LECTURE 6: THE EXPONENTIAL MAP

1. One-parameter Subgroups

Let $G$ be a Lie group, $X_e \in T_e G$ be a tangent vector at the identity element and $X \in \mathfrak{g}$ the left invariant vector field generated by $X_e$. One can show that (exercise) any left invariant vector field on $G$ is complete. So for any $g \in G$ there is a unique integral curve of $X$ defined on the whole real line $\mathbb{R}$, 

$$\gamma_g : \mathbb{R} \to G,$$

so that $\gamma_g(0) = g$. We are interested in the special map $\phi := \gamma_e$, i.e. the integral curve of $X$ that starts at $e$.

**Lemma 1.1.** The map $\phi = \gamma_e$ is a Lie group homomorphism from $\mathbb{R}$ to $G$, i.e.

$$\phi(s + t) = \phi(s)\phi(t)$$

holds for all $s,t \in \mathbb{R}$.

**Proof.** For any $s \in \mathbb{R}$ fixed, consider the curves 

$$\gamma_1 : \mathbb{R} \to G, \quad t \mapsto \gamma_e(s)\gamma_e(t)$$

and 

$$\gamma_2 : \mathbb{R} \to G, \quad t \mapsto \gamma_e(t + s).$$

We claim that both $\gamma_1$ and $\gamma_2$ are integral curves of the vector field $X$ with identical initial condition $\gamma_1(0) = \gamma_2(0)$. In fact, $\gamma_2$ is an integral curve of $X$ since it is the translation-reparametrization of the integral curve $\gamma_e$. For $\gamma_1$, use the left-invariance of $X$ we have

$$\dot{\gamma}_1(t) = (dL_{\gamma_e(s)})\gamma_e(t)\dot{\gamma}_e(t) = (dL_{\gamma_e(s)})\gamma_e(t)X_{\gamma_e(t)} = X_{\gamma_e(s)\gamma_e(t)} = X_{\gamma_1(t)}.$$ 

It follows that $\gamma_1 \equiv \gamma_2$. \hfill $\Box$

**Definition 1.2.** A one-parameter subgroup of a Lie group $G$ is a Lie group homomorphism $\phi : \mathbb{R} \to G$, i.e. $\phi$ is smooth such that $\phi(s + t) = \phi(s)\phi(t)$ for all $s,t \in \mathbb{R}$.

So the arguments above shows that for any $X \in \mathfrak{g}$ (or any for any $X_e \in T_e G$), one can construct a one-parameter subgroup $\phi$ of $G$. Conversely, for any one-parameter subgroup $\phi : \mathbb{R} \to G$, we must have $\phi(0) = e$, and thus construct a left-invariant vector field $X$ on $G$ via the vector

$$X_e = \dot{\phi}(0) = (d\phi)_0\left(\frac{d}{dt}\right) \in T_e G.$$ 

It is not hard to see that different vectors in $T_e G$ give rise to different one-parameter subgroups, and different one-parameter subgroups give rise to different vectors in $T_e G$. 

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As a consequence, we get one-to-one correspondences between

- One-parameter subgroups of $G$.
- Left invariant vector fields on $G$.
- Tangent vectors at $e \in G$.

So we have three different descriptions of the Lie algebra $g$.

2. The Exponential Map

For any $X \in g$, let $\phi_X$ be the one-parameter subgroup of $G$ corresponding to $X$.

**Definition 2.1.** The exponential map of $G$ is the map

$$\exp : g \to G, \quad X \mapsto \phi_X(1).$$

Since $\tilde{\phi}(s) = \phi_X(ts)$ is the one parameter subgroup corresponding to $tX$, we have

$$\exp(tX) = \phi_X(t).$$

**Example.**

1. For $G = \mathbb{R}^*$, we can identify $T_1G = \mathbb{R}$. For any $x \in T_1G = \mathbb{R}$, the map

$$\phi : \mathbb{R} \to G, \quad t \mapsto e^{tx}$$

is the one-parameter subgroup of $G$ with $\dot{\phi}(0) = x$. It follows $\exp(x) = e^x$.

2. For $G = S^1$, we can identify $T_1S^1 = i\mathbb{R}$. The one-parameter subgroup corresponding to $ix \in T_1S^1 = i\mathbb{R}$ is

$$\phi : \mathbb{R} \to S^1, \quad t \mapsto e^{itx}.$$ 

So the exponential map is given by $\exp(ix) = e^{ix}$.

3. For $G = \mathbb{R}$, we identify $T_0G = \mathbb{R}$. The one-parameter subgroup for $x \in \mathbb{R}$ is

$$\phi : \mathbb{R} \to \mathbb{R}, \quad t \mapsto tx.$$ 

So the exponential map is $\exp(x) = x$.

Note that the zero vector $0 \in T_eG$ generates the zero vector field on $G$, whose integral curve through $e$ is the constant curve. So $\exp(0) = e$.

**Lemma 2.2.** The exponential map $\exp : g \to G$ is smooth, and if we identify both $T_0g$ and $T_eG$ with $g$,

$$ (d\exp)_0 = \text{Id}. $$

**Proof.** Consider the map

$$\Phi : \mathbb{R} \times G \times g \to G \times g, \quad (t, g, X) \mapsto (g \cdot \exp(tX), X).$$

One can check that this is the flow on $G \times g$ corresponding to the (left invariant) vector field $(g, X) \mapsto (Xg, 0)$ on $G \times g$, thus it is smooth. It follows that $\exp = \pi_1 \circ \Phi|_{(1) \times (e) \times g}$ is smooth.
Since \( \exp(tX) = \phi_X(t) \), \( \frac{d}{dt}|_{t=0} \exp(tX) = X \). On the other hand,

\[
\frac{d}{dt}|_{t=0} \exp \circ tX = (d\exp)_0 \frac{d(Xt)}{dt} = (d\exp)_0X.
\]

We conclude that \((d\exp)_0\) equals to the identity map. \(\square\)

Since \((d\exp)_0\) is bijective, we have

**Corollary 2.3.** \(\exp\) is a local diffeomorphism near 0, i.e. it is a diffeomorphism from a neighborhood of 0 \(\in T_eG\) to a neighborhood of \(e \in G\).

Recall that for any Lie group homomorphism \(\varphi : G \to H\), its differential at \(e\),

\[d\varphi : g \to h,\]

is a Lie algebra homomorphism.

**Proposition 2.4 (exp is Natural).** Given any Lie group homomorphism \(\varphi : G \to H\), the diagram

\[
\begin{array}{ccc}
g & \xrightarrow{d\varphi} & h \\
\downarrow{\exp_g} & & \downarrow{\exp_h} \\
G & \xrightarrow{\varphi} & H
\end{array}
\]

is commutative, i.e. \(\varphi \circ \exp_g = \exp_h \circ d\varphi\).

**Proof.** Let \(X \in g\), then

\[\varphi \circ \exp_g : \mathbb{R} \to H, \quad t \mapsto \varphi \circ \exp_g(tX)\]

is the one-parameter subgroup of \(H\) associated to the vector

\[
\frac{d}{dt}|_{t=0} \varphi \circ \exp_g(tX) = d\varphi(X).
\]

So \(\varphi \circ \exp_g(tX) = \exp_h \circ t\varphi(X)\). \(\square\)

As an application, one can show that if \(G\) is connected, any Lie group homomorphism \(\varphi : G \to H\) is determined by the induced Lie algebra homomorphism \(d\varphi : g \to h\).

### 3. Different Descriptions of Lie Bracket

Now we have three different descriptions of the Lie algebra \(g\) of \(G\). Consequently we should also have three different descriptions of the Lie bracket operation \([\cdot, \cdot]\):

(a) \(g\) = the set of left invariant vector fields on \(G\): For left invariant vector fields \(X\) and \(Y\) on \(G\),

\[ [X, Y] := XY - YX. \]
(b) $g = T_e G$: For $X, Y \in T_e G$,

$$[X, Y] := \text{ad}(X)Y,$$

where $\text{ad} : T_e G \to \text{End}(T_e G)$ is defined as follows. Each element $g \in G$ gives rise to an automorphism

$$c(g) : G \to G, \quad x \mapsto gxg^{-1}.$$

Notice that $c(g)$ maps $e$ to $e$, its differential at $e$ gives us a linear map

$$\text{Ad}_g = (dc(g))_e : T_e G \to T_e G.$$

In other words, we get a map (the adjoint representation of the Lie group $G$)

$$\text{Ad} : G \to \text{End}(T_e G), \quad g \mapsto \text{Ad}_g.$$

Note that $\text{Ad}(e)$ is the identity map in $\text{End}(T_e G)$. Moreover, since $\text{End}(T_e G)$ is a linear space, its tangent space at $\text{Id}$ can be identified with $\text{End}(T_e G)$ itself in a natural way. Taking derivative again at $e$, we get (the adjoint representation of the Lie algebra $g$)

$$\text{ad} : T_e G \to \text{End}(T_e G).$$

Applying the naturality of $\exp$ to the Lie group homomorphism $\text{Ad} : G \to \text{End}(g)$ and to the conjugation map $c(g) : G \to G$, we have

**Proposition 3.1.** (1) $\text{Ad}(\exp(tX)) = \exp(t\text{ad}(X))$.

(2) $g(\exp tX)g^{-1} = \exp(t\text{Ad}_g X)$.

(c) $g =$ the set of one-parameter subgroups: The one-parameter subgroups generated by $X, Y \in T_e G$ are $\phi_X$ and $\phi_Y$. Define

$$a(t, s) = \phi_X(t)\phi_Y(s)\phi_X(-t).$$

Then

$$[\phi_X, \phi_Y] := \text{the one-parameter subgroup generated by } \frac{\partial}{\partial t} \bigg|_{t=0} \frac{\partial}{\partial s} \bigg|_{s=0} a(t, s),$$

Now we show that the three different Lie bracket described above are equivalent:

**Theorem 3.2.** The three different Lie brackets defined in (a), (b), (c) are equivalent.

**Proof.** First let’s compute $(\text{ad}X)Y$. According to proposition 3.1,

$$(\text{ad}X)Y = \frac{d}{dt} \bigg|_{t=0} (\text{Ad}(\exp tX))Y.$$ 

On the other hand, since $\text{Ad}_g$ is the differential of $c(g)$, we have

$$\text{Ad}(\exp tX)Y = \frac{d}{ds} \bigg|_{s=0} c(\exp tX) \exp sY = \frac{d}{ds} \bigg|_{s=0} \exp(tX) \exp(sY) \exp(-tX).$$
This shows that (b) is equivalent to (c). To show that they are also equivalent to (a), we compute for any $f \in C^\infty(G)$,

$$\begin{align*}
(\text{ad}(X)Y)f &= \left. \left( \frac{d}{dt} \right|_{t=0} \text{Ad}(\exp tX)Y \right) f \\
&= \left. \frac{\partial^2}{\partial s \partial t} \right|_{s=t=0} f(\exp(tX) \exp(sY) \exp(-tX)) \\
&= \left. \frac{\partial^2}{\partial s \partial t} \right|_{s=t=0} f(\exp(tX) \exp(sY)) + \left. \frac{\partial^2}{\partial s \partial t} \right|_{s=t=0} f(\exp(sY) \exp(-tX)) \\
&= XY f(e) - YX f(e).
\end{align*}$$

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