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Abstract. The dynamics of the Poincaré map, associated with a periodic tridiagonal system modeling cooperative-competitive ecological interactions, has been investigated. It is shown that the limit-set is either a fixed point or is contained in the boundary of the positive cone and itself contains a cycle of fixed points. Furthermore, the dynamics is trivial if the number of interactive species is not greater than 4.

1. Introduction. The dynamics of a community of $n \geq 2$ interacting (unstructured) species is described by the following so-called Kolmogorov system of ordinary differential equations:

$$\dot{x}_i = x_i f_i(t, x), \quad x_i \geq 0, \quad 1 \leq i \leq n,$$

(see Kolmogorov [15], Hofbauer and Sigmund [14].) Here $x = (x_1, x_2, \cdots, x_n)$ with $x_i$ denoting the population density of the $i$th species and $f_i$ is the per capita growth rate of the $i$th species. As indicated by the notation, the growth rates are in general time-dependent.

The interaction between two species can basically be of three different kinds. Competition is characterized by a decrease of the growth rate as the density of the other species increases. Mathematically, this can be formulated as $\frac{\partial f_j}{\partial x_i} < 0$, $\frac{\partial f_i}{\partial x_j} < 0$.
0. When two species cooperate, then an increase in the density of one species will enhance the growth of the other \( \frac{\partial f_j}{\partial x_i} > 0, \frac{\partial f_i}{\partial x_j} > 0 \). Finally, in predator-prey (or consumer-resource) interaction consumption of prey results in predator offspring and hence \( \frac{\partial f_j}{\partial x_i} < 0, \frac{\partial f_i}{\partial x_j} > 0 \). The three kinds of systems have been intensively studied under various conditions. We refer to [8, 11, 14, 17, 19] for results on the autonomous case. On the other hand, results on the time-dependent (in particular, the time periodic) case are more rare (see [25] and references therein).

In this paper we consider an ecosystem of \( n (n \geq 2) \) species the interaction of which are of the following nature:

(1): The interactions between species fluctuates periodically with time.

(2): Each species interacts with at most two other species.

(3): If two species interact, then their relationship is either competitive or cooperative.

(4): The community of species cannot be partitioned into two noninteracting subcommunities.

As we know, interactive species often live in a fluctuating environment. For example, physical environmental conditions such as temperature and humidity and the availability of food, water and other resources, usually vary in time with seasonal or daily variations. Therefore, more realistic models should be fluctuate with time. We here consider the data in the model are periodic functions of time with commensurate period. For the interpretation of (2), one can think of a hierarchy of species \( x_1, x_2, \cdots, x_n \), where \( x_i \) is the density or biomass of the \( i \)th species. In this hierarchy, \( x_1 \) only interacts with \( x_2, x_n \) only with \( x_{n-1} \), and for \( i = 2, \cdots, n-1, x_i \) interacts with \( x_{i-1} \) and \( x_{i+1} \). Such a hierarchy may occur in the water column of an ocean or on a steep mountain side or on island groups, where each species dominates a species zone (depth, altitude or different island) but is obliged to interact with other species in the (narrow) overlap of their zones of dominance. We do not allow predator-prey relationship. More specific autonomous models, for instance, the model of competition among parasites of ducks or the model of competitor-mutualist in the Galapagos, can also be found in [1, 2, 6]. The requirement (4) is for convenience, for if it fails then one could simply analyze the two decoupled subsystems independently. The central aim of this paper is to study the long-term behavior of such modeling systems.
Let $f_i$ be the per capita growth rate of species $i$. The Kolmogorov system of differential equations modeling the interaction of $n$ species in a community satisfying (1)-(4) with densities $x_i(t), 1 \leq i \leq n, n \geq 2,$ is given by

$$
\begin{align*}
\dot{x}_1 &= x_1 f_1(t, x_1, x_2), \\
\dot{x}_i &= x_i f_i(t, x_{i-1}, x_i, x_{i+1}), & 2 \leq i \leq n-1; \\
\dot{x}_n &= x_n f_n(t, x_{n-1}, x_n),
\end{align*}
$$

where $f = (f_1, f_2, \cdots, f_n)$ is defined on $\mathbb{R} \times U$, $U$ a nonempty open subset of $\mathbb{R}^n$ containing $\mathbb{R}^n_+$. Here $\mathbb{R}^n_+$ denotes the set of $n$-vectors with nonnegative components.

We assume that the functions $f_i$ and their partial derivatives with respect to the $x_j$ are continuous in $\mathbb{R} \times U$ and

$$
\frac{\partial f_i}{\partial x_i}(t, x) \cdot \frac{\partial f_{i+1}}{\partial x_i}(t, x) > 0, \quad 1 \leq i \leq n-1 
$$

holds for each $(t, x) \in \mathbb{R} \times \mathbb{R}^n_+$. It will be assumed that there exists $T > 0$ such that

$$
f(t + T, x) = f(t, x), \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n_+. \quad (1.3)
$$

Following Smith [20], we call the system (1.1)-(1.3) a periodic tridiagonal competitive-cooperative Kolmogorov system.

We may write periodic system (1.1) as

$$
\begin{align*}
\dot{x}_1 &= F_1(t, x_1, x_2), \\
\dot{x}_i &= F_i(t, x_{i-1}, x_i, x_{i+1}), & 2 \leq i \leq n-1; \\
\dot{x}_n &= F_n(t, x_{n-1}, x_n),
\end{align*}
$$

where $F = (F_1, F_2, \cdots, F_n)$ satisfies

$$
\frac{\partial F_i}{\partial x_{i+1}}(t, x) \cdot \frac{\partial F_{i+1}}{\partial x_i}(t, x) > 0, \quad 1 \leq i \leq n-1 
$$

for each $(t, x) \in \mathbb{R} \times \text{Int} \mathbb{R}^n_+$. Note that condition (1.2') holds only in $\text{Int} \mathbb{R}^n_+$, not in all of $\mathbb{R}^n_+$.

For the periodic system (1.1') quite a few results are known: Smith [20] has shown that every bounded solution is asymptotic to a $T$-periodic solution of (1.1') if the condition (1.2') holds in the region of one’s interests. He used an integer-valued Lyapunov function (nodal properties) to prove his result. Zhao [24] has generalized Smith’s result to asymptotic periodic differential equations. The results of Smith do not apply to the system (1.1) directly. The crucial assumption (1.2') holds only
in the interior of the positive orthant. The question arises whether all bounded solutions are still asymptotic to the set of $T$-periodic solutions in this case. The result is obviously true if the system is persistent (cf. [3, 4, 10]). But in the context of population dynamics, the bounded solution approaching the boundary of a given region are important, since they represent a situation whereby a certain population goes extinct.

In order to state our main results, we need to introduce some notations. Let $S$ be the Poincaré map associated with the periodic system (1.1), i.e., $S(x) = x(T)$ for $x \in \mathbb{R}^n_+$, where $x(t)$ is the unique solution of (1.1) satisfying $x(0) = x$. We denote by Fix($S$) the set of all fixed points of $S$. In terms of $S$, given any $x \in \mathbb{R}^n_+$, the orbit, $\omega$-limit set and $\alpha$-limit set of $x$, denoted by $O(x; S)$, $\omega(x; S)$, and $\alpha(x; S)$, have their usual meaning. Let $E_1$ and $E_2$ be two fixed points. $E_1$ is said to be chained to $E_2$, written $E_1 \to E_2$, if there exists a full orbit through some $x \neq E_1, E_2$ such that $\omega(x; S) = E_2$ and $\alpha(x; S) = E_1$. A finite sequence $\{E_1, \cdots, E_k\}$ of fixed points is called a chain if $E_1 \to E_2 \to \cdots \to E_k$. The chain is called a cycle if $E_k = E_1$. We write $\partial \mathbb{R}^n_+$ for the boundary of $\mathbb{R}^n_+$ and $\text{Int} \mathbb{R}^n_+$ for the interior of $\mathbb{R}^n_+$. A solution $x(t)$ of (1.1) is called positive if $x(t) \in \text{Int} \mathbb{R}^n_+$ for every $t \in [0, l_x)$, which is the maximal existence interval of $x(t)$. Otherwise, $x(t)$ is a non-positive solution. Furthermore, system (1.1) is persistent if, for solutions with initial values in $\text{Int} \mathbb{R}^n_+$, $\liminf_{t \to +\infty} x_i(t) > 0, \forall i = 1, \cdots, n$; is weakly persistent if, for solutions with initial values in $\text{Int} \mathbb{R}^n_+$, $\limsup_{t \to +\infty} \min_{1 \leq i \leq n} \{x_i(t)\} > 0$.

**Theorem 1.** Assume that (1.1) satisfy conditions (1.2)-(1.3) on $\mathbb{R}^n_+$ for some $n \geq 2$. Let $S$ be the Poincaré map associated with the system (1.1). Further, assume

(1): the set of non-positive $T$-periodic solutions is finite,

(2): If $p(t)$ is a non-positive $T$-periodic solution, then one is not a Floquet multiplier of $p(t)$,

Let $L = \omega(x; S)$ be the $\omega$-limit set of a bounded positive orbit of $x \in \text{Int} \mathbb{R}^n_+$. Then either

(a): $L$ is a fixed point, or

(b): $L \subset \partial \mathbb{R}^n_+$ and there exist indices $k, l, 1 \leq k < k + 1 \leq l - 1 < l \leq n$ such that

$$\lim_{n \to \infty} (S^n x)_i = p_i$$
for $1 \leq i \leq k$ and $l \leq i \leq n$. For $j = k + 1$ and $j = l - 1$, we have

$$\lim\inf_{n \to \infty} (S^n x)_j < \lim\sup_{n \to \infty} (S^n x)_j.$$  

In this case $L$ contains a cycle of fixed points:

$$E_1 \to E_2 \to \cdots \to E_r \to E_1, \quad r \geq 1.$$  

In particular, if the bounded solution $x(t)$ is weakly persistent, then $x(t)$ is asymptotic to a positive $T$-periodic solution.

The corresponding result holds for every boundary subsystem of (1.1).

**Theorem 2.** If all assumptions of Theorem 1 are satisfied and $n \leq 4$, then any bounded solution of (1.1) is asymptotic to a $T$-periodic solution of (1.1).

**Remark 1:** Theorems 1 and 2 generalize the results of Freedman and Smith [9] concerning autonomous systems to the periodic tridiagonal Kolmogorov systems modeling the competitive-cooperative ecological interactions. Our basic idea of the proofs is similar with that of [9]. However, it is more difficult to construct the cycles in our discrete-time cases and more techniques are needed to show the convergence of the trajectories in lower dimensional cases. In a certain sense, our results here are also natural generalizations of the results by de Mottoni and Schiaffino [18] and Hale and Somolinos [13], who proved that all bounded solutions of two-dimensional $T$-periodic competitive or cooperative systems are asymptotic to $T$-periodic solutions. See also Smith[21], Wang and Jiang[22, 23] for extensions of this work.

**Remark 2:** Recently, Fiedler and Gedeon [7] cleverly constructed a real Lyapunov function on the entire $\mathbb{R}^n_+$ and, by appealing to Lasalle’s principle, proved that all the bounded orbits approach to the set of equilibria in autonomous tridiagonal competitive-cooperative Kolmogorov system. A careful examination of the proof in [7] suggests that their construction of Lyapunov function strongly depends on the independence of the vector field on the time $t$ and that the characteristic of the equilibrium: $\dot{x}_i = 0$ for all $i = 1, 2, \cdots, n$. For the time-periodic systems, although Lasalle’s principle still hold, the Lyapunov function in [7] cannot work anymore. In our opinion, it’s very difficult to find a new Lyapunov function for the $T$-periodic systems such that every bounded orbit is asymptotic to the set of $T$-periodic solutions. We only solved the problem for $n \leq 4$ in Theorem 2 by the
theory of monotone dynamical systems. The corresponding problem for the case 
$n > 4$ remains open.

2. Notations and Preliminary Results. We will reserve the letter $n, n \geq 2$, for 
the dimension $n$ of the space $\mathbb{R}^n$ and set $N = \{1, 2, \cdots, n\}$. Let $i, j \in N$ and $i \leq j$, 
define an integer segment $[i, j] = \{i, i+1, \cdots, j\} \subset N$. For each nonempty subset $I \subset N$ we may express $I$ as an ordered union of segments in a unique way,

$$I = [i_1, j_1] \cup [i_2, j_2] \cup \cdots [i_r, j_r],$$

where 

$$i_1 \leq j_1 < i_2 \leq j_2 < \cdots < i_{r-1} \leq j_{r-1} < i_r \leq j_r,$$

and

$$j_i + 1 \neq i_{i+1}.$$

Hereafter, when referring to the representation of a subset $I$ of $N$ as an ordered union of segment, we shall always mean that representation described above.

For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n_+$, let $\text{Supp } x = \{i \in N : x_i > 0\}$. From the form of (1.1) it is clear that, given $x(0) \in \mathbb{R}^n_+$, the corresponding solution $x(t)$ satisfies $\text{Supp } x(t) = \text{Supp } x(0)$ for all $t \in \mathbb{R}$. Let $I \subset N$ contain $q$ elements, we write 

$$x_I = P_I x,$$

where $P_I : \mathbb{R}^n \rightarrow \mathbb{R}^q$ is a projection. Then $x_I = (x_{I_1}, x_{I_2}, \cdots, x_{I_r})$, 
where $I = I_1 \cup \cdots \cup I_r$ and each $I_l(l = 1, \cdots, r)$ is a segment. For any such $I$ and $q < n$, we also define a map $Q_I : \mathbb{R}^q \rightarrow \mathbb{R}^n$ by $Q_I y = \sum_{l=1}^q y e_{j(l)}$, where 

$$\{e_1, \cdots, e_n\}$$

is the standard basis for $\mathbb{R}^n$ and $j(l)$ is the $l$-th element of the set $I$, on ordering the elements of $I$ from smallest to largest.

Now, let $I = \text{Supp } x(0)$ have $m$ elements. Then $x_I(t) \equiv (x(t))_I \in \text{Int } \mathbb{R}^m_+$ is a solution of

$$\frac{d}{dt} x_I(t) = F_I(t, x_I(t))$$

(2.1)

where $F_I(t, x_I) = P_I \circ F(t, Q_I x_I)$. If $I$ is expressed as an ordered union of segments 
$I = I_1 \cup \cdots \cup I_r$, then (2.1) can be viewed as $r$ uncoupled subsystems $x_I(t) = (x_{I_1}(t), x_{I_2}(t), \cdots, x_{I_r}(t))$ where

$$\frac{d}{dt} x_{I_l}(t) = F_{I_l}(t, x_{I_l}(t)), \quad 1 \leq l \leq r.$$ 

(2.11)

It is easy to see that each of subsystems (2.11) is of the form (1.1) and that $x_{I_l}(t) \in 
\text{Int } \mathbb{R}^n_{m_l}$, $m_l$ is the cardinality of $I_l$. 

Following [20], we can define a unique continuous function \( \sigma : \Lambda \to \{0, 1, 2, \cdots, n-1\} \) on
\[
\Lambda = \{ v \in \mathbb{R}^n : v_1 \neq 0, v_n \neq 0 \text{ and if } v_i = 0 \text{ for some } i, 2 \leq i \leq n-1, \text{ then } v_{i-1}v_{i+1} < 0 \}
\]
such that for any \( v \in \Lambda_0 = \{ v \in \mathbb{R}^n : v_i \neq 0 \text{ for all } 1 \leq i \leq n \} \),
\[
\sigma(v) = \# \{ i : v_i v_{i+1} < 0 \},
\]
where \( \# \) denotes the cardinality of the set. Clearly, \( \Lambda \) is open and dense in \( \mathbb{R}^n \).

Consider the linear system
\[
\begin{align*}
\frac{dx_1}{dt} &= a_{11}(t)x_1 + a_{12}(t)x_2 \\
\frac{dx_j}{dt} &= a_{jj-1}(t)x_{j-1} + a_{jj}(t)x_j + a_{jj+1}(t)x_{j+1}, & 2 \leq j \leq n-1, \\
\frac{dx_n}{dt} &= a_{n-1}(t)x_{n-1} + a_{nn}(t)x_n,
\end{align*}
\]
where the functions \( a_{ij}(\cdot) \) are continuous and defined on a nontrivial interval \( J \) and
\[
\begin{align*}
a_{jj+1}(t) &> 0 \quad \text{on } J, \quad 1 \leq j \leq n-1, \\
a_{jj-1}(t) &> 0 \quad \text{on } J, \quad 2 \leq j \leq n.
\end{align*}
\]

**Lemma 2.1** ([20], Proposition 1.2). If \( x(t) \) is a nontrivial solution of (2.2)-(2.3) on \( J \) then

(i): \( x(t) \in \Lambda \) except for isolated values of \( t \).

(ii): If \( x(s) \notin \Lambda \) for some \( s \in \text{Int} J \) then \( \sigma(x(s+)) < \sigma(x(s-)) \).

**Lemma 2.2** ([20], Theorem 1.3). Let \( X(t) \) denote the fundamental matrix solution of the linear \( T \)-periodic system (2.2)-(2.3) satisfying \( X(0) = I \), where \( I \) is the identity matrix. Then (2.2)-(2.3) has \( n \) distinct positive Floquet multipliers \( \alpha_1, \cdots, \alpha_n \) satisfying
\[
\alpha_1 > \alpha_2 > \cdots > \alpha_{n-1} > \alpha_n > 0.
\]

**Remark 3:** Lemma 2.2 can also hold if the assumption (2.3) is replaced by
\[
a_{jj+1}(t) \cdot a_{jj+1}(t) > 0 \quad \text{on } J, \quad 1 \leq j \leq n-1. \tag{2.3'}
\]
Indeed, for each \( j \in [1, n-1] \), there is a \( \delta_j \in \{-1, +1\} \) such that \( \delta_j a_{jj+1}(t) > 0 \), \( \delta_j a_{jj+1}(t) > 0 \). We let \( \hat{x}_j = \mu_j x_j, \mu_j \in \{\pm 1\}, 1 \leq i \leq n \), with \( \mu_1 = 1, \mu_{j+1} = \delta_j \mu_j \). Then the system (2.2)-(2.3') transforms into a new system of (2.2)-(2.3). Let \( Y(t) \) is the fundamental matrix solution of the linear \( T \)-periodic system (2.2)-(2.3') satisfying \( Y(0) = I \). Then \( Y(t) = BX(t)B^{-1} \), where \( B = \text{diag}(1, \mu_2, \cdots, \mu_n) \).

Hence, the linear \( T \)-periodic system (2.2)-(2.3') has the same Floquet multipliers.
Let by the main result in [20], we only need to prove the lemma under the distinct bounded positive solutions of (1.1)-(1.3) for $t \geq kT$, $k \geq 1$. Putting $t_k \geq m$ we assume that $(u(t) - u(t + T))_1 \cdot (u(t) - u(t + T))_n \neq 0$ for all sufficiently large $t$. Similar results hold for the periodic systems (2.1).

Proof. $x(t) = y(t) - \dot{y}(t)$ satisfies the linear system (2.2)-(2.3)', where

$$a_{ij}(t) = \int_0^1 u_i(s, t) \frac{\partial f}{\partial x_j}(t, u_{i-1}(s, t), u_i(s, t), u_{i+1}(s, t)) ds$$

with $u_j(s, t) = s y_j(t) + (1 - s) \dot{y}_j(t), j = i - 1, i, i + 1$. Then, by Remark 3, $z(t) = Bx(t)$ satisfies the linear system (2.2)-(2.3). It follows from Lemma 2.1 and the definition of $\Lambda$ that $z(t)_1 \cdot z(t)_n \neq 0$ for all large $t > 0$. Hence $x(t)_1 \cdot x(t)_n \neq 0$ for all large $t > 0$.

3. Proofs of the Main Results.

Lemma 3.1. Let $x(t), (t \geq 0)$, be the bounded, positive solution of (1.1)-(1.3). Let $L = \omega(x; S)$ be the $\omega$-limit set of $x$. Then either

(a) $L$ is a fixed point, or
(b) $L \subset \partial \mathbb{R}^n_+$ and there exist indices $k, l, 1 \leq k < k + 1 \leq l - 1 \leq l \leq n$ such that

$$\lim_{n \to \infty} (S^n x)_i = p_i$$

for $1 \leq i \leq k$ and $l \leq i \leq n$. For $j = k + 1$ and $j = l - 1$, we have

$$\liminf_{n \to \infty} (S^n x)_j < \limsup_{n \to \infty} (S^n x)_j.$$  

The corresponding results hold for periodic system (2.1).

Proof. By the main result in [20], we only need to prove the lemma under the assumption that $L \cap \partial \mathbb{R}^n_+ \neq \emptyset$. Now suppose that $L$ is not a fixed point.

Before we go further into the proof, we should note that $z_1 \equiv p_1$ and $z_n \equiv p_n$ for any $z \in L$. Indeed, since $L$ is not a fixed point, $x(t)$ and $x(t + T)$ are distinct bounded positive solutions of (1.1)-(1.3) for $t \geq 0$. Lemma 2.3 implies that $(x(t) - x(t + T))_1 \cdot (x(t) - x(t + T))_n \neq 0$ for all large $t$. Without loss of generality, we assume that $u(t)_1 < u(t + T)_1$ for all $t \geq mT$, where $m$ is some positive integer. Putting $t = kT, k \geq m$ into this inequality, we find that $(S^k x)_1 < (S^{k+1} x)_1$ for all $k \geq m$. In particular, $\lim_k (S^k x)_1 = p_1$ and $\lim_k (S^k x)_n = p_n$.

We now claim that $L \subset \partial \mathbb{R}^n_+$. If not, then $L \cap \text{Int} \mathbb{R}^n_+ \neq \emptyset$. Firstly, we shall show that $L \cap \text{Int} \mathbb{R}^n_+ \subset \text{Fix}(S)$. To end this, let $u \in L \cap \text{Int} \mathbb{R}^n_+$ such that $u \neq
\( Su = u(T) \). Then, we can repeat the step in the previous paragraph to obtain that 
\((S^m u)_1 < (S^{m+1} u)_1\) for some \( m \in \mathbb{N} \). Note that \( S^m u, S^{m+1} u \in L \), a contradiction.

Next, we will show that \( L \cap \text{Int} \mathbb{R}_+^n \) is indeed a singleton. Otherwise, there exist \( u, v \in L \cap \text{Int} \mathbb{R}_+^n \subset \text{Fix}(S) \), then \( u(t), v(t) \) are distinct bounded positive solutions for \( t \geq 0 \), which implies that \( (S^k u)_1 = u(kT)_1 \neq v(kT)_1 = (S^k v)_1 \) for some \( k \in \mathbb{N} \). Note that \( S^k u, S^k v \in L \), a contradiction. Now we have proved that \( L \cap \text{Int} \mathbb{R}_+^n = \{ p \} \) is a singleton.

Since \( L \cap \partial \mathbb{R}_+^n \neq \emptyset \), \( p \) is an isolated fixed point with respect to \( L \), which contradicts the internal chain transitivity of \( L \) (see, [12, Lemma 2.1]). Thus, we have proved the claim that \( L \subset \partial \mathbb{R}_+^n \).

Let \( k + 1, k \geq 1 \), be the smallest integer such that \( \liminf_{n \to \infty} (S^n x)_{k+1} < \limsup_{n \to \infty} (S^n x)_{k+1} \). Similarly, we can obtain \( l - 1, k + 2 \leq l \leq n \), which is the largest integer such that \( \liminf_{n \to \infty} (S^n x)_{l-1} < \limsup_{n \to \infty} (S^n x)_{l-1} \). This completes the proof.

**Remark 4:** The last statement in Lemma 3.1 implies that Lemma 3.1 holds, as stated, without the assumption that \( x(t) \) is a positive. Indeed, let \( I = \text{Supp} x(t) \subset N \) be represented as an ordered union of segments, \( I = I_1 \cup \cdots \cup I_r \). Then Lemma 3.1 applies to each \((2.1_t)\), together with the invariance of Kolmogorov systems, we can obtain that Lemma 3.1 holds without the assumption that \( x(t) \) is a positive.

**Lemma 3.2.** Let \( L = \omega(x; S) \) be the compact \( \omega \)-limit set of \( x \in \mathbb{R}_+^n \). Then \( L \cap \text{Fix}(S) \neq \emptyset \).

**Proof.** Let the index set \( I \subset N \) satisfy \( \text{Supp} y = I \) for some \( y \in L \) and we can also choose such \( I \) that \( \{ z \in L : \text{Supp} z \subset I \} = \emptyset \). We will use the minimal property of \( I \) to prove this lemma. Indeed, choose any \( z \in L \) such that \( \text{Supp} z = I \). If \( z \in \text{Fix}(S) \), then we have done. Suppose that \( z \notin \text{Fix}(S) \) and \( I = I_1 \cup \cdots \cup I_r \), where \( I_l, l = 1, \cdots, r, \) are the integer segments. Then Lemma 3.1 can apply to \( z_{I_l}(t) \) for each \( l = 1, \cdots, r \), and hence, together with Remark 4, we obtain that either \( S^n z \to p \in \text{Fix}(S) \), or else there exists some \( w \in \omega(z; S) \subset L \) such that \( \text{Supp} w \subset I \). It follows from the minimal property of \( I \) that the second case cannot happen. Thus \( p \in \text{Fix}(S) \cap L \neq \emptyset \).

The following Lemma gives an explicit expression of the linearization of \( S \) at a fixed point on \( \partial \mathbb{R}_+^n \). In order to state this Lemma, we introduce some notations. Let
$w \in \partial \mathbb{R}^n_+$ be a fixed point of the Poincaré map $S$. We write $DS(w) = \left( \frac{\partial S_i}{\partial x_j}(w) \right)_{n \times n}$ as the linearization of $S$ at $w$. Let $I = \text{Supp} w$, $w_I = P_I w$ and $I = I_1 \cup \cdots \cup I_r$ be the representation of $I$ as an ordered union of segments. For each $I_l$, $w_{I_l}$ is the positive fixed point of the Poincaré map, denoted by $S_{I_l}$, associated with periodic system (2.1). Furthermore, it is also easy to see that the linearization $DS_{I_l}(w_{I_l})$ of $S_{I_l}$ at $w_{I_l}$ is equal to $\left( \frac{\partial S_i}{\partial x_j}(w) \right)_{I_l \times I_l}$, $l = 1, \ldots, r$.

**Lemma 3.3.** Let $w$ be a fixed point on $\partial \mathbb{R}^n_+$, $I = \text{Supp} w$ and $I = I_1 \cup \cdots \cup I_r$ be the representation of $I$ as an ordered union of segments. Then the eigenvalues of $DS(w)$ consist of $\frac{\partial S_i}{\partial x_j}(w)$, $i \notin I$, together with the eigenvalues of $DS_{I_l}(w_{I_l})$, $l = 1, \ldots, r$. In particular, they are positive real numbers.

**Proof.** Firstly, it follows from the invariance of the Kolmogorov system that $\frac{\partial S_i}{\partial x_j}(w) = 0$ for any $i \notin I$, $j \neq i$.

Secondly, for any $i \in I_l$, $j \in I_m$ with $1 \leq l \neq m \leq r$, since (2.1) and (2.1.m) are decoupled, we have $(S(w + te_j))_i = (S(w))_i$, which implies that $\frac{\partial S_i}{\partial x_j}(w) = 0$.

Now we can choose a permutation matrix $P$ such that

$$P^{-1}DS(w)P = \begin{bmatrix}
DS_{I_1}(w_{I_1}) & 0 & * \\
0 & DS_{I_r}(w_{I_r}) & 0 \\
0 & & \frac{\partial S_{i_1}}{\partial x_{i_1}} \bigg|_w & \cdots & 0 \\
& & & & \frac{\partial S_{i_s}}{\partial x_{i_s}} \bigg|_w
\end{bmatrix},$$

where $i_m \notin I$, $m = 1, \ldots, s$, $s = n - \# I$. Here $\#$ denotes the cardinality of $I$.

Now we have proved the first statement of this Lemma. Given any $l = 1, 2, \ldots, r$, applying Lemma 2.2 to the variational equations along the positive $T$-periodic solution $w_{I_l}(t)$ of the $T$-periodic system (2.1), we obtain that $DS_{I_l}(w_{I_l})$ has different positive eigenvalues, which implies the second statement of this Lemma.

**Remark 5:** Under the assumption (2) in Theorem 1, Lemma 3.3 implies that all the boundary fixed points are hyperbolic, which is one of the essential hypothesis for the results in [16] on the global trivial dynamics in general periodic Kolmogorov systems which are cooperative or with limited competition. The authors also believe
that the hyperbolic hypothesis could be the weakest hypothesis which can guarantee the global trivial dynamics in general periodic Kolmogorov systems which are cooperative or with limited competition.

**Proof of Theorem 1.** The proof goes by induction on \(n, n \geq 2\). If \(n = 2\) the result follows immediately from the result of Hale and Somolinos [13].

From now on we assume \(n > 2\). The induction hypothesis is that Theorem 1 holds for systems in \(\mathbb{R}^m\) if \(m < n\). Let \(L = \omega(x; S)\) be the compact \(\omega\)-limit set of \(x \in \text{Int}\mathbb{R}_+^m\). If (a) does not hold, then by Lemma 3.1, the first assertions of (b) in Theorem 1 hold. We need only prove that \(L\) contains a cycle of fixed points.

Let \(I = N\setminus(\{1, k\} \cup \{l, n\})\), then for any \(y \in L\), write \(y = (p_1, \cdots, p_k; y_1; p_1, \cdots, p_n)\). It follows from the invariance of \(L\) that \(y(T) \in L\), which implies that \(p_i(t) := (y(t))_i\) is a \(T\)-periodic function for each \(i \in \{1, k\} \cup \{l, n\}\). For simplicity, we write \(p_{[1,k]}(t) = (p_1(t), \cdots, p_k(t))\) and \(p_{[l,n]}(t) = (p_l(t), \cdots, p_n(t))\) for \(t \in [0, T]\). Define \(\dot{S}y_I = \varphi(T, 0, y_I)\), where \(\varphi(t, 0, y_I), t \geq 0\), is the solution of the \(T\)-periodic system of ODEs

\[
\frac{dz_I(t)}{dt} = F_I(t, p_{[1,k]}(t), z(t), p_{[l,n]}(t)) := G_I(t, z(t)),
\]

with the initial value \(y_I\). It is easy to see that

\[
S^jy \equiv (p_{[1,k]}; \dot{S}^jy_I; p_{[l,n]})
\]

for all \(j \in \mathbb{N}\) and \(y \in L\). Furthermore, system (3.1) satisfies all the assumptions of system (1.1).

If there exists a \(y \in L\) such that \(\dot{S}^jy_I\) does not converge, then it follows from the induction hypothesis that \(\omega(y_I; \dot{S})\) contains a cycle of fixed points of \(\dot{S}\). Then, by (3.2), \(\omega(y; S)\) contains a cycle of fixed points of \(S\). Note that \(\omega(y; S) \subset L\), we have done. Therefore, we hereafter assume that \(\dot{S}^jy_I\) converges for every \(y \in L\), and hence, \(S^jy\) converges for every \(y \in L\). Before we go further into the proof, we should note that the assumption (2) in Theorem 1, together with Lemma 3.3, implies that every fixed point on \(\partial\mathbb{R}_+^n\) is hyperbolic. By Lemma 3.2, we can find an \(E_1 \in L \cap \text{Fix}(S)\). Since \(L \neq \{E_1\}\) and \(E_1\) is hyperbolic, the Butler-McGehee Lemma [4, 12] implies that there is a \(y \in L \cap W^u(E_1)\) with \(y \neq E_1\), where \(W^u(E_1)\) denotes the unstable manifold of \(E_1\). Let \(S^jy \to E_2\) as \(j \to \infty\). Since \(L \neq \{E_2\}\) and \(E_2\) is hyperbolic, we again obtain a \(\hat{y} \in L \cap W^u(E_2)\) by the Butler-McGehee
Lemma. Repeating the steps and noticing the assumption (1) in Theorem 1, we obtain that $L$ contains a cycle of fixed points of $S$. □

**Proof of Theorem 2.** If $n = 2$, then (1.1) is a planar periodic competitive or cooperative system and therefore all bounded solutions converge to some $T$-periodic solution (see [13]). If $n = 3$, we only need to observe that case (b) in Theorem 1 would imply the existence of a cycle of fixed points of the Poincaré map associated with the one dimensional $T$-periodic subsystem $\dot{x}_2 = x_2 f_2(t, p_1(t), x_2, x_3)$, which is impossible. Finally, consider that case $n = 4$. It should be clear from the reasoning in the case $n = 3$ that case (b) can be ruled out except possibly when $x(t)$ is positive and $k = 1, l = 4$. Then, in this case, the dynamics of $S$ on $L$ is described by the dynamics of the Poincaré map $\hat{S}$ associated with the following two-dimensional $T$-periodic competitive or cooperative subsystem

$$
\dot{x}_2 = x_2 f_3(t, p_1(t), x_2, x_3) := x_2 g_2(t, x_2, x_3), \\
\dot{x}_3 = x_3 f_3(t, x_2, x_3, p_4(t)) := x_3 g_3(t, x_2, x_3).
$$

(3.3)

From now on, we focus on system (3.3). According to case (b), (3.3) must possess a cycle $E_1 \to E_2 \to \cdots \to E_r \to E_1$ of fixed points of $\hat{S}$. Without loss of generality, we assume that (3.3) is cooperative. Furthermore, it follows from hypothesis (2) and the proof of Lemma 3.3 that each $E_i$ is a saddle point for system (3.3).

Remember that we are focusing on (3.3), so every statement below is on the system (3.3). We first show that none of $E_i$ can be positive. If not, let, say $E_1$, be a positive one, then it follows from the Perron-Frobenius Theorem that, for any $y \in W^u(E_1) \cap \text{Int} \mathbb{R}^2_+$, there exists some $k_0 \in \mathbb{N}$ such that either $\hat{S}^{-(k+1)}y < \hat{S}^{-k}y$ for all $k \geq k_0$, or else $\hat{S}^{-k}y < \hat{S}^{-(k+1)}y$ for all $k \geq k_0$. Here we write “$<$” in case the order relation holds component-wise in $\mathbb{R}^2$. Hence, by the strong monotonicity of $\hat{S}$ (see [5, 19]) in $\text{Int} \mathbb{R}^2_+$, the connecting orbit from $E_1$ to $E_2$ is either monotone increasing or monotone decreasing with respect to the order relation $\prec$. Furthermore, it is easy to see that $E_2$ attracts all points in the order open interval between $E_1$ and $E_2$, which is an open set, This contradicts that $E_2$ is a saddle point. Therefore, all $E_i$ must belong to $\partial \mathbb{R}^2_+$.

Suppose that $E_i = (0, 0)$ for some $i$, then necessarily $E_{i-1}$ and $E_{i+1}$ have exactly one positive component. In other words, there must be at least one fixed point with exactly one positive component. Suppose that $E_1 = (u, 0)$ in $x_2$-axis, where $u > 0$. 

A similar argument applies in the other case. Then $D\hat{S}(E_1)$ has the form
\[
\begin{pmatrix}
  a & b \\
  0 & c
\end{pmatrix}
\]
where $b \geq 0$. Furthermore, since $\frac{\partial g_2}{\partial x_3} > 0$, it is easy to conclude that $b > 0$. Now if $a < 1 < c$, then the eigenvector corresponding to $c$ is $w = (\frac{c-a}{b}, 1) > 0$. Then, as in the previous paragraph, we have $E_2 \in \text{Int} \mathbb{R}^2_+$, a contradiction. Hence, $c < 1 < a$ and consequently, by the invariance of the $x_2$-axis, the unstable manifold of $E_1$ is on $x_2$-axis, and hence, $E_2$ must belong to $x_2$-axis. Suppose that $E_2 \neq (0, 0)$. Then one can repeat as above to obtain that $E_3 \in \text{Int} \mathbb{R}^2_+$, a contradiction. Therefore, $E_2 = (0, 0)$, which implies that $E_3 = (0, v)$ with its global stable manifold being $x_3$-axis. Then $E_4 \in \text{Int} \mathbb{R}^2_+$, a contradiction. Therefore, case (b) of Theorem 1 cannot hold. We have completed the proof. □

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