LECTURE 11: CARTAN’S CLOSED SUBGROUP THEOREM

1. Cartan’s closed subgroup theorem

Suppose $G$ is a Lie group and $H$ a closed subgroup of $G$, i.e. $H$ is subgroup of $G$ which is also a closed subset of $G$. Let

$$\mathfrak{h} = \{ X \in \mathfrak{g} \mid \exp(tX) \in H \text{ for all } t \in \mathbb{R} \}.$$ 

In what follows we will prove the closed subgroup theorem due to E. Cartan. We will need the following lemmas:

**Lemma 1.1.** $\mathfrak{h}$ is a linear subspace of $\mathfrak{g}$.

*Proof.* Clearly $\mathfrak{h}$ is closed under scalar multiplication. It is closed under vector addition because for any $t \in \mathbb{R}$,

$$H \ni \lim_{n \to \infty} \left( \exp \left( \frac{tX}{n} \right) \exp \left( \frac{tY}{n} \right) \right)^n = \lim_{n \to \infty} \left( \exp \left( \frac{t(X + Y)}{n} + O \left( \frac{1}{n^2} \right) \right) \right)^n = \exp(t(X+Y)).$$

□

**Lemma 1.2.** Suppose $X_1, X_2, \cdots$ be a sequence of nonzero elements in $\mathfrak{g}$ so that

1. $X_i \to 0$ as $i \to \infty$.
2. $\exp(X_i) \in H$ for all $i$.
3. $\lim_{i \to \infty} \frac{X_i}{|X_i|} = X \in \mathfrak{g}$.

Then $X \in \mathfrak{h}$.

*Proof.* For any fixed $t \neq 0$, we take $n_i = \lfloor \frac{t}{|X_i|} \rfloor$ be the integer part of $\frac{t}{|X_i|}$. Then

$$\exp(tX) = \lim_{i \to \infty} \exp(n_i X_i) = \lim_{i \to \infty} \exp(X_i)^{n_i} \in H.$$ □

**Lemma 1.3.** The exponential map $\exp : \mathfrak{g} \to G$ maps a neighborhood of $0$ in $\mathfrak{h}$ bijectively to a neighborhood of $e$ in $H$.

*Proof.* Take a vector subspace $\mathfrak{h}'$ of $\mathfrak{g}$ so that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}'$. Let $\Phi : \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}' \to G$ be the map

$$\Phi(X + Y) = \exp(X) \exp(Y).$$

Then as we have seen, $d\Phi_0(X + Y) = X + Y$. So $\Phi$ is a local diffeomorphism from $\mathfrak{g}$ to $G$. Since $\exp|_\mathfrak{h} = \Phi|_\mathfrak{h}$, to prove the lemma, it is enough to prove that $\Phi$ maps a neighborhood of $0$ in $\mathfrak{h}$ bijectively to a neighborhood of $e$ in $H$. 

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Suppose the lemma is false, then we can find a sequence of vectors \( X_i + Y_i \in h \oplus h' \) with \( Y_i \neq 0 \) so that \( X_i + Y_i \to 0 \) and \( \Phi(X_i + Y_i) \in H \). Since \( \exp(X_i) \in H \), we must have \( \exp(Y_i) \in H \) for all \( i \). We let \( Y \) be a limit point of \( \frac{Y_i}{|Y_i|} \)'s. Then according to the previous lemma, \( Y \in h \). Since \( h' \) is a subspace and thus a closed subset, \( Y \in h' \). So we must have \( Y = 0 \), which is a contradiction since by construction, \( |Y| = 1 \).

Now we are ready to prove

**Theorem 1.4 (E. Cartan’s closed subgroup theorem).** Any closed subgroup \( H \) of a Lie group \( G \) is a Lie subgroup (and thus a submanifold) of \( G \).

**Proof.** According to the previous lemma, one can find a neighborhood \( U \) of \( e \) in \( G \) and a neighborhood \( V \) of 0 in \( g \) so that \( \exp^{-1} : U \to V \) is a diffeomorphism, and so that \( \exp^{-1}(U \cap H) = V \cap h \). It follows that \( (\exp^{-1}, U, V) \) is a chart on \( G \) which makes \( H \) a submanifold near \( e \). For any other point \( h \in H \), we can use left translation to get such a chart.

As an immediate consequence, we get

**Corollary 1.5.** If \( \varphi : G \to H \) is Lie group homomorphism, then \( \ker(\varphi) \) is a closed Lie subgroup of \( G \) whose Lie algebra is \( \ker(d\varphi) \).

**Proof.** It is easy to see that \( \ker(\varphi) \) is a subgroup of \( G \) which is also a closed subset. So according to Cartan’s theorem, \( \ker(\varphi) \) is a Lie subgroup. It follows that the Lie algebra of \( \ker(\varphi) \) is given by

\[
\text{Lie}(\ker(\varphi)) = \{ X \in g \mid \exp(tX) \in \ker(\varphi), \forall t \}.
\]

The theorem follows since

\[
\exp(tX) \in \ker(\varphi), \forall t \iff \varphi(\exp(tX)) = e, \forall t
\]

\[
\iff \exp(td\varphi(X)) = e, \forall t
\]

\[
\iff d\varphi(X) = 0.
\]

As an application, we have

**Theorem 1.6.** Any connect abelian Lie group is of the form \( \mathbb{T}^r \times \mathbb{R}^k \).

**Proof.** Let \( G \) be a connect abelian Lie group. Then we have seen that \( \exp : g \to G \) is a surjective Lie group homomorphism, so \( G \) is isomorphic to \( g/\ker(\exp) \).

On the other hand, \( \ker(\exp) \) is a Lie subgroup of \( (g, +) \), and it is discrete since \( \exp \) is a local diffeomorphism near \( e \). By using induction one can show that \( \ker(\exp) \) is a lattice in \( (g, +) \), i.e. there exists linearly independent vectors \( v_1, \ldots, v_r \in g \) so that \( \ker(\exp) = \{ n_1v_1 + \cdots + n_rv_r \mid n_i \in \mathbb{Z} \} \).
Let \( V_1 = \text{span}(v_1, \cdots, v_r) \) and \( V_2 \) be a linear subspace of \( g \) so that \( g = V_1 \times V_2 \). Then
\[
G \simeq g/\ker(\exp) = V_1/\ker(\exp) \times V_2 \simeq T^r \times \mathbb{R}^k.
\]
□

Another important consequence of Cartan’s theorem is

**Corollary 1.7.** Every continuous homomorphism of Lie groups is smooth.

**Proof.** Let \( \phi : G \rightarrow H \) be a continuous homomorphism, then
\[
\Gamma_\phi = \{(g, \phi(g)) \mid g \in G\}
\]
is a closed subgroup, and thus a Lie subgroup of \( G \times H \). The projection
\[
p : \Gamma_\phi \xrightarrow{i} G \times H \xrightarrow{pr_1} G
\]
is bijective, smooth and is a Lie group homomorphism. It follows that \( dp \) is a constant rank map, and thus has to be bijective at each point. So \( p \) is local diffeomorphism everywhere. Since it is globally invertible, \( p \) is also a global diffeomorphism. Thus \( \phi = pr_2 \circ p^{-1} \) is smooth. □

As a consequence, for any topological group \( G \), there is at most one smooth structure on \( G \) to make it a Lie group. (However, it is possible that one group admits two different topologies and thus have different Lie group structures.)

## 2. Simply Connected Lie Groups

Recall that a **path** in \( M \) is a continuous map \( f : [0, 1] \rightarrow M \). It is **closed** if \( f(0) = f(1) \).

**Definition 2.1.** Let \( M \) be a connected Hausdorff topological space.

1. Two paths \( f, g : [0, 1] \rightarrow M \) with the same end points (i.e. \( f(0) = g(0), f(1) = g(1) \)) are **homotopic** if there is a continuous map \( h : [0, 1] \times [0, 1] \rightarrow M \) such that
\[
h(s, 0) = f(s), h(s, 1) = g(s)
\]
for all \( s \), and
\[
h(0, t) = f(0), h(1, t) = f(1)
\]
for all \( t \).
2. \( M \) is **simply connected** if any two paths with the same ends are homotopic.
3. A continuous surjection \( \pi : X \rightarrow M \) is called a **covering** if each \( p \in M \) has a neighborhood \( V \) whose inverse image under \( \pi \) is a disjoint union of open sets in \( X \) each homeomorphic with \( V \) under \( \pi \).
4. A simply connected covering space is called the **universal cover**.
For example, $\mathbb{R}^n$ is simply connected, $\mathbb{T}^n$ is not simply connected. The map
$$\mathbb{R}^n \to \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n, x \mapsto x + \mathbb{Z}^n$$
is a covering map. The following results are well known:

**Facts from topology:**

- Let $\pi : X \to M$ is a covering, $Z$ a simply connected space. Suppose $\alpha : Z \to M$ be a continuous map, such that $\alpha(z_0) = m_0$. Then for any $x_0 \in \pi^{-1}(m_0)$, there is a unique “lifting” $\tilde{\alpha} : Z \to X$ such that $\pi \circ \tilde{\alpha} = \alpha$ and $\tilde{\alpha}(z_0) = x_0$.
- Any connected manifold has a simply connected covering space.
- If $M$ is simply connected, any covering map $\pi : X \to M$ is a homeomorphism.

**Theorem 2.2.** The universal covering space of a connected Lie group admits a Lie group structure such that the covering map is a Lie group homomorphism.

**Proof.** Since $G$ is connected, it has a universal covering $\tilde{\pi} : \tilde{G} \to G$. One can use the charts on $G$ and the lifting map to define charts on $\tilde{G}$ so that $\tilde{G}$ becomes a smooth manifold. Moreover, one can check that under this smooth structure, the lifting of a smooth map is also smooth.

To define a group structure on $\tilde{G}$, and show $\pi$ is a Lie group homomorphism, we consider the map
$$\alpha : \tilde{G} \times \tilde{G} \to G, \quad (\tilde{g}_1, \tilde{g}_2) \mapsto \pi(\tilde{g}_1)\pi(\tilde{g}_2)^{-1}.$$ Choose any $\tilde{e} \in \pi^{-1}(\epsilon)$. Since $\tilde{G} \times \tilde{G}$ is simply connected, there is a lifting map $\tilde{\alpha} : \tilde{G} \times \tilde{G} \to \tilde{G}$ such that $\pi \circ \tilde{\alpha} = \alpha$ and such that $\tilde{\alpha}(\tilde{e}, \tilde{e}) = \tilde{e}$. Now for any $\tilde{g}_1, \tilde{g}_2 \in \tilde{G}$ we define
$$\tilde{g}^{-1} := \tilde{\alpha}(\tilde{e}, \tilde{g}), \quad \tilde{g}_1 \cdot \tilde{g}_2 = \tilde{\alpha}(\tilde{g}_1, \tilde{g}_2^{-1}).$$ By uniqueness of lifting, we have $\tilde{g}\tilde{e} = \tilde{e}\tilde{g} = \tilde{g}$ for all $\tilde{g} \in \tilde{G}$, since the maps
$$\tilde{g} \mapsto \tilde{g}\tilde{e}, \quad \tilde{g} \mapsto \tilde{e}\tilde{g}, \quad \tilde{g} \mapsto \tilde{g}$$
are all lifting of the map $\tilde{g} \mapsto \pi(\tilde{g})$. Similarly $\tilde{g}\tilde{g}^{-1} = \tilde{g}^{-1}\tilde{g} = \tilde{e}$, and $(\tilde{g}_1\tilde{g}_2)\tilde{g}_3 = \tilde{g}_1(\tilde{g}_2\tilde{g}_3)$. So $\tilde{G}$ is a group. One can check that the group operations are smooth under the smooth structure chosen above. So $\tilde{G}$ is actually a Lie group.

Finally by definition $\pi(\tilde{g}^{-1}) = \pi(\tilde{g})^{-1}$ and $\pi(\tilde{g}_1\tilde{g}_2) = \pi(\tilde{g}_1)\pi(\tilde{g}_2)$. So $\pi$ is a continuous group homomorphism between Lie groups, and thus a Lie group homomorphism. \qed