$ ▷ set of non-pivoted, transverse $(n+d)$-simplices of $\mathcal{T}$
    Find a transverse starting $(n+d)$-simplex $\sigma \in \mathcal{T}$
    $S \leftarrow S \cup \{\sigma\}$
    while $S \neq \varnothing$ **do**
        Get $\sigma \in S$
        Label vertices of σ
        Determine F ▷ set of non-pivoted, transverse facets of σ
        while $F \neq \varnothing$ **do**
            Choose a pivot facet $\tau \in F(\sigma)$
            Determine the $(n+d)$-simplex $\hat{\sigma}$ by pivoting σ across τ
            $S \leftarrow S \cup \{\hat{\sigma}\}$
            $S \leftarrow S \setminus \{\sigma\}$
            $\mathcal{T} \leftarrow \mathcal{T} \cup \{\sigma\}$
            $F \leftarrow F \setminus \{\tau\}$
        end while
    end while
end procedure

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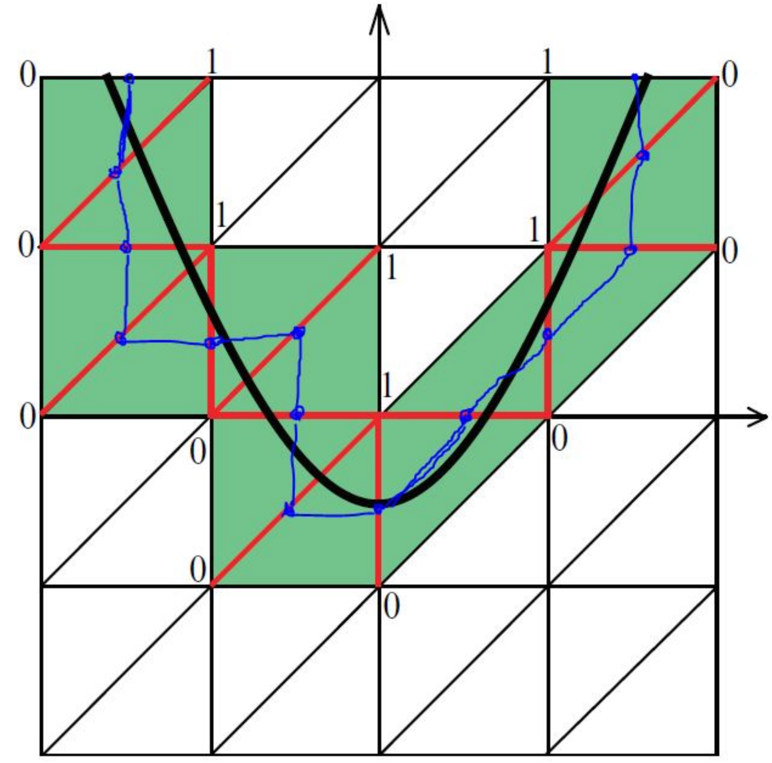

Fig. 6.6. Integer labelling of a triangulation in $\mathbb{R}^2$.

6.4 Vector Labelling-based PL Algorithms

In comparison to integer labelling,vector labelling provides a finer PL zero set of f than integer labelling.Here,we still consider the Freudenthal's triangulation of $\Omega \subset R^2$ used before.

Now, we are able to compute the PL zero set on each transverse simplex using linear interpolation. This computation can be done using one of the following two alternatives:

- Computing the convex hull of the zero points on the edges of the simplex.
- Computing the equation of the hyperplane that contains the convex hull.

(1) Using the first alternative, we can determine the contour that passes through a triangle by computing the points where the line intersects the edges of such triangle. In this case, any transverse edge with vertices p_0 and p_1 can be written as

$$p = p_0 + t(p_1 - p_0) \qquad (6.5)$$

with $t \in [0, 1]$, and the linear interpolant over such an edge is given by

$$F(p) = f(p_0) + t[f(p_1) - f(p_0)] \qquad (6.6)$$

Setting $F(p) = 0$, we get $t = -\frac{f(p_0)}{f(p_1)-f(p_0)}$ from Equation (6.6). Substituting the value of t in Equation (6.5), we obtain the zero point on the transverse edge.

(2) Uses the interpolant over the triangle as a whole, not over its edges. Analogously, the values of f at the corners of a triangle $[p_0, p_1, p_2]$ define a unique piecewise linear interpolant $F(p)$ to $f(p)$ over each triangle, which can be written in terms of the equations

$$p = p_0 + s(p_1 - p_0) + t(p_2 - p_0) \qquad (6.7)$$

and

$$F(p) = f(p_0) + s[f(p_1) - f(p_0)] + t[f(p_2) - f(p_0)] \qquad (6.8)$$

where $s \geqslant 0, t \geqslant 0$, and $s + t = 1$.
The interpolant F is piecewise linear because its contour (or PL zero set) across an individual triangle is a line segment, whose line equation can be easily determined by solving the system of Equations (6.7) and (6.8), after setting $F(p) = 0$.

Using linear interpolation amounts to use barycentric coordinates. In fact, Equation (6.7) can be written as

$$p = \alpha_0 p_0 + \alpha_1 p_1 + \alpha_2 p_2 \qquad (6.9)$$

and

$$F(p) = \alpha_0 f(p_0) + \alpha_1 f(p_1) + \alpha_2 f(p_2) \qquad (6.10)$$

where $\alpha_0 = 1 - (s+t), \alpha_1 = s, \alpha_2 = t, \alpha_0 + \alpha_1 + \alpha_2 = 1 \; and \; 0 \leqslant s, t \leqslant 1.$

In general, every point $p \in R^{n+d}$ of an $(n+d)$-simplex $\sigma = [p_0, \ldots, p_{n+d}] \subset R^{n+d}$ can be expressed in barycentric coordinates

$$p = \alpha_0 p_0 + \cdots + \alpha_{n+d} p_{n+d} \qquad (6.11)$$

with $\alpha_i \geqslant 0 \ \ (i = 0, \ldots, n+d)$ and $\ \alpha_0 +, \ldots, + alpha_{n+d} = 1.$
Similarly, we have

$$F(p) = \alpha_0 f(p_0) +, \ldots, + \alpha_{n+d} f(p_{n+d}) \qquad (6.12)$$

or, equivalently,

$$\mathbf{L_f}(\sigma) \cdot \alpha = \begin{pmatrix} 1 \\ F(p) \end{pmatrix} \qquad (6.13)$$

where $\alpha = (\alpha_0, \ldots, \alpha_{n+d})^T$are the barycentric coordinates of a point $p \in R^{n+d}$ and

$$\mathbf{L_f}(\sigma) = \begin{pmatrix} 1 & \ldots & 1 \\ f(p_0) & \ldots & f(p_{n+d}) \end{pmatrix}$$

The matrix $L_f(\sigma)$ is known as labelling matrix of a $(n+d)$-simplex $\sigma = [p_0, \ldots, p_{n+d}] \subset R^{n+d}$. It consists of $n+d+1$ labelling column vectors, each vector storing the value of f, which works as a label, at each vertex. In general, the standard vector labelling induced by $f : R^{n+d} \to R^n$ is then

$$\mathbf{l_f}(\mathbf{p}) = \begin{pmatrix} 1 \\ f(p) \end{pmatrix}$$

where $p \in R^{n+d}$.

Intuitively, this means that f changes sign at vertices of τ , consequently, any $(n+d)$-simplex σ having τ as a n-face is said to be transverse to the zero set of f.

From Equation (6.13), the PL zero set across the simplex σ is the set of points whose barycentric coordinates satisfy

$$\mathbf{L_f}(\sigma)\cdot\alpha = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad (6.16)$$

with $\alpha_i \geqslant 0 \ \ (i = 0, \ldots, n+d)$, that is

$$\alpha = L_f(\sigma)^{-1}\cdot e_1 \qquad (6.17)$$

This gives us the barycentric coordinates of at least a point b_0 in the d-dimensional hyperplane that approximates the zero set inside a given (n+d)-simplex σ. The parametric equation in barycentric coordinates corresponding to the general Equation (6.16) of such hyperplane can be written as follows

$$b(t_1, \ldots, t_d) = b_0 + \sum_{i=0}^{d} t_i b_i \qquad 6.18$$

where $t_i \in R$ is the real parameter on the line defined by the vector $b_i - b_0$, and $\{b_i\}(i = 1, \ldots, d)$ is a linearly independent set of points. This linear independence implies that the barycentric coordinates of a nonzero $(n + d)$- tuple bi have sum zero. Thus, computing bi in the zero set hyperplane inside σ reduces to determine a nontrivial solution of the homogeneous equation

$$\mathbf{L_f}(\sigma) \cdot \mathbf{b_i} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad (6.16)$$

The hyperplane passes through the facets of σ, the completely labelled facets, opposite to vertices for which $t_i = -\frac{b_0}{b_1}$ is negative.The remaining facet is the exit or pivoting facet.

Found the index i of the vertex p_i opposite to the exit facet, pi is pivoted into a new vertex $\hat{p}_i$, and the labelling matrix $L_f(\hat{\sigma})$ is obtained by replacing the i^{th} label or column of the $L_f(\sigma)$ by

$$\mathbf{l_f}(\hat{\mathbf{p}}_\mathbf{i}) = \begin{pmatrix} 1 \\ f(\hat{p}_i) \end{pmatrix} \qquad (6.20)$$

Example 6.9. Let $f : \Omega \subset R^2 \to R$ a real function in two real variables defined by $f(x, y) = -2x + y + 1/4$ (see Figure 6.9). In this case, the zero set of f is the straight line$-2x + y + 1/4 = 0$ in R^2, so it coincides with its PL zero set. Let us also consider that the domain $\Omega \subset R^2$ is to be triangulated according to Freudenthal's pivoting rule, where the coordinates of the vertices are all integer. For brevity, we let us consider the 2-simplex or triangle $\sigma = [p_0, p_1, p_2] \subset R_2$, with $p_0 = (0, 0), p_1 = (1, 0) \;\; and \;\; p_2 = (1, 1)$. The labelling matrix is then

$$\mathbf{L_f}(\sigma) = \begin{pmatrix} 1 & 1 & 1 \\ \frac{1}{4} & -\frac{7}{4} & -\frac{3}{4} \end{pmatrix} \qquad (6.22)$$

$$\mathbf{L_f}(\sigma)^{-1} = \frac{1}{6} \begin{pmatrix} \frac{68}{16} & 3 \\ -\frac{1}{4} & -3 \\ 2 & 0 \end{pmatrix} \qquad (6.23)$$

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$$\mathbf{b_0} = \mathbf{L_f}(\sigma)^{-\mathbf{1}} \cdot \mathbf{e_1} = \begin{pmatrix} \frac{17}{24} \\ -\frac{1}{24} \\ \frac{1}{3} \end{pmatrix} \tag{6.24}$$

$$\mathbf{b_1} = \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix} \tag{6.25}$$

So,we have

$$\mathbf{0} = \begin{pmatrix} \frac{17}{24} \\ -\frac{1}{24} \\ \frac{1}{3} \end{pmatrix} + t \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix} \tag{6.26}$$

or

$$t = \begin{cases} t & = & \frac{17}{12} \\ t & = & -\frac{1}{12} \\ t & = & -\frac{1}{3} \end{cases} \tag{6.27}$$

For $t = -\frac{1}{12}$:

$$\alpha = \begin{cases} \alpha_0 & = & \frac{3}{4} \\ \alpha_1 & = & 0 \\ \alpha_2 & = & \frac{1}{4} \end{cases} \qquad (6.28)$$

So,we get

$$\mathbf{p} = \frac{3}{4}\begin{pmatrix} 0 \\ 0 \end{pmatrix} + 0\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{4} \end{pmatrix} \qquad (6.29)$$

Analogously,for $t = -\frac{1}{3}$,we have: So,we get

$$\mathbf{p} = \begin{pmatrix} \frac{1}{8} \\ 0 \end{pmatrix} \qquad (6.30)$$

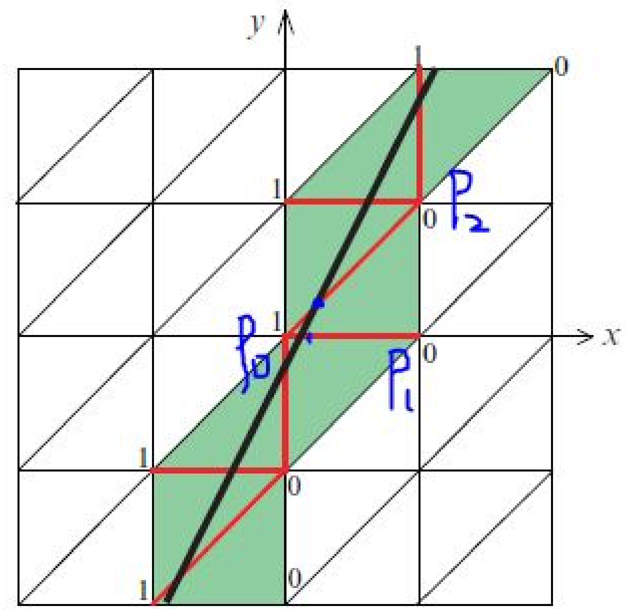

Fig. 6.9. Vector labelling of a triangulation in $\mathbb{R}^2$.

Algorithm 13 Vector-labelling PL Algorithm for Curves

procedure ALLGOWER-GEORG-GNUTZMANN(f,Ω,$\mathcal{T}$)
    Find a transverse $(n+1)$-simplex $\sigma \in \mathcal{T}$ with c.l. n-face τ opposite to $\mathbf{p}_i$.
    Calculate labelling matrix $L_\sigma = \begin{pmatrix} 1 & \cdots & 1 \\ f(\mathbf{p}_0) & \cdots & f(\mathbf{p}_{n+1}) \end{pmatrix}$.
    repeat
        Solve $L_\sigma\, \alpha = \mathbf{e}_1$, with $\alpha_i = 0$. ▷ first hyperplane point
        if $\alpha \not\geq \mathbf{0}$ **then**
            stop
        end if
        Solve $L_\sigma\, \beta = \mathbf{0}$. ▷ find other hyperplane points
        Find index j of the next pivoting vertex. ▷ door-in-door-out step
        Pivot $\mathbf{p}_j$ into $\hat{\mathbf{p}}_j$. ▷ pivoting step
        $\mathbf{p}_j \leftarrow \hat{\mathbf{p}}_j$
        Update j-component of σ with the new $\mathbf{p}_j$. ▷ adjacent $(n+1)$–simplex
        Calculate new label $l_j = \begin{pmatrix} 1 \\ f(\mathbf{p}_j) \end{pmatrix}$.
        $L_\sigma \leftarrow L_\sigma + (l_j - L_\sigma\, e_j)e_j^T$ ▷ update labelling matrix
        $i \leftarrow j$
    until
end procedure

THANKS YOU ALL!