The General Properties of
Discrete-Time Competitive Dynamical Systems *

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For strongly competitive discrete-time dynamical systems on a strongly ordered topological vector space, it is proved that any \( \alpha \)- or \( \omega \)-limit set is unordered and lies on some invariant hypersurface with codimension one, which generalizes Hirsch's results for competitive autonomous systems of ordinary differential equations to competitive maps in a very general framework and implies the Sarkovskii's Theorem for planar strongly competitive and monotone maps. We also show that the definition of competitiveness gives more restriction on homeomorphic property.

Key Words: competitive map, order decomposition, invariant \( d \)-hypersurface, Sarkovskii's Theorem

1. INTRODUCTION

In the biological science and population ecology, there are large quantities of mathematical models of competition in which an increase of competitor’s population size or density can only have a negative effect on a species per capita growth rate due. Frequently, such models are described by ordinary differential equations, reaction-diffusion equations or difference equations. For ordinary differential equations and difference equations, we refer the reader to the books [3,14] which contain a large amount of competitive systems, and for partial differential equations, we refer him to the book [5] and references therein. Although there is a very long history for studying competitive systems, applied mathematicians before 1980's only worked with

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various particular examples on their own interest. Not until the paper by S. Smale [17] and a series of important papers [6-9] was the research on competitive systems and monotone systems fully integrated with dynamical systems ideas. Hereafter, the research in this branch has received considerable attention and been carried out in a very general and abstract context. Smale showed that any vector field on a standard \((n-1)\)-simplex in \(\mathbb{R}^n\) can be embedded into a smooth competitive vector field on \(\mathbb{R}^n\) for which the simplex is an attractor. On the positive side, Hirsch [6,7] established that the \(\alpha\)- or \(\omega\)-limit sets of competitive systems can be no more complicated than those of general systems in one fewer dimension, that is, the flow on a compact limit set of an \(n\)-dimensional competitive system is topologically equivalent to the flow of a Lipschitz \((n-1)\)-dimensional system restricted to a compact invariant set. This then leads to the Poincaré-Bendixson theorem for three dimensional systems, that is, three competitive systems behave like general planar systems. Under additional assumption, he [8] proved that there is a canonically defined countable (generically finite) family of disjoint invariant \((n-1)\)-dimensional Lipschitz manifolds which are unordered and attract all non-convergent persistent trajectories. These papers of Hirsch are especially important for introducing some useful techniques to treating monotone flows (see [9,18]).

The above-mentioned work focuses on autonomous systems. If one wants to model systems with day-night and seasonal variant, he needs to study competitive systems with time periodic (the normalized period is assumed to \(2\pi\)). The important results for periodic competitive systems were obtained by de Mottoni and Schiaffino [15], Hale and Somolinos [11] and Smith [19,20]. The authors in [11,15,19,20] studied the discrete dynamical system generated by the Poincaré map \(T\), defined by \(T : x(0) \mapsto x(2\pi)\), which has the property that if \(Tx < Ty\) then \(x < y\) (Following Smith [19,20], we call such a map competitive), where the inequalities between vectors are to be understood as holding componentwise and is implied by the comparison principle. It was proved in [11,15] that every bounded solution of a periodic two-dimensional competitive system is asymptotic to some \(2\pi\)-periodic solution. Smith [19,20] investigated the geometrical properties of any finite dimensional competitive dynamical system. Under a set of hypotheses which general \(n\)-dimensional competitive systems satisfy, he proved that every compact \(\omega\)-limit set is unordered and came very close to concluding that the \(\omega\)-limit set of every nontrivial orbit lies on a certain lower dimensional manifold which is homeomorphic to an \((n-1)\)-simplex and contains all periodic orbits. Although there are many open problems remaining to help understand where \(\omega\)-limit sets lie exactly, his results (see [20, Propositions 3.7 and 3.8]) and their proofs strongly suggest that the \(\omega\)-limit sets lie in the one lower dimensional manifolds he formulated.
Takáč is the first researcher to investigate the asymptotic behavior of
discrete-time strongly monotone dynamical systems on a strongly ordered
metrizable topological space. In the papers [23, 24], he introduced the
concepts of order decomposition and $d$-hypersurface which are powerful tools
to treat discrete-time monotone dynamical systems and proved that every
nonempty, unordered invariant subset of state space is contained in some
invariant $d$-hypersurface which is a Lipschitz submanifold with codimension
one if the state space is an open set of a strongly ordered Banach space
and presented many interesting properties for asymptotic behavior, such
as $\omega$-stability.

The purpose of the present paper is to study the general properties of
competitive dynamical systems on a strongly ordered metrizable topological
space. We are going to generalize the same results obtained by Hirsch for
$n$-dimensional autonomous competitive ordinary differential equations to
discrete-time competitive dynamical systems on a strongly ordered topo-
logical vector space. More precisely, under the assumption that $x \ll y$
whenever $Tx \ll Ty$ and $x, y \in X$, we shall prove that every nonempty,
unordered with respect to $\ll$, totally invariant subset for the competitive
system $\{T^n\}_{n \geq 1}$ is contained in some invariant $d$-hypersurface which is a
Lipschitz submanifold with codimension one if the state space is an open
set of a strongly ordered topological vector space. The main tool to prove
this result is to employ order decomposition and $d$-hypersurface introduced
by Takáč [24], but it is more difficult to construct order decomposition and
needs more technical trick to prove the invariance of $d$-hypersurface. For
$n$-dimensional competitive Poincaré map $T$ satisfying Kamke condition,
we shall prove that any compact $\alpha$- or $\omega$-limit set is on an invariant Lipschitz
submanifold whose dimension is $n - 1$. Therefore, the dynamical behavior
for planar competitive and monotone maps is like that of one dimensional
dynamical systems. In particular, the well known Sarkovskii’s theorem
holds for planar competitive and cooperative maps, which is one of very
few results for planar maps (see [1, 4] for such kind of results). Moreover,
we shall show that the definition of competitiveness gives more restric-
tion on homeomorphic property and verify that any local homeomorphic
competitive map is a global homeomorphism.

This paper is organized as follows. In section 2 we agree on some nota-
tions, give important definitions and state some known results which will
be important to our proofs. The main results and their proofs are given in
section 3. In section 4 we show the relation between local homeomorphism
and global homeomorphism. Finally, in section 5 we discuss the dynamics
of strongly competitive planar maps.
2. DEFINITIONS AND PRELIMINARY RESULTS

We start with some notations and a few definitions.

The space $X$ is called an ordered space if it is a metrizable topological space together with a closed partial order relation $R \subseteq X \times X$. We write $x \cdot y$ if $(x, y) \in R$; $x < y$ if $x \cdot y$ and $x \neq y$; $x \ll y$ if $(x, y) \in \text{Int}R$, where Int indicates the interior of a set. Notations such as $x > y$ have the natural meanings.

If $A, B \subseteq X$ are subsets of $X$ then $A < B$ means $a < b$ for all $a \in A, b \in B$; and similarly for $A \leq B, A \ll B$, etc.

The ordered space $X$ is called strongly ordered if every open set $U$ of $X$ satisfies:

(SO1) : If $x \in U$ then $a \ll x \ll b$ for some $a, b \in U$.

It is easy to see that this implies

(SO2) : If $a, b \in U$ and $a \ll b$ then $a \ll x \ll b$ for some $x \in U$.

Suppose that $V$ is a metrizable topological vector space together with a closed convex cone $V_+$ such that $V_+ \cap V_- = \emptyset$. Then an ordering is defined by $x \leq y$ if and only if $y - x \in V_+$. Notice that $V$ is strongly ordered (shortly, strongly ordered vector space) if and only if $V_+$ is solid, i.e., $\text{Int}V_+ \neq \emptyset$. In this case $x \ll y$ if and only if $y - x \in \text{Int}V_+$. Therefore, every nonempty, open subset of $V$ is a strongly ordered space.

Suppose $T : X \to X$ is a continuous map from $X$ into $X$. We call it competitive, if $x < y$ whenever $Tx < Ty$ and $x, y \in X$. We call it strongly competitive, if $x \ll y$ whenever $Tx < Ty$ and $x, y \in X$. Sometimes we also assume that $T$ satisfies

(S) : $x \ll y$ whenever $Tx \ll Ty$ and $x, y \in X$.

Obviously, strongly competitiveness implies the assumption (S).

The positive semi-orbit of any $x \in X$ is defined by

$$O^+(x) = \{T^m x : m \in \mathbb{Z}_+\},$$

where $\mathbb{Z}_+$ denotes the set of nonnegative integers. The closure of $O^+(x)$, denoted by $\overline{O^+(x)}$, is called the orbit closure of $x$. The $\omega$-limit set of $x$ is
defined by
\[ \omega(x) = \{ y \in X : T^{n_k}x \rightarrow y(k \rightarrow \infty) \text{ for some sequence } n_k \rightarrow \infty \text{ in } \mathbb{Z}_+ \}. \]

Note that if \( \overline{O^+(x)} \) is compact in \( X \), then \( \omega(x) \neq \emptyset \) and is totally invariant, i.e., \( T \omega(x) = \omega(x) \).

For \( x \in \cap_{n \geq 0} T^n X \), let
\[ \lim(X, T, x) = \left\{ (x_n)_{n=0}^{\infty} \in \prod X : T x_{n+1} = x_n \text{ for } n \geq 0 \text{ and } x_0 = x \right\}. \]

Then usually the space \( X_T = \cup \lim(X, T, x) : x \in \cap_{n \geq 0} T^n x \) is called the inverse limit of \( T \) according to William [26]. We define a negative semi-orbit of \( x \), denoted by \( O^-(x) \), is an element of \( \lim(X, T, x) \). The \( \alpha \)-limit set of \( O^-(x) \) is defined by
\[ \alpha(x) = \{ y \in X : x_{n_k} \rightarrow y(k \rightarrow \infty) \text{ for some sequence } n_k \rightarrow \infty \text{ in } \mathbb{Z}_+ \}, \]

where \( \{ (x_n)_{n=0}^{\infty} \} = O^-(x) \). If \( \overline{O^-(x)} \) is compact, then \( \alpha(x) \neq \emptyset \) and is totally invariant. Obviously, \( O^-(x) \) is unique if and only if \( T \) is a homeomorphism. In this case, we call \( \alpha(x) \) the \( \alpha \)-limit set of \( x \).

The point \( p \) is called a \((k-) \) periodic point of \( T \) if \( T^k p = p \) and \( T^l p \neq p \) for \( 0 < l < k \). We call \( O^+(p) \) a cycle, or a \( k \)-cycle. If \( Tp = p \), then we say \( p \) is a fixed point.

Given \( a, b \in X \), the set \( [a, b] = \{ x \in X : a \leq x \leq b \} \) is called a closed order interval, and \( ]a, b[ = \{ x \in X : a < x < b \} \) is called an open order interval in \( X \). We write \( [a, \infty[ = \{ x \in X : x \geq a \} \), and similarly for \( ]-\infty, b[ \), etc. A subset \( Y \) of \( X \) is called lower closed if \( ]-\infty, b[ \subset Y \) whenever \( b \in Y \); and upper closed if \( [a, \infty[ \subset Y \) whenever \( a \in Y \). We denote by \( Y^c \) the complement of \( Y \) in \( X \).

Now, we are ready to introduce our crucial concepts contained in Takáč [23,24].

**Definition 2.1.** A pair of \((A, B)\) of subsets \( A, B \) of \( X \) is called an order decomposition of \( X \) if it has the following properties:
(i) \( A \neq \emptyset \) and \( B \neq \emptyset \);
(ii) \( A \) and \( B \) are closed;
(iii) \( A \) is lower closed and \( B \) is upper closed;
(iv) \( A \cup B = X \); and
(v) \( \text{Int}(A \cap B) = \emptyset \).

An order decomposition \((A, B)\) of \( X \) is called invariant if \( TA \subset A \) and \( T(B) \subset B \). The set \( H = A \cap B \) (possibly empty) is called the boundary of
the order decomposition $(A, B)$ of $X$. A \textit{d-hypersurface} is any nonempty subset $H$ of $X$ such that $H = A \cap B$ for some order decomposition $(A, B)$ of $X$.

Note that the boundary $H$ of an order decomposition $(A, B)$ of $X$ satisfies $H = \partial A = \partial B$, where \(\partial\) is the boundary symbol in $X$, and $H$ is invariant whenever $(A, B)$ is invariant. It is also easy to see that a \textit{d-hypersurface} $H$ never contains two distinct points $x, y$ related by $\ll$.

Finally, we state several known results.

**Proposition 2.1.** Let $X$ be a strongly ordered space. If $F \subseteq X$ is lower closed (upper closed, resp.), then so is its closure $\overline{F}$. Its complement $X - F$ is upper closed (lower closed, resp.). The union and intersection of any family of lower (upper) closed sets are also lower (upper) closed.

**Proposition 2.2.** Let $X$ be a nonempty, open subset of $V$, and let $(A, B)$ be an order decomposition of $X$ with the boundary $H = A \setminus B$. Fix any vector $v \in \text{Int}(V_+)$, and denote by $R = \text{lin}\{v\}$ the linear subspace of $V$ spanned by $v$. Let $Q$ be a positive continuous projection of $V$ onto $R$, which always exists, and set $P = I - Q$ with $W = P(V)$, the range of $P$, so that $V = W \oplus R$ is the direct algebraic and topological sum of $W$ and $R$. Then we have the following statements:

(i) The restriction $P|_H$ of $P$ to $H$ is one-to-one, and both $P|_H$ and its inverse $\pi = (P|_H)^{-1} : P(H) \to H$ are Lipschitz continuous in the ordered norm $\|\cdot\|_v$ with a common Lipschitz constant.

(ii) $P|_H$ is a homeomorphism of $H$ onto $P(H)$ in the topologies induced by that on $V$.

Propositions 2.1 and 2.2 are due to Takáč and can be found in [23, 24]. Notice that the condition that $H$ is unordered in the proof of Proposition 2.2 can be replaced by the weaker one that $H$ never contains two strongly ordered points.

### 3. THE MAIN RESULTS AND THEIR PROOFS

Before proceeding to the proof of our main results, we present some lemmas and propositions.

First some notations are required. Let $O^+(x)$ be the positive semiorbit of $x$ and $m < n$, an integer segment $[m, n] = \{m, m + 1, \ldots, n\} \subseteq \mathbb{N} \cup \{0\}$. $[m, n]$ is called \textit{rising segment} of $O^+(x)$ if $T^m x < T^n x$ and a \textit{falling segment} if $T^m x > T^n x$. 
**Lemma 3.1.** Assume that $X$ is an ordered space and $T$ is competitive. Then $O^+(x)$ cannot have both a rising segment and a falling segment that are disjoint.

**Remark 3. 1.** The following proof is originated from the nonoscillation principle due to Hirsch in [6,9], which deals with the continuous-time systems.

**Proof.** Suppose that $O^+(x)$ contains the falling segment $[m,n]$ and the rising segment $[k,l]$, where $0 \leq m < n < k < l$. The other case can be treated similarly. In $[m,n]$, decreasing $n$ if necessary, we can suppose

\[ T^s x \not\prec T^n x \quad \text{for any integer } s \in [m, n-1]. \]  

(a) If $l-k \leq n-m$, then from competitive property and $T^k x < T^l x$, we get $T^{n-l-k} x < T^n x$. Moreover, $m \leq n+k-l \leq n-1$, contradicting (1).

(b) If $l-k > n-m$, then we have $m < n < m+l-k < l$. From $T^k x < T^l x$, we get $T^m x < T^{m+l-k} x$. Let $j = \min \{i \in [n, m+l-k] : T^m x < T^i x \}$. Then, obviously, $n < j \leq m+k-l$, $T^m x < T^j x$ and

\[ T^m x \not\prec T^s x \quad \text{for any integer } s \in [n, j-1]. \]  

Now, we consider the segments $[m,n]$ and $[n,j]$.

(a1) If $j - n < n - m$, then from $T^n x < T^j x$, we get $T^{2n-j} x < T^n x$, but $m < 2n - j < n$, which contradicts (1).

(b1) If $j - n \geq n - m$, then also from $T^n x < T^j x$, we get $T^m x < T^{j+m-n} x$, but $n \leq j + n - m \leq j - 1$, which contradicts (2).

The proof of lemma 3.1 is complete. 

**Proposition 3.1.** Assume that $X$ is a strongly ordered metrizable topological space and $T$ is competitive. Then any $\alpha$- or $\omega$-limit set $L$ never contains two points related by $\ll$.

**Proof.** First we suppose that $L = \omega(x)$ contains points $y$ and $z$ satisfying $y \ll z$. Then we obtain positive integers $m_i (i = 1, 2, 3, 4)$ satisfying $m_1 < m_2 < m_3 < m_4$, $T^{m_1} x \ll T^{m_2} x$ and $T^{m_4} x \ll T^{m_3} x$, contradicting Lemma 3.1.

Secondly, let $L = \alpha(x)$ contain points $y$ and $z$ satisfying $y \ll z$. Then we obtain $x_{m_1} \in O^-(x)$, for $i = 1, 2, 3, 4$, satisfying $0 < m_1 < m_2 < m_3 < m_4$, $x_{m_1} \ll x_{m_2}$ and $x_{m_3} \gg x_{m_4}$. Let $y = x_{m_4}$, then $y \ll T^{m_4-m_3} y$ and $T^{m_4-m_2} y \gg T^{m_4-m_3} y$, contradicting Lemma 3.1.
Proposition 3.2. Let $X$ be a strongly ordered metrizable topological space and $T : X \to X$ be a competitive map satisfying (S). Assume that $G$ is a nonempty, unordered with respect to $\ll$, totally invariant subset of $X$. Then there exists an invariant order decomposition $(Y, Z)$ of $X$ such that $G \subset H = Y \cap Z$.

Proof. We define

$$G_- = \{ x \in X : x \leq y \text{ for some } y \in G \}$$

and

$$G_+ = \{ x \in X : x \geq y \text{ for some } y \in G \}.$$ 

Since $G \subset G_- \cap G_+, G_+ \neq \emptyset$. We can also see that $G_+$ is upper closed and $G_-$ is lower closed. Define

$$\mathcal{F} = \{ Y \subset X : Y \text{ is closed and upper closed, } Y \supset G_+, \text{ satisfying (1), (2) below} \}$$

(1) : $T(Y^c) \subset (\text{Int}Y)^c$;

(2) : $Y \cap \text{Int}(G_-) = \emptyset$.

1°: First we show $\mathcal{F} \neq \emptyset$. Let $Y = \overline{G_+}$, obviously, $\overline{G_+}$ is closed, upper closed and satisfies $\overline{G_+} \supset G_+$. Suppose that there exists $x \notin \overline{G_+}$, but $Tx \in \text{Int(\overline{G_+})}$. Then there is a neighborhood $U$ of $Tx$, such that $U \subset \overline{G_+}$.

From (SO1), we get $y \ll Tx$ for some $y \in U$, thus we can find $z \in G_+$ such that $z \ll Tx$. It follows from the total invariance of $G$ that there is a $z_1 \in G$ such that $Tz_1 \leq z$. Hence, we obtain $Tz_1 \ll Tx$, which implies that $z_1 \ll x$ by (S), that is, $x \in G_+$ which contradicts our hypothesis that $x \notin \overline{G_+}$. We conclude that (1) holds. Next suppose that $x \in \overline{G_+} \cap \text{Int}(G_-) \neq \emptyset$. From the lower closed property of $G_-$ and (SO1), we obtain that there is an open order neighborhood $[[-\infty, b')]$ of $x$, satisfying $[[-\infty, b')] \subset G_-$. Then (SO2) implies that there is $b$ satisfying $x \ll b \ll b'$ and $b \in G_-$. Therefore, we can find $d \in G$ such that $b \leq d$. On the other hand, since $x \in \overline{G_+}$, we get $w \ll b$ for some $w \in G_+$. It follows from the definition of $G_+$ that $c \leq w$ for some $c \in G$. Hence, $c \leq w \ll b \leq d$, i.e., $c \ll d$, which contradicts our hypothesis that $G$ is unordered with respect to $\ll$. Therefore, (2) holds. We conclude $\mathcal{F} \neq \emptyset$.

2°: Second we show that the ordered set $\mathcal{F}$ endowed with the “$\subset$” ordering possesses a maximal element. Consider a nonempty, simply ordered
apply Zorn’s Lemma to conclude that $A$ is closed, which implies that $y \leq Tx$. Then it follows from the fact that $Y_{i_0}$ is upper closed that $Tx \in \text{Int}(Y_{i_0})$. On the other hand, because $x \notin A$, $x \notin Y_{i_0}$. This contradicts $T((Y_{i_0})^c) \subset (\text{Int}Y_{i_0})^c$, thus (1) holds for $A$. Furthermore, if $x \in A \cap \text{Int}(G_-) \neq \emptyset$, then we can find $b \in \text{Int}(G_-)$ such that $x \ll b$. Note that $x \in A$ implies that there exist some $j_0 \in \Gamma$ and some $y \in Y_{j_0}$ such that $y \ll b$, so $b \in Y_{j_0}$ follows from the fact $Y_{j_0}$ is upper closed, which implies that $Y_{j_0} \cap \text{Int}(G_-) \neq \emptyset$, a contradiction. Thus (2) holds for $A$. This proves $A \in \mathcal{F}$ is an upper bound of $\mathcal{F}_1$. Hence, we may apply Zorn’s Lemma to conclude that $\mathcal{F}$ possesses a maximal element, say $Y$.

3°: Finally, let $Z = (X \setminus Y)$. Obviously, $\text{Int}(Y \cap Z) = \emptyset$. Since $G_- \subset Z$, $Z \neq \emptyset$. We claim that $T(Z^c) \subset (\text{Int}Z)^c$.

In fact, if there exists $x \notin Z$, but $Tx \in \text{Int}Z$, then $x \in \text{Int}(Y)$. Let $\bar{Y} = Y \cup [Tx, \infty)$. It is easy to see that $\bar{Y}$ is closed, upper closed and $\bar{Y} \supset G_+$. To show that (2) holds for $\bar{Y}$, it is sufficient to prove that $[Tx, \infty] \cap \text{Int}(G_-) = \emptyset$. If not, let $y \in [Tx, \infty] \cap \text{Int}(G_-)$, then $y \ll z$ for some $z \in G$, which implies that $Tx \ll Tz$ for some $z_1 \in G$ by the total invariance of $G$. Therefore, we obtain $x \ll z_1$ from the fact that $T$ satisfies (S), i.e., $x \in \text{Int}(G_-)$. This shows that $Y \cap \text{Int}(G_-) \supset \text{Int}(Y) \cap \text{Int}(G_-) \neq \emptyset$, a contradiction.

We can also show that (1) holds for $\bar{Y}$. If not, then there exists $y \notin \bar{Y}$, but $Ty \in \text{Int}(\bar{Y}) = \text{Int}(Y \cup [Tx, \infty])$. From the upper closed property of $Y$ and $[Tx, \infty]$, we know $\text{Int}(\bar{Y}) = [[Tx, \infty]] \cup \text{Int}(Y)$.

(i) If $Ty \in [[Tx, \infty]]$, then $x \ll y$ by the condition (S). It follows from $x \in \text{Int}(Y)$ and $\text{Int}(Y)$ is upper closed that $y \in \text{Int}(Y)$, contradicting $y \notin \bar{Y}$.

(ii) Assume that $Ty \in \text{Int}(Y)$ in this case. Then since $y \notin Y$, this produces a contradiction that $Y \in \mathcal{F}$.

Thus we obtain $Y \in \mathcal{F}$ and $Y \subset \bar{Y}$. This contradicts the maximality of $Y$. The claim is proved.

From all above, we get a pair $(Y, Z)$ of subsets $Y, Z$ of $X$, they satisfy:

(a) $Y \neq \emptyset, Z \neq \emptyset$;
(b) $Y$ is upper closed and $Z$ is lower closed;
(c) $Y \cup Z = X$;
(d) $Y, Z$ are closed;
(e) $\text{Int}(Y \cap Z) = \emptyset$. 

subset $\mathcal{F}_1 = \{Y_i\}_{i \in \Gamma}$ of $\mathcal{F}$, where $\Gamma$ is an index set. Set

$$A = (\bigcup_{i \in \Gamma} Y_i).$$
They also satisfy $T(Y') \subset (\text{Int}Y')$, $T(Z') \subset (\text{Int}Z')$, i.e., $T(\text{Int}Z) \subset Z$; $T(\text{Int}Y) \subset Y$. For $x \in \partial Y \subset Y$, it follows from the strongly ordered property of $X$ that there exists a sequence $x_n \gg x$ and $x_n \to x(n \to \infty)$, so $Tx_n \to Tx(n \to \infty)$. Since $Y$ is upper closed, $x_n \in \text{Int}Y$. Then $Tx \in Y$ follows from $Tx_n \in Y$ and (d). So we obtain $T(Y) \subset Y$, similarly, $T(Z) \subset Z$.

We observe that $(Y, Z)$ is an invariant order decomposition of $X$ satisfying $G_- \subset Z$ and $G_+ \subset Y$. Finally, we have

$$G = G_+ \cap G_- \subset Y \cap Z = H.$$ 

This completes the proof.

We are now able to prove the main result in this section.

**Theorem 3.1.** Let $X$ be a nonempty, open subset of a strongly ordered metrizable topological vector space $V$, and let $T : X \to X$ be a competitive map satisfying (S). Assume that $L$ is a nonempty $\alpha$- or $\omega$-limit set. Then there exists an invariant Lipschitz submanifold $H$ of $X$, whose codimension is 1, such that $L \subset H$.

**Remark 3.2.** Theorem 3.1 shows that the dynamics of strongly competitive maps, without homeomorphism hypothesis, is essentially 1-codimensional. It generalizes the results obtained by Hirsch for $n$-dimensional autonomous competitive ordinary differential equations to discrete-time competitive systems on a strongly ordered topological vector space. In the case of finite dimension, it also strengthens the similar results by Smith [20].

**Proof.** From Proposition 3.1, we know that $L$ is an unordered set with respect to $\ll$ because $L$ is an $\alpha$- or $\omega$-limit set. In Proposition 3.2, let $G = L$, we obtain corresponding $H$ which is an invariant Lipschitz submanifold of $X$ by Proposition 2.2. Furthermore, the codimension of $H$ is 1 and $L \subset H$.

In the following, we shall apply Theorem 3.1 to the system of differential equations

$$\dot{x} = -F(t, x) \quad \text{for } x \in X \subset \mathbb{R}^n. \tag{3}$$

A vector function $f(x) = (f_1(x), \ldots, f_n(x))$ of a vector variable $x = (x_1, \ldots, x_n)$ will be said to be of type $K$ in a set $X$ if for each subscript $i = 1, \ldots, n$ we have $f_i(a) \leq f_i(b)$ for any two points $a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n)$ in $X$ with $a_i = b_i$ and $a_k \leq b_k (k = 1, \ldots, n; k \neq i)$. 

Kamke’s Theorem. Let \( f(t, x) \) be continuous in an open set \( X \subset \mathbb{R}^n \) and of type \( K \) for each fixed value of \( t \). Assume that \( x(t) \) and \( y(t) \) are the solutions of
\[
\dot{x} = f(t, x) \quad \text{for } x \in X \subset \mathbb{R}^n
\]
on \([a, b]\). Then \( x(b) <_r y(b) \) whenever \( x(a) <_r y(a) \). Here, \( <_r \) stands for one of the three relations \( \leq, <, \ll \).

Suppose that \( F(t, x) \) in (3) satisfies all the assumptions of Kamke’s Theorem. In addition, we assume that \( F(t, x) \) is 2\( \pi \)-periodic in \( t \) for each fixed value of \( x \) and that all solutions of (3) are continuable in the future and the system (3) has the uniqueness property for initial problems.

We denote by \( \Phi(t; s, x) \) the solution of (3) which satisfies \( \Phi(s; s, x) = x \).

With this notation it follows from the above assumptions that the Poincaré map
\[
T(x) = \Phi(2\pi; 0, x) \quad \text{for } x \in X
\]
is well defined as a continuous map \( T : X \to X \). By Kamke’s Theorem, we can conclude that \( T \) satisfies the assumptions in Theorem 3.1. Thus, applying Theorem 3.1 to the system (3), we obtain the following:

**Corollary 3.1.** Let \( X \subset \mathbb{R}^n \) be a nonempty, open set. Assume that
\[
F(t, x) : \mathbb{R} \times X \to \mathbb{R}^n
\]
is continuous, 2\( \pi \)-periodic in \( t \) and of type \( K \) for each fixed value of \( t \). Let \( T \) be the Poincaré map of (3). Then every compact \( \alpha \)- or \( \omega \)-limit set of system \( \{T^n\}_{n \geq 1} \) lies on some invariant Lipschitz invariant submanifold whose dimension is \( n - 1 \).

**Remark 3.3.** Assume that \( X \subset \mathbb{R}^n \) is \( p \)-convex and \( F(t, x) \) is continuously differentiable with respect to \( x \in X \). If every Jacobian matrix \( D_x F(t, x) \) is cooperative for every \( (t, x) \), then the Poincaré map \( T \) of the system (3) is competitive satisfying (8) by Kamke’s Theorem. Therefore, only assume the cooperation condition, we conclude that every \( \alpha \)- or \( \omega \)-limit set lies on an invariant Lipschitz submanifold with codimension one. However, similar results are obtained by Hirsch [8] and Smith [20] under the irreducibility and a set of additional hypotheses. In particular, if the dimension of the system (3) is two, then every \( \omega \)-limit is located in a Lipschitz curve which is unordered with respect to \( \ll \). Since the Poincaré map \( T \) is an order-preserving homeomorphism and the dynamics of one-dimensional homeomorphism is trivial, we also obtain the trivial dynamical property of planar competitive system (3) in a different way from that in [11,15,19,20].
4. THE RELATION BETWEEN LOCAL HOMEOMORPHISM AND GLOBAL HOMEOMORPHISM

**Theorem 4.1.** Let $X$ be a strongly ordered metrizable topological space and $T : X \rightarrow T(X) \subset X$ be a competitive map. If $T$ is a local homeomorphism, then $T$ is a global one onto the image $T(X)$.

**Proof.** Let $Tx = Ty = z \in T(X)$. Since $T$ is a local homeomorphism, there exist neighborhoods $U \subset X$ of $x$, $V \subset X$ of $y$ and $W \subset X$ of $z$, such that both $T : U \rightarrow W$ and $T : V \rightarrow W$ are homeomorphisms. Consequently, we can find a sequence $\{x_n\}_{n=1}^{\infty} \subset U$ satisfying $x_n \in U$ and $Tx_n \in W$ such that $x_n \rightarrow x$, $Tx_n < z = Ty$. The competitive property of $T$ implies that $x_n < y$, letting $n \rightarrow \infty$, we get $x \leq y$. In the same way we also obtain $y \leq x$. Hence $x = y$, which shows that $T$ is injective, thus $T$ is a homeomorphism from $X$ onto its image. 

We denote by $C^1(X, X)$ the space of all continuously-differentiable maps from $X$ into $X$, and $DT(x)$ the derivative of $T$ at $x \in X$. Let

$$M = \{x \in X : DT(x) \text{ is invertible}\}$$

and $M_0 = X \setminus M$.

**Proposition 4.1.** Let $T \in C^1(X, X)$ be a competitive map on a strongly topological space and $x, y \in X$ satisfying $Tx = Ty$. If either $x$ or $y$ belongs to $M$, then $x = y$.

**Proof.** Without loss of generality we may assume that $x \in M$. Let $z = Tx = Ty$. Then there exist neighborhoods $U \subset X$ of $x$ and $W \subset X$ of $z$, such that $T : U \rightarrow W$ is a homeomorphism. Thus we can choose $\{v_n\}_{n=1}^{\infty} \subset W$ and $\{z_n\}_{n=1}^{\infty} \subset W$ satisfying

1. $v_n < z < z_n$,  
2. $v_n, z_n \rightarrow z (n \rightarrow \infty)$

such that there exist $\{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty} \subset U$ satisfying $Tx_n = v_n, Ty_n = z_n$. Since $T^{-1}|W$ is continuous, $x_n, y_n \rightarrow x(n \rightarrow \infty)$ and $x_n < y < y_n$. Letting $n \rightarrow \infty$, we obtain $x \leq y \leq x$, i.e., $x = y$.

**Remark 4.1.** For $T \in C^1(X, X)$, if $M_0 = \emptyset$, then $T$ is a homeomorphism to its image by Theorem 4.1. Therefore, the theory of monotone
maps and semiflows can be applied to the competitive maps by considering $T^{-1}$. Especially, when the $d$-hypersurface $H$ is totally invariant and $T$ takes values in Banach space, we can use the results by Tereščák [25] to obtain that $H$ is $C^1$.

If $T$ is not injective, then Proposition 4.2 means that the only possibility for $Tx = Ty$ with different $x, y$ is $x, y \in M_0$.

5. THE PLANAR COMPETITIVE AND COOPERATIVE MAPS

It is well known that the flow restricted to a compact $\alpha$- or $\omega$-limit set of a competitive or cooperative system in $\mathbb{R}^n$ is topologically equivalent to a flow on a compact invariant set of a Lipschitz system of differential equations in $\mathbb{R}^{n-1}$. In other words, the long term dynamics of an $n$-competitive or cooperative system can be no more badly behaved than that of a general system with one lower dimension. Applying this result and the classification of limit sets for two-dimensional general systems, Hirsch [6,8] and Smith [18,21] established that the Poincaré-Bendixson Theorem holds for three dimensional competitive and cooperative systems.

Correspondingly, the results in the section 3 and Proposition 1.2 in Takáč [23] imply that $n$-dimensional discrete-time competitive and monotone dynamical systems can behave no worse than general discrete-time dynamical systems in one lower dimension. Since our understanding of one-dimensional discrete-time dynamical systems is much more than that of higher dimensional ones, we like to show that some well-known results for one-dimensional dynamical systems, such as Sarkovskii’s Theorem, hold for planar competitive or monotone maps.

Theorem 5.1. Let $T : X \to X$ be strongly competitive or monotone with respect to the order $\preceq_K$, where $X \subset \mathbb{R}^2$ is open and $K \subset \mathbb{R}^2$ is a cone. Then Sarkovskii’s Theorem holds for such a planar map, that is, if we define the order of the positive integers in the following form:

$$1 \triangleleft 2 \triangleleft 4 \triangleleft \ldots \triangleleft 2^k \triangleleft 2^{k+1} \triangleleft \ldots$$

$$\ldots \triangleleft 2^{k+1} (2l + 1) \triangleleft 2^{k+1} (2l - 1) \triangleleft \ldots \triangleleft 2^{k+1} \cdot 5 \triangleleft 2^{k+1} \cdot 3 \triangleleft \ldots$$

$$\ldots \triangleleft 2^k (2l + 1) \triangleleft 2^k (2l - 1) \triangleleft \ldots \triangleleft 2^k \cdot 5 \triangleleft 2^k \cdot 3 \triangleleft \ldots$$

$$\ldots$$
\[ \cdots \triangleright 2(2l + 1) \triangleright 2(2l - 1) \triangleright \cdots \triangleright 2 \cdot 5 \triangleright 2 \cdot 3 \cdots \]
\[ \cdots \triangleright (2l + 1) \triangleright (2l - 1) \triangleright \cdots \triangleright 5 \triangleright 3, \]
and \( T \) has a periodic point of period \( p \) and \( q \triangleright p \) in this ordering, then \( T \) has a periodic point of period \( q \).

**Proof.** Suppose that \( T \) has a periodic orbit \( O^+(x) \) of period \( p \). Then \( \omega(x) = \{x, Tx, \cdots, T^{p-1}x\} \) is an unordered invariant subset in \( X \). Applying Theorem 3.1 in section 3 and Proposition 1.2 in Takác [23], we obtain that there is a Lipschitz curve \( C \) in \( X \) which contains \( \omega(x) \) and is invariant for \( T \). Therefore, we reduce the dynamical system

\[ T|_C : C \to C. \]

Obviously, \( x \) is a periodic orbit of period \( p \) for \( T|_C \). Thus applying the Sarkovskii’s Theorem for one-dimensional continuous maps to \( T|_C \), we obtain that for any \( q \in \mathbb{N} \) such that \( q \triangleright p \), \( T|_C \) has a periodic orbit with period \( q \). This proves the theorem.

We have known that the dynamics of discrete-time strongly monotone dynamical systems in a strongly ordered space cannot be arbitrarily chaotic because every attractor contains a stable periodic orbit (see Hirsch [10, Theorem 6.3]). This implies that any attractor cannot contain a dense orbit (unless it is a cycle), nor can periodic points in the attractor be dense. However, if we adopt Li-Yorke’s chaos definition which was first presented by them in 1975 (see [13]) or Devaney’s chaos definition in [2], then the planar monotone can be chaotic in this sense.

**Definition 5.1.** Let \( f : X \to X \) be a dynamical system when \( X \) is metric space with metric \( d \). We say \( f \) is chaotic in the sense of Li-Yorke, if there is an uncountable subset \( S \subset X \) such that

\[
\lim_{n \to \infty} \inf d(f^n(x), f^n(y)) = 0
\]

and

\[
\lim_{n \to \infty} \sup d(f^n(x), f^n(y)) > 0,
\]

for any different points \( x, y \in S \).

**Definition 5.2.** Let \( f : X \to X \) be continuous where \( X \) is metric space with metric \( d \). \( f \) is called chaotic in the sense of Devaney if there is a closed invariant subset \( D \) satisfying:
(1) \( f \mid_{D} \) is transitive;
(2) The set of periodic points of \( f \mid_{D} \) is dense in \( D \);
(3) \( f \mid_{D} \) is sensitive dependence on initial conditions in the sense that there is an \( \varepsilon > 0 \) such that for each \( x \) and each neighborhood \( U(x) \) in \( D \) of \( x \) there exist \( y = y(x) \in U(x) \) and an integer \( n > 0 \) such that \( d(f^n(x), f^n(y)) > \varepsilon \).

Since Y. Oono [16] proved that \( f : I \to I \) is chaotic in the sense of Li-Yorke if \( f \) has a periodic orbit whose period is not the power of two, the following corollary follows from Theorem 4.1:

**Corollary 5.1.** Let \( T : X \to X \) be a strongly competitive or monotone map. If \( T \) has a periodic orbit whose period is not the power of two, then \( T \) is chaotic in the sense of Li-Yorke.

For any one-dimensional system generated by the continuous map \( h : \mathbb{R} \to \mathbb{R} \) given by
\[ u_{n+1} = h(u_n); \]
Smith [22] proved the following:

**Theorem 5.2 (Smith).** If \( h \) is of bounded variation on any closed interval of \( \mathbb{R} \), then \( h \) can be imbedded into a monotone map in \( \mathbb{R}^2 \) which can be modified into strongly monotone map without affecting any dynamics property.

In the recent quite interesting paper [12], Huang and Ye have proved that if \( D \) is compact then Devaney’s chaos implies Li-Yorke’s, that is, chaos in the sense of Devaney is stronger than that of Li-Yorke. Consequently, if a planar monotone dynamical system is dissipative and is chaos in the sense of Devaney, then it is also chaotic in the sense of Li-Yorke. Therefore, if we choose \( h(u) = 4u(1 - u) \) in Smith [22] and embed it into a strongly monotone map in \( \mathbb{R}^2 \), we know that both Devaney’s chaos and Li-Yorke’s chaos can occur in planar monotone dynamical systems. More concrete example is as follows:

**Example.**
\[
\begin{align*}
u_{n+1} &= b u_n + v_n^3 + \varepsilon v_n \\
v_{n+1} &= u_n^3 + \varepsilon u_n + b v_n
\end{align*}
\]
where $b$ and $\varepsilon$ are positive parameters. It is easy to check such a dynamical system is strongly monotone in $\mathbb{R}^2$. On the unordered “negative” diagonal

$$\Gamma = \{(u, v) : v = -u\},$$

this map is conjugate to one-dimensional map

$$h(u) = (b - \varepsilon)u - u^3.$$  

Therefore, we can choose suitable parameters $b, \varepsilon > 0$ such that the strongly monotone dynamical system (4) has a periodic orbit with period three. Hence, (4) is chaotic in the sense of Li-Yorke or Devaney for such parameters. Besides, if (4) has a nondegenerate periodic orbit whose period is not a power of two for the parameters $b_0$ and $\varepsilon_0$, then we can choose a suitable bounded subset $X \subset \mathbb{R}^2$ which contains such a periodic orbit. On $X$ we can perturbs the system (4.1) such that it is also strongly monotone in $X$. Therefore, as long as the perturbation is small enough, the perturbed system is chaotic in the sense of Li-Yorke or Devaney.

The above arguments show that although strongly monotone dynamical systems cannot possess any strange attractor in the sense of Hirsch [10], they can be chaotic and complicated in the sense of Li-Yorke or Devaney. A natural and very interesting question is whether Li-Yorke or Devaney chaos occurs for planar discrete-time strongly competitive systems. One way to show this is to embed an one-dimensional map which is chaotic in the sense of Li-Yorke or Devaney in a strongly competitive map on $\mathbb{R}^2$. Unfortunately, we fail to achieve this purpose. Although we can provide the example

$$T \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}(u-v)^2 + 1 \\ \frac{1}{2}(u-v)^2 - 1 \end{pmatrix}$$

which leaves invariant the unordered “negative” diagonal $\Gamma$ on which the dynamics (5) are described by $h(u) = 1 - 2u^2$ that has unstable periodic orbit of every period in $[-1, 1]$ and has an ergodic measure on that interval, it is unsatisfactory because $T$ maps $\mathbb{R}^2$ into $\Gamma$ and we cannot find a pair of points $x, y$ such that $Tx < Ty$. In our opinion, such a map could not be viewed as competitiveness.

As shown in section 4, the definition of competitiveness gives more restriction to homeomorphic property of map. So it is more difficult to embed one-dimensional map into the planar competitive map than the monotone map. In the following, we shall work in the opposite direction and try to study under what conditions the planar competitive maps have simple dynamics property, that is, every bounded orbit is asymptotic to either some
fixed point or some 2-cycle. One result says that if $T$ is a $C^1$ strongly competitive map and its Jacobian matrix at every point has non zero entries then its dynamics is simple. The other result implies that the dynamics is simple if $T^{-1}(\{x\})$ is discrete for every point $x$ and the image of $T$ has some property.

First we introduce some notations. Let $Q_i, i = 1, 2, 3, 4$ denote the usual open quadrants in $\mathbb{R}^2$ in counter clockwise order with increasing $i$, e.g., $Q_1 = \{(x_1, x_2) : x_i > 0, i = 1, 2\}$. For $x \in \mathbb{R}^2$ denote by $Q_i(x)$ the set $x + Q_i$, that is, the portion of the $i$-th quadrant centered at $x$. Let $Y$ be a connected subset of $\mathbb{R}^2$, we denote by

$$\partial_- Y = \{x \in \partial Y : x < y_n \in Y \text{ for some sequence } y_n \to x\}$$

the lower boundary of $Y$, and by

$$\partial_+ Y = \{x \in \partial Y : x > y_n \in Y \text{ for some sequence } y_n \to x\}$$

the upper boundary of $Y$. Then

$$Y = (\text{Int} Y) \cup (\partial_+ Y) \cup (\partial_- Y) \cup Fr(Y)$$

where we denote by

$$Fr(Y) = \partial Y \setminus (\partial_+ Y \cup \partial_- Y)$$

the frame of $Y$.

**Lemma 5.1.** Assume that $T$ is a planar competitive map. Then $T(Q_1(x)) \cap (Q_2(Tx) - Tx) = \emptyset$; $T(Q_3(x)) \cap (Q_1(Tx) - Tx) = \emptyset$; and $T(Q_i(x)) \subseteq Q_2(Tx) \cup Tx \cup Q_4(Tx), i = 2, 4$ for all $x \in \mathbb{R}^2$.

**Proof.** The proof of this lemma is similar to that of Smith (Lemma 4.1 of [20]).

Define

$$W^+ = \{x \in \mathbb{R}^2 : T(Q_i(x)) \subseteq \{Tx\} \cup Q_i(Tx), \quad i = 2, 4\};$$

$$W^- = \{x \in \mathbb{R}^2 : T(Q_2(x)) \subseteq \{Tx\} \cup Q_4(Tx), T(Q_4(x)) \subseteq \{Tx\} \cup Q_2(Tx)\};$$

$$V^+ = \{x \in \mathbb{R}^2 : T(Q_i(x)) \subseteq \{Tx\} \cup Q_2(Tx), \quad i = 2, 4\};$$

$$V^- = \{x \in \mathbb{R}^2 : T(Q_i(x)) \subseteq \{Tx\} \cup Q_4(Tx), \quad i = 2, 4\}.$$
Lemma 5.2. Let $T$ be a planar competitive map. If $\mathbb{R}^2 = W^+$, then the dynamics of $\{T^n\}_{n \geq 1}$ is simple. The same result holds in the case that $\mathbb{R}^2 = W^-$, or $\mathbb{R}^2 = V^+(V^-)$.

Proof. The proof of this Lemma is similar to that of Smith (Theorem 4.2 of [20]).

Now, we define the set

$$B_{\{x\}}^1 = \{ y \in T\mathbb{R}^2 : y < Tx \text{ or } y > Tx \}$$

where $x \in \mathbb{R}^2$, and the sets

$$B_{\{x\}}^{n+1} = B_{\{x\}}^1 = \{ y \in T\mathbb{R}^2 : y < y_1 \text{ or } y > y_1 \text{ for some } y_1 \in B_{\{x\}}^n \}$$

for $n = 1, 2, 3, \ldots$. Obviously, if $Fr(T\mathbb{R}^2) = \emptyset$, then $B_{\{x\}}^n \neq \emptyset$ for $n = 1, 2, 3, \ldots$. Furthermore, we can obtain

Lemma 5.3. Assume that $T^{-1}(\{x\})$ is discrete for every point $x$. Let $Fr(T\mathbb{R}^2) = \emptyset$. Then $T\mathbb{R}^2 = \cup_{n=1}^{\infty} B_{\{x\}}^n$.

Proof. Denote $\cup_{n=1}^{\infty} B_{\{x\}}^n$ by $M$. Suppose that there exists some $y = (y_1, y_2) \in T\mathbb{R}^2 \setminus M$. It is easy to see that $Tx \in Q_2(y) \cup Q_4(y)$, otherwise $y \in M$. We may assume that $Tx \in Q_2(y)$. We claim that $B_{\{x\}}^1 \subset Q_2(y)$.

Suppose that there is a $z \in B_{\{x\}}^1 \setminus Q_2(y)$. Then $z \in Q_1(y) \cup Q_3(y)$ or $z \in Q_4(y)$. If the former holds, then $z \leq y$ or $z \geq y$, thus $y \in M$, a contradiction. If the latter holds, then we obtain that $z \in Q_4(Tx)$, which contradicts $z \in B_{\{x\}}^1$. Hence, we have proved the claim. Using the same method, we can get $B_{\{x\}}^n \subset Q_2(y)$ for $n \geq 1$. Thus, $M \subset Q_2(y)$.

Let $P_i$ be the continuous orthogonal projection of $\mathbb{R}^2$ onto the $i$-th axis, $i = 1, 2$. We define the sets

$$A = \{ x \in \mathbb{R} : x > P_1 M \},$$

$$B = \{ x \in \mathbb{R} : x < P_2 M \}.$$

Since $y_1 \in A$ and $y_2 \in B$, $A, B \neq \emptyset$. Let $a_1 = \inf A, a_2 = \sup B$. Then we get $(a_1, a_2) = a \leq_K y$, where $\leq_K$ means that $a_1 \leq y_1$ and $a_2 \geq y_2$. We can also see that $M \subset Q_2(a)$.

(i) If $a <_K y$, then we find an $x_0 \in M$. Since $T\mathbb{R}^2$ is connected, there exists a continuous curve $C \subset T\mathbb{R}^2 : \gamma = \gamma(t)$ for $t \in [0, 1]$ such that
Let $\gamma(0) = x_0$ and $\gamma(1) = y$. It is easy to see that $C \cap \partial Q_2(a) \neq \emptyset$. Then there exists some $t_0 \in [0, 1)$ such that $\gamma(t_0) \in \partial Q_2(a)$ and $\gamma(t) \notin Q_2(a)$ for $t > t_0$ (if not, there must exist a sequence $t_n \to 1$ such that $\gamma(t_n) \in Q_2(a)$, thus $y \in Q_2(a)$, a contradiction). Let $b = (b_1, b_2) = \gamma(t_0)$. We claim that $a \in C$. If it is not the case, then $b \notin a$. Thus without loss of generality, we may assume that $b_1 = a_1$ and $b_2 > a_2$. It follows from the definition of $b$ that there must be some $t > t_0$, which is sufficiently close to $t_0$ such that $\tilde{b} = (b_1, b_2) = \gamma(t)$ satisfies $b_1 > a_1$ and $b_2 > a_2$. Then the definition of $a_2$ implies that there exists some $u \in M$ such that $u_2 = P_u < b_2$. Combining this with the fact that $u_1 = P_1u \leq a_1 < b_1$, we obtain that $u \ll \tilde{b}$. Hence $\tilde{b} \in M$. Note that $a \ll \tilde{b}$ which contradicts the fact $M \subset Q_2(a)$. Thus we have proved that $a \in C$, hence $a \in T \mathbb{R}^2$.

(ii) If $a = y$, then $a = y \in T \mathbb{R}^2$. It follows from $Fr(T \mathbb{R}^2) = \emptyset$ that there is some $z$ of $T \mathbb{R}^2$ such that $z < a$ or $z > a$. Then it follows from the property of $a$ that $z \in \partial Q_2(a) \subset Q_2(a)$.

From the above, we obtain that $a \in T \mathbb{R}^2$, $Q_2(a) \setminus \{a\} \cap T \mathbb{R}^2 \neq \emptyset$ and $M \subset \overline{Q_2(a)}$.

On the other hand, we claim that $T \mathbb{R}^2 \setminus (Q_1(a) \cup Q_3(a)) = \emptyset$. If not, we may assume that there exists $z \ll a$ such that $z \in T \mathbb{R}^2$. Using the method in the proof of (i), we can get $z \in M$. But $P_2z < a_2$, which contradicts the definition of $a_2$.

Thus we have proved that $T \mathbb{R}^2 \setminus \{a\}$ is not connected. Note that $T^{-1}(\{a\})$ is discrete. Then $T \mathbb{R}^2 \setminus \{a\} = T(\mathbb{R}^2 \setminus T^{-1}(\{a\}))$ is connected, a contradiction. This completes the proof. $

\textbf{Theorem 5.3.}$ Let $T$ be a planar competitive map and $Fr(T \mathbb{R}^2) = \emptyset$. Assume that $T^{-1}(\{x\})$ is discrete for every point $x$. Then

(i) if $\mathbb{R}^2 \cap W^+ \neq \emptyset$, then $\mathbb{R}^2 = W^+$;
(ii) if $\mathbb{R}^2 \cap W^- \neq \emptyset$, then $\mathbb{R}^2 = W^-$;
(iii) if $\mathbb{R}^2 \cap V^+ \neq \emptyset$, then $\mathbb{R}^2 = V^+$;
(iv) if $\mathbb{R}^2 \cap V^- \neq \emptyset$, then $\mathbb{R}^2 = V^-$. 

Moreover, the dynamics of $\{T^n\}_{n \geq 1}$ is simple in any case of (i) $\sim$ (iv).

\textbf{Proof.} First we assume that $x \in \mathbb{R}^2 \cap W^+ \neq \emptyset$. Since $Fr(T \mathbb{R}^2) = \emptyset$, $B_{1|x}^1 \neq \emptyset$. We claim that $T^{-1}B_{1|x}^1 \subset W^+$. In fact, for $z \in T^{-1}B_{1|x}^1$, we may assume $Tz < Tx$, hence $z < x$. Thus $T(Q_2(x) \cap Q_2(z)) \subset Q_2(Tx) \cup \{Tz\} \subset Q_2(Q_2(z)) \subset Q_2(Tz) \cup \{Tz\}$, but the competitive property implies that only $T(Q_2(x) \cap Q_2(z)) \subset Q_2(Tz) \cup \{Tz\}$. Combining this with the fact

$$T(Q_2(z)) \subset \{Tz\} \cup Q_4(Tz)$$ or $$\{Tz\} \cup Q_2(Tz),$$
which is implied by hypothesis that the preimage of every point is discrete we obtain 
\( T(Q_2(z)) \subset Q_2(Tz) \cup \{ Tz \} \). Similarly, we also obtain 
\( T(Q_4(z)) \subset Q_4(Tz) \cup \{ Tz \} \), i.e., \( z \in W^+ \), which proves the claim.

In the same way we also obtain 
\( T^{-1}B^2_{\{x\}} \subset W^+ \). Repeating the above, we get \( B^n_{\{x\}} \), satisfying 
\( T^{-1}B^n_{\{x\}} \subset W^+ \) for \( n = 1, 2, \cdots \). From Lemma 5.3, we obtain that 
\( \mathbb{R}^2 = \bigcup_{n=1}^{\infty} T^{-1}B^n_{\{x\}} \subset W^+ \). Thus \( \mathbb{R}^2 = W^+ \), and it follows

from that Lemma 5.2 the dynamics of \( \{ T^n \}_{n \geq 1} \) is simple. The proof of (ii),

(iii), (iv) is similar to that of (i).

Next we shall discuss the strongly competitive map \( T \) under the assumption that 
\( T \in C^1(\mathbb{R}^2, \mathbb{R}^2) \). We first need to classify the planar linear map.

**Lemma 5.4.** Assume that \( A \) is a planar linear map. Then \( A \) has the following forms:

(I)

\[
\begin{pmatrix}
  a & -b \\
  -c & d
\end{pmatrix}
\]

or

\[
\begin{pmatrix}
  a & 0 \\
  0 & d
\end{pmatrix}
\]

where \( ad > bc \), and \( a, b, c, d \) are positive real numbers.

(II)

\[
\begin{pmatrix}
  -a & b \\
  c & -d
\end{pmatrix}
\]

or

\[
\begin{pmatrix}
  0 & b \\
  c & 0
\end{pmatrix}
\]

where \( ad < bc \), and \( a, b, c, d \) are positive real numbers.

(III)

(i)

\[
\begin{pmatrix}
  a & -b \\
  -c & d
\end{pmatrix},
\begin{pmatrix}
  a & 0 \\
  -c & 0
\end{pmatrix},
\begin{pmatrix}
  0 & -b \\
  0 & d
\end{pmatrix}
\]

or

\[
\begin{pmatrix}
  0 & 0 \\
  0 & 0
\end{pmatrix}
\]

(ii)

\[
\begin{pmatrix}
  -a & b \\
  c & -d
\end{pmatrix},
\begin{pmatrix}
  -a & 0 \\
  c & 0
\end{pmatrix},
\begin{pmatrix}
  0 & b \\
  0 & -d
\end{pmatrix}
\]

or

\[
\begin{pmatrix}
  0 & 0 \\
  0 & 0
\end{pmatrix}
\]

(iii)

\[
\begin{pmatrix}
  a & b \\
  -c & -d
\end{pmatrix},
\begin{pmatrix}
  a & -b \\
  -c & d
\end{pmatrix}
\]

where \( ad = bc \), and \( a, b, c, d \) are positive real numbers. In this case, there

never exists \( \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^2 \), such that \( A \begin{pmatrix} u \\ v \end{pmatrix} < 0 \).

(IV) In other cases, there always exists \( \begin{pmatrix} u \\ v \end{pmatrix} \notin (-\mathbb{R}^+)^2 \) such that

\( A \begin{pmatrix} u \\ v \end{pmatrix} \ll 0. \)
The idea of this proof is very easy, while the process is very tedious. We omit it.

**Proposition 5.1.** Let $T \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ be a strongly competitive map. Then the case (IV) never exists for $DT(x), \forall x \in \mathbb{R}^2$.

**Proof.** If not, there exists $x_0 \in \mathbb{R}^2$, such that $DT(x_0)$ is of type (IV).

Then there is $\begin{pmatrix} u \\ v \end{pmatrix} \notin (-\mathbb{R}_+^2)$ satisfying $DT(x_0) \begin{pmatrix} u \\ v \end{pmatrix} \ll 0$. Note that

$$
T\left(x_0 + t \begin{pmatrix} u \\ v \end{pmatrix}\right) - T(x_0) = t DT(x_0) \begin{pmatrix} u \\ v \end{pmatrix} + o(t) = t \left( DT(x_0) \begin{pmatrix} u \\ v \end{pmatrix} + o(1) \right).
$$

Hence when $t$ is a sufficiently small positive real number, we obtain

$$
T\left(x_0 + t \begin{pmatrix} u \\ v \end{pmatrix}\right) - T(x_0) \ll 0.
$$

Since $T$ is strongly competitive, we have

$$
x_0 + t \begin{pmatrix} u \\ v \end{pmatrix} \ll x_0,
$$

that is, $\begin{pmatrix} u \\ v \end{pmatrix} \ll 0$, a contradiction. $\blacksquare$

Let

$$
N_+ = \{ x \in \mathbb{R}^2 : \det DT(x) \text{ is of type (I)} \},
N_- = \{ x \in \mathbb{R}^2 : \det DT(x) \text{ is of type (II)} \},
N_0 = \{ x \in \mathbb{R}^2 : \det DT(x) \text{ is of type (III)} \}.
$$

**Theorem 5.4.** Let $T \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ be a strongly competitive map. Then

(1) If $\mathbb{R}^2 = N_+ \cup N_0$, and when $x \in N_0$, $DT(x)$ satisfies (i) of (III), then the dynamics of $(T^n)_{n \geq 1}$ is simple.

(2) If $\mathbb{R}^2 = N_- \cup N_0$, and when $x \in N_0$, $DT(x)$ satisfies (ii) of (III), then the dynamics of $(T^n)_{n \geq 1}$ is simple.
Proof. We first consider the case (1). Let \( x \in \mathbb{R}^2 \) and \( y \in Q_2(x) \). Then \( Ty \in Q_2(Tx) \cup Q_4(Tx) \cup \{Tx\} \). Furthermore,

\[
Ty - Tx = \int_0^1 DT(\eta(s))ds \cdot (y - x) = \begin{pmatrix}
\int_0^1 \frac{\partial T_1}{\partial x_1}(\eta(s))ds & \int_0^1 \frac{\partial T_1}{\partial x_2}(\eta(s))ds \\
\int_0^1 \frac{\partial T_2}{\partial x_1}(\eta(s))ds & \int_0^1 \frac{\partial T_2}{\partial x_2}(\eta(s))ds
\end{pmatrix} \begin{pmatrix}
y_1 - x_1 \\
y_2 - x_2
\end{pmatrix}
\]

where \( \eta(s) = sx + (1 - s)y \). The condition of (1) and Lemma 5.4 imply that

\[
\int_0^1 \frac{\partial T_i}{\partial x_i}(\eta(s))ds \geq 0, \quad i = 1, 2,
\]

and

\[
\int_0^1 \frac{\partial T_i}{\partial x_j}(\eta(s))ds \leq 0, \quad i \neq j.
\]

Then we can obtain that \( T(Q_i(x)) \subset \{Tx\} \cup Q_i(Tx), \ i = 2, 4 \). Hence, we can complete the proof of the case (1) by Lemma 5.2. The proof of the case (2) is similar to that of (1) by considering the map \( T^2 \).

Corollary 5.2. Let \( T \in C^1(X, X) \) be a strongly competitive planar map which is not a constant. Assume that the Jacobian matrix \( DT(x) \) only has nonzero entries for each \( x \in \mathbb{R}^2 \). Then the dynamics of \( \{T^n\}_{n \geq 1} \) is simple.

Proof. In view of Theorem 5.4, it is sufficient to show that (iii) of (III) in Lemma 5.4 does not occur. If not, it follows from the assumption that \( DT(x) \) only has nonzero entries for all \( x \in \mathbb{R}^2 \) that \( T \) is a constant, contradicting the assumption.

Before ending this paper, we note that we have defined a map to be competitive in the sense of “time reversal” in this paper, while many researches use the competitive definition in the following way (see [18]):

Let \( T = (T_1, T_2) : \mathbb{R}^2 \to \mathbb{R}^2 \) be a mapping. Then \( T \) is competitive if and only if

\[
x_1 \leq y_1 \text{ and } x_2 \geq y_2 \quad \implies \quad T_1(x_1, x_2) \leq T_1(y_1, y_2) \text{ and } T_2(x_1, x_2) \geq T_2(y_1, y_2).
\]
It is easy to see that it is a planar monotone map with respect to the $K$-ordering, where $K = \{(x_1, x_2) : x_1 \leq 0, x_2 \geq 0\}$. We call such a map to be competitive in the sense of "competitive ordering". By Theorem 5.1, we obtain that the dynamics of planar competitive maps in sense of the "competitive ordering" is analogous to that of planar monotone maps. Therefore, chaotic dynamics in the sense of Li-Yorke or Devaney can occur in "competitive ordering" planar competitive dynamical systems.

Now we discuss the relation of these two definitions.

Firstly, we claim that without further restriction any one cannot imply the other. The following two examples can show this.

Let $T = \begin{pmatrix} -a & b \\ c & -d \end{pmatrix}$, where $ad < bc$ and $a, b, c, d$ are positive real numbers. Then from Lemma 5.4(II), $T$ is competitive in the sense of "time reversal". But a direct calculation shows that $T$ is not competitive in that sense of "competitive ordering".

On the other hand, let $T' = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$, where $ad < bc$ and $a, b, c, d$ are positive real numbers. Then $T'$ is competitive in the sense of "competitive ordering", but not in the sense of "time reversal".

Secondly, these two definitions are equivalent if $T$ is an orientation preserving homeomorphism.

We remark that the "time reversal" definition is originated from the property of flow (or Poincaré map) of the competitive ODE and can be generalized to higher dimension naturally and easily, while the "competitive ordering" definition cannot.

It seems to us that the dynamics of planar competitive maps in the sense of "time reversal" is simpler than that of planar monotone maps. However, dynamics of higher dimensional competitive dynamical systems is more complicated than that of higher dimensional monotone dynamical systems because the former can have a strange attractor but the latter cannot. An open problem is whether chaotic dynamics in the sense of Li-Yorke or Devaney can occur in "time reversal" planar competitive dynamical systems.

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