LECTURE 12: LIE'S FUNDAMENTAL THEOREMS

1. Lie Group Homomorphism v.s. Lie Algebra Homomorphism

Lemma 1.1. *Suppose G, H are connected Lie groups, and $\Phi : G \to H$ is a Lie group homomorphism. If $d\Phi : \mathfrak{g} \to \mathfrak{h}$ is bijective, then Φ is a covering map.*

Proof. Since H is connected and Φ is a Lie group homomorphism and is a local diffeomorphism near onto a neighborhood of $e \in H$, Φ is surjective. By group invariance, it suffices to check the covering property at $e \in H$. Since $d\Phi : T_eG \to T_eH$ is bijective, Φ maps a neighborhood $\mathcal{U}$ of e in G bijectively to a neighborhood $\mathcal{V}$ of e in H.

Let $\Gamma = \Phi^{-1}(e) \subset G$. Then Γ is a subgroup of G. Moreover, for any $a \in \Gamma$,

$$\Phi \circ L_a(g) = \Phi(ag) = \Phi(a)\Phi(g) = \Phi(g).$$

So $\Phi^{-1}(\mathcal{V}) = \cup_{a\in\Gamma} L_a\mathcal{U}$. The lemma is proved if we can show $L_{a_1}\mathcal{U} \cap L_{a_2}\mathcal{U} = \emptyset$ for $a_1 \neq a_2 \in \Gamma$. We show this by contradiction. Let $a = a_1^{-1}a_2$. If $L_{a_1}\mathcal{U} \cap L_{a_2}\mathcal{U} \neq \emptyset$, then $L_a\mathcal{U} \cap \mathcal{U} \neq \emptyset$. Consider $p_2 = ap_1 \in L_a\mathcal{U} \cap \mathcal{U}$, where $p_1, p_2 \in \mathcal{U}$. Then $\Phi(p_2) = \Phi(ap_1) = \Phi(p_1)$. However, Φ is one-to-one on $\mathcal{U}$. So $p_1 = p_2$. It follows $a = e$, and $a_1 = a_2$. Contradiction. □

Corollary 1.2. *Let $\Phi : G \to H$ be a Lie group homomorphism with $d\Phi : \mathfrak{g} \to \mathfrak{h}$ bijective. Suppose G, H are connected and H is simply connected. Then Φ is a Lie group isomorphism.*

Proof. Since Φ is a covering map and H is simply connected, Φ is homeomorphism. In particular, Φ and Φ^{-1} are both continuous Lie group homomorphisms. If follows that both Φ and Φ^{-1} are smooth, and thus Φ is a diffeomorphism. □

The main theorem in this section is the following "lifting" property:

Theorem 1.3. *Let G, H be Lie groups with G connected and simply connected, $\mathfrak{g}, \mathfrak{h}$ the Lie algebras of G, H. If $\rho : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism, then there is a unique Lie group homomorphism $\Phi : G \to H$ such that $d\Phi = \rho$.*

Proof. Let

$$\mathfrak{k} = \mathrm{graph}(\rho) = \{(g, h) \in \mathfrak{g} \oplus \mathfrak{h} : h = \rho(g)\}.$$

Obviously $\mathfrak{k}$ is a vector space. It is actually a Lie subalgebra of $\mathfrak{g} \oplus \mathfrak{h}$. In fact, if $h_i = \rho(g_i)$, $i = 1, 2$, then the fact ρ is a Lie algebra homomorphism implies

$$[h_1, h_2] = [\rho(g_1), \rho(g_2)] = \rho([g_1, g_2]).$$

It follows

$$[(g_1, h_1), (g_2, h_2)] = ([g_1, g_2], [h_1, h_2]) = ([g_1, g_2], \rho([g_1, g_2])).$$

In other words, $\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{g} \oplus \mathfrak{h}$.

By the theorem we proved in last time, there exists a unique connected Lie subgroup K of $G \times H$ with $\mathfrak{k}$ as its Lie algebra. Consider the composition map

$$\varphi : K \overset{\iota_K}{\hookrightarrow} G \times H \overset{\mathrm{pr}_1}{\to} G.$$

This is a Lie group homomorphism, so $d\varphi = d\mathrm{pr}_1 \circ d\iota_K$ is a Lie algebra homomorphism. Since $d\mathrm{pr}_1 : \mathfrak{g} \times \mathfrak{h} \to \mathfrak{g}$ is the projection map, $d\varphi : \mathfrak{k} \to \mathfrak{g}$ is a bijection. It follows that $\varphi : K \to G$ is a Lie group isomorphism.

Now let $\Phi : G \to H$ be the composition

$$G \xrightarrow{\varphi^{-1}} K \xrightarrow{\mathrm{pr}_2} H.$$

Then this is a Lie group homomorphism with $d\Phi = \rho$. This completes the proof. □

Corollary 1.4. *If connected and simply connected Lie groups G and H have isomorphic Lie algebra, then G and H are isomorphic.*

2. Lie's Fundamental Theorems

We have seen that associated to each Lie group G there is a god-given Lie algebra $\mathfrak{g}$. A natural question is: To what extend will the Lie algebra determine this Lie group? On one hand, we have seen that the Lie algebras of S^1 and $\mathbb{R}^1$ are the same, so Lie groups are not determined by its Lie algebra. On the other hand, according to the B-C-H formula, the Lie group product near the identity e is totally determined by the Lie bracket. So at least the Lie algebra $\mathfrak{g}$ provides the local information for G. The subtle relations between Lie groups and Lie algebras are described by the following theorems observed by S. Lie at the beginning of the whole subject.

Recall that for any Lie group homomorphism $f : G_1 \to G_2$, there is an induced Lie algebra homomorphism $df : \mathfrak{g}_1 \to \mathfrak{g}_2$. The assignment $f \rightsquigarrow df$ satisfies the following *functorial properties*:

- For $f = \mathrm{Id} : G \to G$, $df = \mathrm{Id} : \mathfrak{g} \to \mathfrak{g}$.
- If $f_i : G_i \to G_{i+1}$, $i = 1, 2$, are Lie group homomorphisms, then $d(f_2 \circ f_1) = df_2 \circ df_1$.

To state the theorems, let's first give a definition.

Definition 2.1. A *local homomorphism* between two Lie groups G, H is a smooth map f from a neighborhood U of e_G in G to a neighborhood V of e_H in H such that if $g_1, g_2 \in U$ and $g_1 g_2 \in U$, then $f(g_1 g_2) = f(g_1) f(g_2)$. f is called a *local isomorphism* if it is a diffeomorphism from U to V such that both f and f^{-1} are local homomorphisms.

We know the Lie algebra, as the tangent space at e, is determined by the Lie group structure on any neighborhood of e in G. So any local homomorphism determines a Lie algebra homomorphism as well. Moreover, the functorial properties above also holds for local homomorphisms.

Theorem 2.2 (Lie's first fundamental theorem)**.** *If G and H are locally isomorphic Lie groups, then $\mathfrak{g}$ and $\mathfrak{h}$ are isomorphic Lie algebras.*

Proof. Let f be the local isomorphism between G and G. Since $df_e : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism, We only need to show df_e is a bijection. However, this follows from the fact that exp is locally diffeomorphism. □

Conversely,

Theorem 2.3 (Lie's second fundamental theorem)**.** *If $\mathfrak{g}$ and $\mathfrak{h}$ are isomorphic Lie algebras, then G and H are locally isomorphic Lie groups.*

Proof. Let $\rho : \mathfrak{g} \to \mathfrak{h}$ be the Lie algebra isomorphism map. As in the proof of theorem 1.3, there exists a connected Lie subgroup K of $G \times H$ whose Lie algebra is

$$\mathfrak{k} = \{(x, \rho(x)) \mid x \in \mathfrak{g}\}.$$

We still have the facts that the composition map

$$\varphi : K \hookrightarrow G \times H \xrightarrow{\mathrm{pr}_1} G$$

is a Lie group homomorphism whose differential $d\varphi : \mathfrak{k} \to \mathfrak{g}$ is bijective. It follows that φ is a local isomorphism and a diffeomorphism, i.e. there is a neighborhood U of $e \in K$, a neighborhood V of $e \in G$ and a diffeomorphism $\psi : V \to U$ such that $\psi \circ \varphi = 1_U$ and $\varphi \circ \psi = 1_V$.

Similarly the map

$$\phi : K \hookrightarrow G \times H \xrightarrow{\mathrm{pr}_2} H$$

is also a local diffeomorphism, and a local isomorphism. Now the composition $\phi \circ \psi$ is the local isomorphism we want. □

Theorem 2.4 (Lie's third fundamental theorem)**.** *For any finite dimensional Lie algebra $\mathfrak{g}$, there is a unique simply connected Lie group G whose Lie algebra is $\mathfrak{g}$.*

The proof is based on the following amazing theorem whose proof is beyond the scope of this course can can be found in books on Lie algebra representation theory.

Theorem 2.5 (Ado)**.** *Every finite dimensional Lie algebra is a Lie subalgebra of $\mathfrak{gl}(n, \mathbb{R})$ for n large enough.*

Proof of Lie's third fundamental theorem. According to Ado's theorem $\mathfrak{g}$ is a Lie subalgebra of some $\mathfrak{gl}(n, \mathbb{R})$. So there is a connected linear Lie group G_1 whose Lie algebra is $\mathfrak{g}$. Let G be the simply connected covering of G_1. Then G is a simply connected Lie group. Its Lie algebra is $\mathfrak{g}$ since any covering map is a local isomorphism. The uniqueness follows from corollary 1.4. □