LECTURE 13-14: ACTIONS OF LIE GROUPS AND LIE ALGEBRAS

1. Smooth actions of Lie groups

Definition 1.1. Let M be a smooth manifold and G a Lie group.

(1) An *action* of G on M is a homomorphism of groups $\tau : G \to \mathrm{Diff}(M)$, where $\mathrm{Diff}(M)$ is the group of diffeomorphisms on M. In other words, τ associates to any $g \in G$ a diffeomorphism $\tau(g) : M \to M$ such that for any $g_1, g_2 \in G$,
$$\tau(g_1 g_2) = \tau(g_1)\tau(g_2).$$
We will denote $\tau(g)(m)$ by $g \cdot m$ for brevity.

(2) An action τ of G on M is *smooth* if the evaluation map
$$\mathrm{ev} : G \times M \to M, \quad (g, m) \mapsto g \cdot m$$
is smooth.

In what follows by an action we always means a smooth action.

Remark. The Lie group action we defined above is the *left action*. One can also define a *right action* of G on M to be an *anti*-homomorphism $\hat{\tau} : G \to \mathrm{Diff}(M)$, i.e. for any $g_1, g_2 \in G$ we require
$$\hat{\tau}(g_1 g_2) = \hat{\tau}(g_2)\hat{\tau}(g_1).$$
Any left action τ can be converted to a right action $\hat{\tau}$ by letting
$$\hat{\tau}_g(m) = m \cdot g := \tau(g^{-1})m = g^{-1} \cdot m.$$

Example. S^1 acts on $\mathbb{R}^2$ by rotations.

Example. $A \cdot \vec{v} := A\vec{v}$ defines a left action of $\mathrm{GL}(n, \mathbb{R})$ on $\mathbb{R}^n$, and $A \cdot \vec{v} := (\vec{v}^T A)^T = A^T \vec{v}$ defines a right action of $\mathrm{GL}(n, \mathbb{R})$ on $\mathbb{R}^n$.

By the remark above, $A \cdot \vec{v} := A^{-1}\vec{v}$ is also a right action of $\mathrm{GL}(n, \mathbb{R})$ on $\mathbb{R}^n$, and $A \cdot v := (A^T)^{-1} \cdot \vec{v}$ is a left action of $\mathrm{GL}(n, \mathbb{R})$ on $\mathbb{R}^n$.

Example. If X is a complete vector field on M, then
$$\rho : \mathbb{R} \to \mathrm{Diff}(M), \quad t \mapsto \rho_t = \exp(tX)$$
is a smooth action of $\mathbb{R}$ on M, where $\exp(tX)(m) := \gamma_m^X(t)$ is defined via the integral curves of X. Conversely, every smooth action of $\mathbb{R}$ on M is given by this way.

Example. Any Lie group G acts on itself by many ways, e.g. by left multiplication, by right multiplication and by conjugation. More generally, any Lie subgroup H of G can act on G by left multiplication, right multiplication and conjugation.

Example. Let $\mathfrak{g}$ be the Lie algebra of G. Then the *adjoint action* of G on $\mathfrak{g}$ is
$$\mathrm{Ad}: G \to \mathrm{Aut}(\mathfrak{g}) \subset \mathrm{Diff}(\mathfrak{g}), \quad g \mapsto \mathrm{Ad}_g.$$
For example, the adjoint action of $\mathrm{GL}(n, \mathbb{R})$ on $\mathfrak{gl}(n, \mathbb{R})$ is
$$\mathrm{Ad}_C X = CXC^{-1}.$$
Similarly if $\mathfrak{g}^*$ is the dual of $\mathfrak{g}$, then the *coadjoint action* of G on $\mathfrak{g}^*$ is
$$\mathrm{Ad}^*: G \to \mathrm{Aut}(\mathfrak{g}^*) \subset \mathrm{Diff}(\mathfrak{g}^*), \quad g \mapsto \mathrm{Ad}^*_g,$$
where Ad^*_g is defined by
$$\langle \mathrm{Ad}^*_g \xi, X\rangle = \langle \xi, \mathrm{Ad}_{g^{-1}} X\rangle$$
for all $\xi \in \mathfrak{g}^*, X \in \mathfrak{g}$.

A smooth manifold M together with a G-action is called a *G-manifold*.

Definition 1.2. Let M, N be G-manifolds. A smooth map $f: M \to N$ is called *G-equivariant*, or a *G-map*, if it commutes with the group actions, i.e.
$$f(g \cdot m) = g \cdot f(m)$$
for all $g \in G$ and $m \in M$.

Example. Let G, H be Lie groups, and $f: G \to H$ a Lie group homomorphism. G acts on G itself by left translation. Define a left G-action on H by
$$g \cdot h := f(g)h.$$
Then f is equivariant with respect to these actions.

Note that if one or both of the actions are right actions, the equivariance condition above should be suitably modified.

2. Infinitesimal actions of Lie algebras

The group $\mathrm{Diff}(M)$ is in some sense an *infinite-dimensional Lie group*. What is its "Lie algebra", i.e. what is its "tangent space" at the identity element? Well, one can regard a (local) flow on M as a "path" in $\mathrm{Diff}(M)$ that pass the identity map. The "derivative" of such a path at the identity map is the vector field that generates the flow. So one can regard $\Gamma^\infty(M)$ as the "Lie algebra" of $\mathrm{Diff}(M)$.

Now suppose Lie group G acts smoothly on M. Then $\tau: G \to \mathrm{Diff}(M)$ is a "Lie group homomorphism". So it's very natural to study its linearization, i.e. its differential as a map between the corresponding "Lie algebras". We pick an arbitrary $X \in \mathfrak{g}$. Then the corresponding flow on M induced by τ is
$$\Phi_X: \mathbb{R} \times M \to M, \quad (t, m) \mapsto \exp(tX) \cdot m.$$
So the differential of τ is the map
$$d\tau: \mathfrak{g} \to \Gamma^\infty(M), \quad X \mapsto X_M$$

with

$$X_M(m) = \left.\frac{d}{dt}\right|_{t=0} \exp(tX) \cdot m$$

Definition 2.1. The map $d\tau$ is called the *infinitesimal action* of $\mathfrak{g}$ on M.

Lemma 2.2. *Let $\tau : G \to \mathrm{Diff}(M)$ be a smooth action. Then*

(1) *For any $X \in \mathfrak{g}$, $\tau(\exp tX) = \exp(tX_M)$.*
(2) *For any $g \in G$ and $X \in \mathfrak{g}$, $(\mathrm{Ad}_g X)_M = \tau(g^{-1})^* X_M$.*

Proof. (1) We need to check that

$$\gamma_m(t) := \exp(tX) \cdot m$$

is the integral curve of X_M. This follows from the definition of X_M, since

$$\dot{\gamma}_m(t) = \frac{d}{dt}(\exp tX \cdot m) = \left.\frac{d}{ds}\right|_{s=0} (\exp sX \circ \exp tX \cdot m) = X_M(\exp tX \cdot m).$$

(2) For $m \in M$,

$$\begin{aligned}
(\mathrm{Ad}_g X)_M(m) &= \left.\frac{d}{dt}\right|_{t=0} \exp(t\mathrm{Ad}_g X) \cdot m \\
&= \left.\frac{d}{dt}\right|_{t=0} (g \exp(tX) g^{-1}) \cdot m \\
&= d\tau(g)_{g^{-1}\cdot m} \left.\frac{d}{dt}\right|_{t=0} \exp(tX) \cdot (g^{-1} \cdot m) \\
&= d\tau(g)_{g^{-1}\cdot m} X_M(g^{-1} \cdot m) \\
&= \tau(g^{-1})^* X_M(m).
\end{aligned}$$

□

We will leave it as exercise for you to show that the map $d\tau$ is linear. It turns out that it also behaves well under the Lie bracket operations:

Proposition 2.3. *The infinitesimal action $d\tau$ is an anti-homomorphism from $\mathfrak{g}$ to $\Gamma^\infty(M)$, i.e.*

$$[X, Y]_M = -[X_M, Y_M], \qquad \forall X, Y \in \mathfrak{g}.$$

Proof. From Lemma 2.2 we see

$$(\mathrm{Ad}_{\exp(tY)} X)_M = \tau(\exp(-tY))^* X_M = \exp(-tY_M)^* X_M.$$

In view of Theorem 3.2 in lecture 6, the derivative of the left hand side at $t = 0$ is $[Y, X]_M$. On the other hand, by theorem 3.5 in Lecture 3, the derivative of the right hand side at $t = 0$ is $[X_M, Y_M]$. This completes the proof. □

Given an action of a Lie group G on M, in view of lemma 2.2 (1), near e one can integrate the infinitesimal action to recover the Lie group action. Since any connected Lie group is generated by group elements near e, we conclude

Proposition 2.4. *An action of a connected Lie group on a manifold M is uniquely determined by its infinitesimal action.*

A natural question is, which Lie algebra anti-homomorphism can be integrate to Lie group actions? Suppose the Lie algebra anti-homomorphism is induced by a G-action on M, then $G \times M$ decompose into submanifolds

$$\mathcal{L}_m = \{(g, g \cdot m) \mid g \in G\},$$

and each $\mathcal{L}_m$ projects diffeomorphically to G. So if we let $\mathcal{L}_m$ be the "leaf" containing (e, m), then the point on $\mathcal{L}_m$ that projects to g must be $(g, g \cdot m)$. In other words, the "leaves" determine the Lie group action.

Theorem 2.5 (Palais)**.** *Let G be a connected and simply connected Lie group, $\varphi : \mathfrak{g} \to \Gamma^\infty(M)$ a Lie algebra anti-homomorphism such that each $X_M := \varphi(X)$ is complete. Then there exists a unique smooth action $\tau : G \to \mathrm{Diff}(M)$ such that $d\tau = \varphi$.*

We first prove

Lemma 2.6. *Let τ be the left multiplication action of G on G. Then for any $X \in T_eG$, $X_G = d\tau(X)$ is the right invariant vector field X^R generated by X_e.*

Proof. Since

$$X_G(e) = \left.\frac{d}{dt}\right|_{t=0} \exp(tX) \cdot e = X_e,$$

it is enough to check X_G is right invariant:

$$X_G(gh) = \left.\frac{d}{dt}\right|_{t=0} \exp(tX)gh = (dR_h)_g \left.\frac{d}{dt}\right|_{t=0} \exp(tX)g = (dR_h)_g X_G(g).$$

□

Proof of Palais's theorem. Consider the Lie algebra anti-homomorphism

$$\mathfrak{g} \to \Gamma^\infty(G \times M), \quad X \mapsto (X^R, X_M),$$

where X^R is the right-invariant vector field on G generated by X_e. The image of this map is an integrable distribution of dimension $\dim G$ on $G \times M$. Let $\mathcal{L}_m$ be the maximal connected integrable submanifold containing the point (e, m). Projecting to the first factor, we get a smooth map $\pi_m : \mathcal{L}_m \to G$ whose tangent map takes (X^R, X_M) to X^R. Here we identify T_eG, and thus $\mathfrak{g}$, with the set of right-invariant vector fields on G. Since the tangent map is an isomorphism, the map π_m is a local diffeomorphism. We claim that π_m is a diffeomorphism.

Since G is simply connected, it is enough to show that π_m is a covering map. Let $U_0 \subset \mathfrak{g}$ be a star-like neighborhood of 0 over which the exponential map is a diffeomorphism, and let $U = \exp(U_0)$. Given $(g, m') \in \mathcal{L}_m$ and $X \in U_0$, the curve $t \mapsto \exp(tX)g$ is an integral curve of X^R. Let Φ_t^X be the flow of X_M, then it follows that $t \mapsto (\exp(tX)g, \Phi_t^X(m'))$ is an integral curve of (X^R, X_M), which has to lie in the leaf $\mathcal{L}_m$. Thus the set

$$\{(\exp(X)g, \Phi_1^X(m')) \mid X \in U_0\}$$

is an open neighborhood of (g, m') in $\mathcal{L}_m$ and is mapped homeomorphically under π_m onto the right translation Ug. This also implies π_m is surjective, since G is generated by U. Finally such open neighborhoods does not intersect: in fact if (g, m') and $((g, m'')$ are two points in $\mathcal{L}_m$ so that such neighborhood intersect, then one can find X and X' in U_0 so that $\exp(X)g = \exp(X')g$ and $\Phi_1^X(m') = \Phi_1^{X'}(m'')$. Then we get $\exp(X) = \exp(X')$ and thus $X = X'$, and thus $m' = m''$. In conclusion, we prove that π_m is a covering map, and thus a diffeomorphism.

Finally we define the G action on M by

$$g \cdot m = \mathrm{pr}_2(\pi_m^{-1}(g)),$$

where pr_2 denotes the projection to the second factor map. The previous arguments also shows that this is a Lie group action. In fact, suppose $g = g_1 g_2 \cdot g_k$, where $g_i = \exp(X_i)$ with $X_i \in U_0$. Then both $(g, g \cdot m)$ and $(g_1 g_2 \cdots g_k, g_1 \cdot g_2 \cdot \cdots \cdot g_k \cdot m)$ are points in $\mathcal{L}_m$ that is maped to $g = g_1 g_2 \cdots g_k$ under the map π_m. So we must have $g \cdot m = g_1 \cdot g_2 \cdot \cdots \cdot g_k \cdot m$. This implies the map τ defined $\tau(g)(m) := g \cdot m$ is smooth in g and satisfies the group law $\tau(gh) = \tau(g)\tau(h)$. The smoothness in m follows from the smooth dependence of integral curve with respect to initial data. So τ is a Lie group action of G on M. By construction $d\tau = \varphi$. The uniqueness follows from 2.4. This completes the proof. $\square$

3. Orbits and Stabilizers

Definition 3.1. Let $\tau : G \to \mathrm{Diff}(M)$ be a smooth action.

(1) The *orbit* of G through $m \in M$ is

$$G \cdot m = \{g \cdot m \mid g \in G\}.$$

(2) The *stabilizer* (also called the *isotropic subgroup*) of $m \in M$ is the subgroup

$$G_m = \{g \in G \mid g \cdot m = m\}.$$

Example. Let S^1 acts on $\mathbb{R}^2$ by rotations centered at the origin. Then the orbit of this action through the origin is the orign itself, and the orbit through any other point is the circle centered at the origin passing that point. The stabilizer of the origin is S^1 itself, while the stabilizer of any other point is the trivial group $\{e\}$.

Proposition 3.2. *Let $\tau : G \to \mathrm{Diff}(M)$ be a smooth action, $m \in M$. Then*

(1) *The orbit $G \cdot m$ is an immersed submanifold.*
(2) *The stabilizer G_m is a Lie subgroup of G, with Lie algebra*

$$\mathfrak{g}_m = \{X \in \mathfrak{g} \mid X_M(m) = 0\}.$$

Proof. (1) Consider the evaluation map

$$\mathrm{ev}_m : G \to M, g \mapsto g \cdot m.$$

It is equivariant with respect to the left G action on G and the G action on M: $\mathrm{ev}_m \circ L_g = \tau_g \circ \mathrm{ev}_m$. Taking derivatives of both sides, we get

$$(d\mathrm{ev}_m)_{gh} \circ (dL_g)_h = (d\tau_g)_{h \cdot m} \circ (d\mathrm{ev}_m)_h.$$

Since $(dL_g)_h$ and $(d\tau_g)_{h \cdot m}$ are both bijective, the rank of $(d\mathrm{ev}_m)_{gh}$ equals the rank of $(d\mathrm{ev}_m)_h$ for any g and h. It follows that the map ev_m is of constant rank. By the constant rank theorem, its image, $\mathrm{ev}_m(G) = G \cdot m$, is an immersed submanifold of M.

(2) Again consider the evaluation map

$$ev_m : G \to M, g \mapsto g \cdot m,$$

then $G_m = ev_m^{-1}(m)$, so it is a closed set in G. It is a subgroup since τ is a group homomorphism. It follows that G_m is a Lie subgroup of G. As a consequence, the Lie algebra $\mathfrak{g}_m$ of G_m is

$$\mathfrak{g}_m = \{X \in \mathfrak{g} \mid \exp(tX) \in G_m, \forall t \in \mathbb{R}\}.$$

It follows that $\exp(tX) \cdot m = m$ for $X \in \mathfrak{g}_m$. Taking derivative at $t = 0$, we get

$$\mathfrak{g}_m \subset \{X \in \mathfrak{g} \mid X_M(m) = 0\}.$$

Conversely, if $X_M(m) = 0$, then $\gamma(t) \equiv m$, $t \in \mathbb{R}$, is an integral curve of the vector field X_M passing m. It follows that $\exp(tX) \cdot m = \gamma(t) = m$, i.e. $\exp(tX) \in G_m$ for all $t \in \mathbb{R}$. So $X \in \mathfrak{g}_m$. $\square$