LECTURE 10: LIE SUBGROUPS

1. Lie Subgroups v.s. Lie Subalgebras

Let's first recall the definition.

Definition 1.1. A subgroup H of a Lie group G is called a *Lie subgroup* if it is a Lie group (with respect to the induced group operation), and the inclusion map $\iota_H : H \hookrightarrow G$ is a smooth immersion (and therefore a Lie group homomorphism).

Note that we don't require H to be a smooth submanifold of G.

Example. Consider $G = \mathbb{T}^2 = S^1 \times S^1$. Then $S^1 \times \{0\}$ and $\{0\} \times S^1$ are Lie subgroups. Moreover, for any co-prime pair of integers (p, q),

$$H^{p,q} := \{(e^{ipt}, e^{iqt}) \mid t \in \mathbb{R}\}$$

is a Lie subgroup of $\mathbb{T}^2$. These are submanifolds as well. However, there are also many Lie subgroups of $\mathbb{T}^2$ which are not submanifolds. In fact, for any irrational number α,

$$H^\alpha := \{(e^{it}, e^{i\alpha t}) \mid t \in \mathbb{R}\}$$

is a Lie subgroup of $\mathbb{T}^2$. But $\bar{H}^\alpha = \mathbb{T}^2$, so they are not submanifolds. (In particular, we see that compact Lie groups may have noncompact subgroups!)

Suppose H is a Lie subgroup of G, and $\mathfrak{h}$ be the Lie algebra of H. As we have explained, we can think of $\mathfrak{h}$ as a *Lie subalgebra* (= a linear subspace that is closed under Lie bracket) of $\mathfrak{g}$. We first prove the following theorem that we have used in determining the Lie algebra of many linear Lie groups:

Proposition 1.2. *Suppose H is a Lie subgroup of G. Then as a Lie subalgebra of $\mathfrak{g}$, the Lie algebra of H is*

$$\mathfrak{h} = \{X \in \mathfrak{g} \mid \exp_G(tX) \in H \text{ for all } t \in \mathbb{R}\}.$$

Proof. First suppose $X \in \mathfrak{h}$, then by naturality of exp, for any $t \in \mathbb{R}$,

$$\exp_G(tX) = \iota_H(\exp_H(tX)) \in \iota_H(H) = H.$$

Conversely suppose $X \notin \mathfrak{h}$. Consider the map

$$\varphi : \mathbb{R} \times \mathfrak{h} \to G, \quad (t, Y) \mapsto \exp(tX)\exp(Y).$$

Since $d\exp_0$ is the identity map,

$$d\varphi_{0,0}(t, Y) = tX + Y.$$

Since $X \notin \mathfrak{h}$, $d\varphi_{0,0}$ is injective. It follows that there exists a small $\varepsilon > 0$ and a neighborhood U of 0 in $\mathfrak{h}$ such that φ maps $(-\varepsilon, \varepsilon) \times U$ injectively into G. Shrinking U

if necessary, we may assume that $\exp_H$ maps U diffeomorphically onto a neighborhood $\mathcal{U}$ of e in H. Choose a smaller neighborhood $\mathcal{U}_0$ of e in H such that $\mathcal{U}_0^{-1}\mathcal{U}_0 \subset \mathcal{U}$. We pick a countable collection $\{h_j \mid j \in \mathbb{N}\} \subset H$ such that $h_j\mathcal{U}_0$ cover H. (This is always possible since H is the union of countable many compact sets.)

For each j denote $T_j = \{t \in \mathbb{R} \mid \exp(tX) \in h_j\mathcal{U}_0\}$. We claim that T_j is a countable set. In fact, if $|t-s| < \varepsilon$ and $t, s \in T_j$, then

$$\exp(t-s)X = \exp(-sX)\exp(tX) \in \mathcal{U}.$$

So $\exp(t-s)X = \exp(Y)$ for a unique $Y \in U$. It follows $\varphi(t-s,0) = \varphi(0,Y)$. Since φ is injective, we conclude $Y = 0$ and $t = s$.

Now each T_j is a countable set. So one can find $t \in \mathbb{R}$ such that $t \notin T_j$ for all j. It follows that $\exp(tX) \notin \cup_j h_j\mathcal{U}_0 = H$. So

$$X \notin \{X \in \mathfrak{g} \mid \exp_G(tX) \in H \text{ for all } t \in \mathbb{R}\}.$$

This completes the proof. □

Conversely, any Lie subalgebra gives rise to some Lie subgroup:

Theorem 1.3. *If $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$, then there is a unique connected Lie subgroup H of G with Lie algebra $\mathfrak{h}$.*

Proof. Let $X_1, \cdots, X_k$ be a basis of $\mathfrak{h} \subset \mathfrak{g}$. Since $X_i's$ are left invariant vector fields on G, linearly independent at e, they are linearly independent at all $g \in G$. In other words,

$$\mathcal{V}_g = \mathrm{span}\{X_1(g), \cdots, X_k(g)\}$$

gives us a k-dimensional distribution on G. Since $[X_i, X_j] \in \mathfrak{h}$ for all i, j, $\mathcal{V}$ is integrable. By Frobenius theorem, there is a unique maximal connected integral manifold of $\mathcal{V}$ through e. Denote this by H.

To show that H is a subgroup, note that $\mathcal{V}$ is a left invariant distribution. So the left translation of any integral manifold is an integral manifold. Now suppose $h_1, h_2 \in H$. Since

$$h_1 = L_{h_1}e \in H \cap L_{h_1}H,$$

and since H is maximal, we have $L_{h_1}H \subset H$. So in particular $h_1h_2 = L_{h_1}h_2 \in H$. Similarly, $h_1^{-1} \in H$ since $L_{h_1^{-1}}(h_1) = e \in H$ implies $L_{h_1^{-1}}H \subset H$. It follows that H is a subgroup of G. Since the group operations on H are the restriction of group operations on G, they are smooth. So H is a Lie group.

For uniqueness, let K be another connected Lie subgroup of G with Lie algebra $\mathfrak{h}$. Then K is also an integral manifold of $\mathcal{V}$. So we have $K \subset H$. Since $T_eK = T_eH$ the inclusion has to be a local isomorphism. In other words, K coincide with H near e. Since any connected Lie group is generated by any open set containing e, we conclude that $K = H$. □

2. Closed Lie Subgroups

We are more interested in those Lie subgroups that are submanifolds as well.

Definition 2.1. A Lie subgroup H of G is said to be a *closed Lie subgroup* if H is both a Lie subgroup and also a submanifold of G.

Although in general a submanifold doesn't have to be a closed subset, a closed Lie subgroup must be. This explains its name.

Lemma 2.2. *Suppose G is a Lie group, H is a subgroup of G which is a submanifold as well. Then H is closed in the sense of topology.*

Proof. Since H is a submanifold of G, it is *locally closed* everywhere. In particular, one can find an open neighborhood U of e in G such that $U \cap H = U \cap \overline{H}$. Now take any $h \in \overline{H}$. Since hU is an open neighborhood of h in G, $hU \cap H \neq \emptyset$. Let $h' \in hU \cap H$, then $h^{-1}h' \in U$. On the other hand, since $h \in \overline{H}$, there is a sequence h_n in H converging to h. It follows that the sequence $h_n^{-1}h' \in H$ converges to $h^{-1}h'$. In other words, $h^{-1}h' \in U \cap \overline{H} = U \cap H$. So $h \in H$, i.e. $\overline{H} \subset H$. Therefore, H is closed. □

In other words, a closed Lie subgroup is must be a closed subgroup. A remarkable theorem due to E. Cartan claims that the inverse is also true, i.e. any closed subgroup must be a Lie subgroup.

Theorem 2.3 (E. Cartan's closed subgroup theorem)**.** *Any closed subgroup H of a Lie group G is a Lie subgroup (and thus a submanifold) of G.*