LECTURE 3: SMOOTH VECTOR FIELDS

1. Tangent and Cotangent Vectors

Let M be an n-dimensional smooth manifold.

Definition 1.1. A *tangent vector* at a point $p \in M$ is a linear map $X_p : C^\infty(M) \to \mathbb{R}$ satisfying the Leibnitz law

$$X_p(fg) = f(p)X_p(g) + X_p(f)g(p) \tag{1}$$

It is easy to see that the set of all tangent vectors of M at p is a vector space. We will call it the *tangent space* of M at p, and denote it by T_pM. Its dual space is called the *cotangent space* of M at p, and is denoted by T_p^*M.

Fact: Both T_pM and T_p^*M are n-dimensional vector spaces.

Locally let $\{\varphi, U, V\}$ be a chart around p with $\varphi(p) = 0$. Then the maps

$$\partial_i : C^\infty(U) \to \mathbb{R}, \quad f \mapsto \frac{\partial f \circ \varphi^{-1}}{\partial x^i}(0), \quad i = 1, 2, \cdots, n$$

are tangent vectors at p. One can check that they are linearly independent and form a basis of T_pM. To describe the cotangent space T_p^*M, we need to introduce

Definition 1.2. Let $\varphi : M \to N$ be a smooth map. Then for each $p \in M$, the *differential* of φ is the linear map $d\varphi_p : T_pM \to T_{\varphi(p)}N$ defined by

$$d\varphi_p(X_p)(f) = X_p(f \circ \varphi)$$

for all $X_p \in T_pM$ and all $g \in C^\infty(N)$.

In the special case $f : M \to \mathbb{R}$ is a smooth function, we can identify $T_{f(p)}\mathbb{R}$ with $\mathbb{R}$. Then we have

$$X_p(f) = df_p(X_p).$$

In other words, $df_p \in T_p^*M$ is a *cotangent vector* at p. Locally in a coordinate chart $\{\varphi, U, V\}$ the dual basis of $\{\partial_1, \cdots, \partial_n\}$ in T_p^*M is $\{dx_1, \cdots, dx_n\}$, and we have

$$df_p = (\partial_1 f)dx_1 + \cdots + (\partial_n f)dx_n.$$

We now set $TM = \cup_p T_pM$, the disjoint union of all tangent vectors. It is called the *tangent bundle* of M. There is a natural projection map

$$\pi : TM \to M, \ (p, X_p) \mapsto p.$$

Obviously we have $T_pM = \pi^{-1}(p)$.

Proposition 1.3. *TM is a smooth manifold of dimension $2n$, and π is smooth.*

Sketch of proof. Suppose $\{\varphi, U, V\}$ is a chart of M, then $\{T\varphi, \pi^{-1}(U), V \times \mathbb{R}^n\}$ is a chart of TM, where the trivialization map $T\varphi : \pi^{-1}(U) \to V \times \mathbb{R}^n$ is given by

$$T\varphi(p, X_p) = (\varphi(p), d\varphi_p(X_p)).$$

□

Similarly $T^*M = \cup_p T_p^*M$, the *cotangent bundle* of M, is also a smooth manifold of dimension $2n$, with the natural projection map a smooth map.

2. Vector Fields

Definition 2.1. A *vector field* on M is a section of the tangent bundle TM, i.e. a map $X : M \to TM$ such that $\pi \circ X = Id_M$. It is *smooth* if for any $f \in C^\infty(M)$, the function

$$Xf(p) = X_p(f)$$

is a smooth function on M. The set of all smooth vector fields on M is denoted by $\Gamma^\infty(TM)$.

From now on when we say "vector fields", we always mean *smooth* vector fields. We can think of a vector field X as a map

$$X : C^\infty(M) \to C^\infty(M), \quad f \mapsto Xf.$$

Locally in a chart $\{\varphi, U, V\}$ any smooth vector field can be represented by

$$X = X^1\partial_1 + \cdots + X^n\partial_n =: X^i\partial_i,$$

where X^i's are smooth functions on U. So X is actually a 1^{st} order differential operator.

Now consider two smooth vector fields X and Y on M. Locally we can write $X = X^i\partial_i$ and $Y = Y^i\partial_i$. Using the fact $\partial_i\partial_j f = \partial_j\partial_i f$, a direct computation yields

$$[X, Y]f := X(Yf) - Y(Xf) = X^i\partial_i(Y^j\partial_j f) - Y^i\partial_i(X^j\partial_j f) = (X^i\partial_i Y^j - Y^i\partial_i X^j)\partial_j f.$$

So the commutator $[X, Y] = XY - YX$ is again a smooth vector field on M.

Definition 2.2. We call the commutator $[X, Y]$ the Lie bracket of X and Y.

It is easy to see that the Lie bracket $[X, Y]$ satisfies the following properties:

Proposition 2.3. (a)(skew symmetry)$[X, Y] = -[Y, X]$.
(b)($\mathbb{R}$ linearity)$[aX + bY, Z] = a[X, Z] + b[Y, Z]$.
(c)(Jacobi identity)$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$.

Proof. (a), (b) are obvious, and (c) follows from direct computation. □

A vector space together with a binary operation $[\cdot, \cdot]$ satisfying these three conditions is called a *Lie algebra*. So $\Gamma^\infty(TM)$ together with the Lie bracket operation above is an (infinite dimensional) Lie algebra.

3. Geometry and Dynamical System of Vector Fields

Recall that a smooth curve in a smooth manifold M is a smooth injective map $\gamma : I \to M$, where I is an interval in $\mathbb{R}$. For any $a \in I$, the derivative of γ at $t = a$ gives a tangent vector on M at $\gamma(a)$ by $\frac{d\gamma}{dt}(a) = d\gamma_a(\frac{d}{dt})$.

Definition 3.1. We say that $\gamma : I \to M$ is an *integral curve* of a vector field X if for any $a \in I$, $\frac{d\gamma}{dt}(a) = X(\gamma(a))$.

Remark. Locally the equation $\frac{d\gamma}{dt}(a) = X(\gamma(a))$ is a system of first order differential equations. By the Picard theorem, locally an integral curve always exists and is unique, and depends on the initial data smoothly.

Definition 3.2. A vector field X on M is *complete* if for any $p \in M$, there is an integral curve $\gamma : \mathbb{R} \to M$ such that $\gamma(0) = p$.

As in the case of functions, we can define the *support* of a vector field by

$$\mathrm{supp}(X) = \overline{\{p \in M \mid X(p) \neq 0\}}.$$

One can prove

Theorem 3.3. *If X is a compactly supported vector field on M, then it is complete.*

In particular, any smooth vector field on a compact manifold is complete. As another example, we will see later that any left-invariant vector field on a Lie group is complete.

Now suppose X is a complete vector field on M. Then for any $p \in M$, there is a unique integral curve $\gamma_p : \mathbb{R} \to M$ such that $\gamma_p(0) = p$. From this one can, for any $t \in \mathbb{R}$, define a map

$$\phi_t : M \to M, \quad p \mapsto \gamma_p(t).$$

Notice that for any $p \in M$ and any $t, s \in \mathbb{R}$, $\phi_t \circ \phi_s(p)$ and $\phi_{t+s}(p)$ are both integral curves for X with initial conditions $\phi_s(p)$ at $t = 0$. By uniqueness, we have

$$\phi_t \circ \phi_s = \phi_{t+s}.$$

Since $\phi_0 = \mathrm{Id}$, we conclude that $\phi_t : M \to M$ is bijective and $\phi_t^{-1} = \phi_{-t}$.

Definition 3.4. We will call $\Phi : \mathbb{R} \times M \to M$, $(t, p) \mapsto \phi_t(p)$ the *flow* of X.

One can show that the map Φ is smooth. It follows that ϕ_t are smooth maps. In other words, the family of maps $\{\phi_t\}$ is a one-parameter group of diffeomorphisms of M.

Example: The vector field ∂_θ is complete on S^1, and the flow generated by ∂_θ is

$$\Phi : \mathbb{R} \times S^1 \to S^1, \quad (t, e^{i\theta}) \mapsto e^{i(\theta+t)}.$$

Recall that if $\varphi : M \to N$ is smooth, then it induces a "pull-back" map

$$\varphi^* : C^\infty(N) \to C^\infty(M), \quad f \mapsto \varphi^* f = f \circ \varphi,$$

and if φ is a diffeomorphism, then it also induces a "push-forward" map

$$\varphi_* : \Gamma^\infty(TM) \to \Gamma^\infty(TN), \quad X \mapsto \varphi_* X, \text{ where } (\varphi_* X)(q) = (d\varphi_{\varphi^{-1}(q)})(X_{\varphi^{-1}(q)}).$$

Finally let's give the dynamical system description of X and $[X, Y]$:

Theorem 3.5. *Suppose $X \in \Gamma^\infty(TM)$ is complete.Then*
(1) *For any $f \in C^\infty(M)$, $Xf = \frac{d}{dt}|_{t=0}\phi_t^* f$.*
(2) *For any $Y \in \Gamma^\infty(TM)$, $[X, Y] = \frac{d}{dt}|_{t=0}(\phi_{-t})_* Y_{\phi_t}$.*

Remark. Suppose X is a complete vector field on M. Let $\{\phi_t\}$ be the family of diffeomorphisms generated by X. Sometimes we will denote $\phi_t = \exp(tX)$ to emphasis the X-dependence, and so that the group law $\phi_t \circ \phi_s = \phi_{t+s}$ reads

$$\exp(tX)\exp(sX) = \exp((s+t)X).$$

Note that in general $\exp(tX)\exp(sY) \neq \exp(sY)\exp(tX)$. In fact, we have

Theorem 3.6. *Let X, Y be complete vector fields on M. Then $\exp(tX)$ commutes with $\exp(sY)$ for all t and s if and only if $[X, Y] = 0$.*