LECTURE 3: SMOOTH VECTOR FIELDS

1. TANGENT AND COTANGENT VECTORS

Let $M$ be an $n$-dimensional smooth manifold.

**Definition 1.1.** A tangent vector at a point $p \in M$ is a linear map $X_p : C^\infty(M) \to \mathbb{R}$ satisfying the Leibnitz law

$$X_p(fg) = f(p)X_p(g) + X_p(f)g(p)$$

It is easy to see that the set of all tangent vectors of $M$ at $p$ is a vector space. We will call it the tangent space of $M$ at $p$, and denote it by $T_pM$. Its dual space is called the cotangent space of $M$ at $p$, and is denoted by $T^*_pM$.

**Fact:** Both $T_pM$ and $T^*_pM$ are $n$-dimensional vector spaces.

Locally let $\{\varphi, U, V\}$ be a chart around $p$ with $\varphi(p) = 0$. Then the maps

$$\partial_i : C^\infty(U) \to \mathbb{R}, \quad f \mapsto \frac{\partial f \circ \varphi^{-1}}{\partial r^i}(0), \quad i = 1, 2, \ldots, n$$

are tangent vectors at $p$. One can check that they are linearly independent and form a basis of $T_pM$. To describe the cotangent space $T^*_pM$, we need to introduce

**Definition 1.2.** Let $\varphi : M \to N$ be a smooth map. Then for each $p \in M$, the differential of $\varphi$ is the linear map $d\varphi_p : T_pM \to T_{\varphi(p)}N$ defined by

$$d\varphi_p(X_p)(f) = X_p(f \circ \varphi)$$

for all $X_p \in T_pM$ and all $g \in C^\infty(N)$.

In the special case $f : M \to \mathbb{R}$ is a smooth function, we can identify $T_{f(p)}\mathbb{R}$ with $\mathbb{R}$. Then we have

$$X_p(f) = df_p(X_p).$$

In other words, $df_p \in T^*_pM$ is a cotangent vector at $p$. Locally in a coordinate chart $\{\varphi, U, V\}$ the dual basis of $\{\partial_1, \ldots, \partial_n\}$ in $T^*_pM$ is $\{dx_1, \ldots, dx_n\}$, and we have

$$df_p = (\partial_1 f)dx_1 + \cdots + (\partial_n f)dx_n.$$

We now set $TM = \bigcup_p T_pM$, the disjoint union of all tangent vectors. It is called the tangent bundle of $M$. There is a natural projection map

$$\pi : TM \to M, \ (p, X_p) \mapsto p.$$

Obviously we have $T_pM = \pi^{-1}(p)$.

**Proposition 1.3.** $TM$ is a smooth manifold of dimension $2n$, and $\pi$ is smooth.
**Sketch of proof.** Suppose \( \{ \varphi, U, V \} \) is a chart of \( M \), then \( \{ T\varphi, \pi^{-1}(U), V \times \mathbb{R}^n \} \) is a chart of \( TM \), where the trivialization map \( T\varphi : \pi^{-1}(U) \to V \times \mathbb{R}^n \) is given by
\[
T\varphi(p, X_p) = (\varphi(p), d\varphi_p(X_p)).
\]
\( \square \)

Similarly \( T^*M = \bigcup_p T^*_p M \), the **cotangent bundle** of \( M \), is also a smooth manifold of dimension \( 2n \), with the natural projection map a smooth map.

## 2. Vector Fields

**Definition 2.1.** A **vector field** on \( M \) is a section of the tangent bundle \( TM \), i.e., a map \( X : M \to TM \) such that \( \pi \circ X = Id_M \). It is smooth if for any \( f \in C^\infty(M) \), the function
\[
Xf(p) = X_p(f)
\]
is a smooth function on \( M \). The set of all smooth vector fields on \( M \) is denoted by \( \Gamma^\infty(TM) \).

From now on when we say “vector fields”, we always mean smooth vector fields.

We can think of a vector field \( X \) as a map
\[
X : C^\infty(M) \to C^\infty(M), \quad f \mapsto Xf.
\]
Locally in a chart \( \{ \varphi, U, V \} \) any smooth vector field can be represented by
\[
X = X^1 \partial_1 + \cdots + X^n \partial_n =: X^i \partial_i,
\]
where \( X^i \)'s are smooth functions on \( U \). So \( X \) is actually a 1\(^{st}\) order differential operator.

Now consider two smooth vector fields \( X \) and \( Y \) on \( M \). Locally we can write
\[
X = X^i \partial_i \quad \text{and} \quad Y = Y^i \partial_i.
\]
Using the fact \( \partial_i \partial_j f = \partial_j \partial_i f \), a direct computation yields
\[
[X,Y]f := X(Yf) - Y(Xf) = X^i \partial_i(Y^j \partial_j f) - Y^i \partial_i(X^j \partial_j f) = (X^i \partial_i Y^j - Y^i \partial_i X^j) \partial_j f.
\]
So the commutator \([X,Y] = XY - YX\) is again a smooth vector field on \( M \).

**Definition 2.2.** We call the commutator \([X,Y]\) the **Lie bracket** of \( X \) and \( Y \).

It is easy to see that the Lie bracket \([X,Y]\) satisfies the following properties:

**Proposition 2.3.** (a)(skew symmetry)\([X,Y] = -[Y,X]\).
(b)(\( \mathbb{R} \) linearity)\([aX + bY,Z] = a[X,Z] + b[Y,Z]\).
(c)(Jacobi identity)\([X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0\).

**Proof.** (a), (b) are obvious, and (c) follows from direct computation. \( \square \)

A vector space together with a binary operation \([\cdot,\cdot]\) satisfying these three conditions is called a **Lie algebra**. So \( \Gamma^\infty(TM) \) together with the Lie bracket operation above is an (infinite dimensional) Lie algebra.
3. Geometry and Dynamical System of Vector Fields

Recall that a smooth curve in a smooth manifold \( M \) is a smooth injective map \( \gamma : I \to M \), where \( I \) is an interval in \( \mathbb{R} \). For any \( a \in I \), the derivative of \( \gamma \) at \( t = a \) gives a tangent vector on \( M \) at \( \gamma(a) \) by \( \frac{d\gamma}{dt}(a) = d\gamma_a(\frac{d}{dt}) \).

**Definition 3.1.** We say that \( \gamma : I \to M \) is an integral curve of a vector field \( X \) if for any \( a \in I \), \( \frac{d\gamma}{dt}(a) = X(\gamma(a)) \).

**Remark.** Locally the equation \( \frac{d\gamma}{dt}(a) = X(\gamma(a)) \) is a system of first order differential equations. By the Picard theorem, locally an integral curve always exists and is unique, and depends on the initial data smoothly.

**Definition 3.2.** A vector field \( X \) on \( M \) is complete if for any \( p \in M \), there is an integral curve \( \gamma : \mathbb{R} \to M \) such that \( \gamma(0) = p \).

As in the case of functions, we can define the support of a vector field by

\[
\text{supp}(X) = \{ p \in M \mid X(p) \neq 0 \}.
\]

One can prove

**Theorem 3.3.** If \( X \) is a compactly supported vector field on \( M \), then it is complete.

In particular, any smooth vector field on a compact manifold is complete. As another example, we will see later that any left-invariant vector field on a Lie group is complete.

Now suppose \( X \) is a complete vector field on \( M \). Then for any \( p \in M \), there is a unique integral curve \( \gamma_p : \mathbb{R} \to M \) such that \( \gamma_p(0) = p \). From this one can, for any \( t \in \mathbb{R} \), define a map

\[
\phi_t : M \to M, \quad p \mapsto \gamma_p(t).
\]

Notice that for any \( p \in M \) and any \( t, s \in \mathbb{R} \), \( \phi_t \circ \phi_s(p) \) and \( \phi_{t+s}(p) \) are both integral curves for \( X \) with initial conditions \( \phi_s(p) \) at \( t = 0 \). By uniqueness, we have

\[
\phi_t \circ \phi_s = \phi_{t+s}.
\]

Since \( \phi_0 = \text{Id} \), we conclude that \( \phi_t : M \to M \) is bijective and \( \phi_t^{-1} = \phi_{-t} \).

**Definition 3.4.** We will call \( \Phi : \mathbb{R} \times M \to M, (t, p) \mapsto \phi_t(p) \) the flow of \( X \).

One can show that the map \( \Phi \) is smooth. It follows that \( \phi_t \) are smooth maps. In other words, the family of maps \( \{ \phi_t \} \) is a one-parameter group of diffeomorphisms of \( M \).

**Example:** The vector field \( \partial_\theta \) is complete on \( S^1 \), and the flow generated by \( \partial_\theta \) is

\[
\Phi : \mathbb{R} \times S^1 \to S^1, \quad (t, e^{i\theta}) \mapsto e^{i(\theta + t)}.
\]

Recall that if \( \varphi : M \to N \) is smooth, then it induces a “pull-back” map

\[
\varphi^* : C^\infty(N) \to C^\infty(M), \quad f \mapsto \varphi^* f = f \circ \varphi,
\]
and if $\varphi$ is a diffeomorphism, then it also induces a “push-forward” map

$$\varphi_* : \Gamma^\infty(TM) \to \Gamma^\infty(TN), \quad X \mapsto \varphi_* X,$$

where $(\varphi_* X)(q) = (d\varphi_{\varphi^{-1}(q)})(X_{\varphi^{-1}(q)})$.

Finally let’s give the dynamical system description of $X$ and $[X,Y]$:

**Theorem 3.5.** Suppose $X \in \Gamma^\infty(TM)$ is complete. Then

1. For any $f \in C^\infty(M)$, $Xf = \frac{d}{dt}\big|_{t=0} \varphi^*_t f$.
2. For any $Y \in \Gamma^\infty(TM)$, $[X,Y] = \frac{d}{dt}\big|_{t=0} (\varphi_{-t})_* Y_{\varphi t}$.

**Remark.** Suppose $X$ is a complete vector field on $M$. Let $\{\phi_t\}$ be the family of diffeomorphisms generated by $X$. Sometimes we will denote $\phi_t = \exp(tX)$ to emphasize the $X$-dependence, and so that the group law $\phi_t \circ \phi_s = \phi_{t+s}$ reads

$$\exp(tX) \exp(sX) = \exp((s+t)X).$$

Note that in general $\exp(tX) \exp(sY) \neq \exp(sY) \exp(tX)$. In fact, we have

**Theorem 3.6.** Let $X, Y$ be complete vector fields on $M$. Then $\exp(tX)$ commutes with $\exp(sY)$ for all $t$ and $s$ if and only if $[X,Y] = 0$. 