LECTURE 15-16: PROPER ACTIONS AND ORBIT SPACES

1. Proper actions

Suppose G acts on M smoothly, and $m \in M$. Then the orbit of G through m is

$$G \cdot m = \{g \cdot m \mid g \in G\}.$$

If m, m' lies in the same orbit, i.e. $m' = g \cdot m$ for some $g \in M$, then obviously $G \cdot m = G \cdot m'$. So M can be decomposed into a disjoint union of G-orbits. We will denote the set of G-orbits by M/G. We will always equip with the set M/G the quotient topology. This topology might be very bad in general, e.g. non-Hausdorff.

Example. Consider the natural action of $\mathbb{R}_{>0}$ on $\mathbb{R}$ by multiplication. Then there are exactly three orbits, $\{+, 0, -\}$. The open sets of the set of orbits in quotient topology are $\{+\}, \{-\}, \{+, 0, -\}$ and the empty set. So the quotient is not Hausdorff.

In what follows we will put conditions on the action to make the quotient Hausdorff, and even a manifold.

Definition 1.1. An action τ of Lie group G on M is *proper* if the action map

$$F : G \times M \to M \times M, \quad (g, m) \mapsto (g \cdot m, m)$$

is proper, i.e. the pre-image of any compact set is compact.

Proposition 1.2. *If G acts on M properly, the quotient M/G is Hausdorff.*

Proof. Suppose to the contrary, the orbits of p and q cannot be separated in M/G. Let U_n and V_n be balls in M of radius $1/n$ around p and q respectively. Then $G \cdot U_n$ intersects $G \cdot V_n$ for all n. In other words, there exists $p_n \in U_n$, $q_n \in V_n$ and $g_n \in G$ such that $p_n = g_n \cdot q_n$. Now the sequence $(p_n, q_n) = (g_n \cdot q_n, q_n)$ lies in the image of the action map F which is proper. It follows that its limit point (p, q) also lies in the image of F, i.e. $p = g \cdot q$ for some $g \in G$. In particular, the orbits of p and q coincide. $\square$

Proposition 1.3. *A smooth action of G on M is proper if and only if for every compact subset $K \subset M$, the set*

$$G_K = \{g \in G \mid (g \cdot K) \cap K \neq \emptyset\}$$

is compact in G.

Proof. By definition,

$$G_K = \mathrm{pr}_1(F^{-1}(K \times K))$$

where pr_1 is the projection $G \times M \to G$. So if the action is proper and $K \subset M$ is compact, G_K must be compact.

Conversely, suppose G_K is compact for any compact set K. If $L \subset M \times M$ is compact, then $K = \pi_1(L) \cup \pi_2(L) \subset M$ is compact. It follows that

$$F^{-1}(L) \subset F^{-1}(K \times K) \subset G_K \times K$$

is a closed set in a compact set $G_K \times K$, and thus compact. $\square$

By definition it is easy to see that for any compact set $K \subset M$, G_K is always closed in G. Thus we have

Corollary 1.4. *If G is compact, any smooth G-action is proper.*

Obviously if the G acts on M properly, then for any $m \in M$, the evaluation map

$$ev_m : G \to M, g \mapsto g \cdot m$$

is proper, since for any compact subset $K \subset M$, we have

$$\mathrm{ev}_m^{-1}(K) \subset F^{-1}(K \times \{m\}).$$

In particular, the stabilizer $G_m = \{g \in G \mid g \cdot m = m\}$ is compact.

Theorem 1.5. *Suppose G acts on M properly. Then each orbit $G \cdot m$ is an embedded closed submanifold of M, with*

$$T_m(G \cdot m) = \{X_M(m) \mid X \in \mathfrak{g}\}.$$

Proof. Since the evaluation map ev_m is proper, its image $G \cdot m$ is closed in M. (Basic topology: any continuous proper map to a Hausdorff topological space that is also first countable must be a closed map.)

We have already seen that $G \cdot m$ is an immersed submanifold of M, so there is a neighborhood U in of e in G such that $\mathrm{ev}_m(U)$ is an embedded submanifold of M near m. Since $\mathrm{ev}_m(U) = \mathrm{ev}_m(UG_m)$, we may replace U by UG_m so that $U \supset G_m$. Since $G \cdot m = G \cdot m'$ for any $m' \in G \cdot m$, to show $G \cdot m = \mathrm{ev}_m(G)$ is an embedded submanifold of M, it suffices to show that there exists a small neighborhood W of m in M such that $W \cap \mathrm{ev}_m(G) = W \cap \mathrm{ev}_m(U)$. We proceed by contradiction. Suppose there is no small neighborhood W of m in M such that $W \cap \mathrm{ev}_m(G) = W \cap \mathrm{ev}_m(U)$, then there exists a sequence of points $g_k \cdot m$ converging to m with $g_k \notin U$. The set $\{(m, g_k \cdot m)\} \cup \{(m, m)\}$ is compact. So by properness, the sequence $\{(g_k, m)\}$ is contained in a compact set. By passing to a subsequence we may assume $g_k \to g_\infty$. Since $g_k \cdot m \to m$, we see $g_\infty \cdot m = m$, i.e. $g_\infty \in G_m \subset U$. Since U is open, we conclude that $g_k \in U$ for large k, contradiction.

Finally since $G \cdot m$ is a submanifold, its the tangent space should be the image of the tangent map, i.e. $T_m(G \cdot m) = (d\mathrm{ev}_m)_e(\mathfrak{g})$. However, for any $X \in \mathfrak{g}$ and any $f \in C^\infty(M)$,

$$(d\mathrm{ev}_m)_e(X)(f) = X_e(f \circ \mathrm{ev}_m) = \frac{d}{dt}\bigg|_{t=0} (f \circ \mathrm{ev}_m)(\exp tX) = \frac{d}{dt}\bigg|_{t=0} f(\exp tX \cdot m) = X_M(m)(f),$$

so the image of the tangent map $(d\mathrm{ev}_m)_e$ is $\{X_M(m) \mid X \in \mathfrak{g}\}$. $\square$

2. Proper Free Actions

Now suppose G acts on M properly. Then M/G is Hausdorff. Still it is possible that the orbit space M/G is not a manifold. For example, the orbit space of the S^1-action on S^2 by rotations is a line segment with end points, while the orbit space of the S^1-action on $\mathbb{R}^2$ by rotations is a ray with end point. Note that in both cases, the end points correspond to a point in the manifold whose stabilizer is not $\{e\}$.

Definition 2.1. Suppose G acts on M smoothly. We say the action is *free* if $G_m = \{e\}$ for all $m \in M$.

The major theorem we want to prove is

Theorem 2.2. *Suppose $\tau : G \to \mathrm{Diff}(M)$ is a proper free action, then the orbit space M/G is a manifold and the quotient map $\pi : M \to M/G$ is a submersion.*

Proof. Since the action is free, the map ev_m is bijective. Moreover, $(d\mathrm{ev}_m)_e$ is injective, since

$$\ker(d\mathrm{ev}_m)_e = \{X \in \mathfrak{g} \mid X_M(m) = 0\} = \mathrm{Lie}(G_m) = \{0\}.$$

Since ev_m is a constant rank map, according to Sard's theorem $(d\mathrm{ev}_m)_g$ has to be surjective for all g. It follows $(d\mathrm{ev}_m)_e$ is invertible, and in particular, $T_m(G \cdot m) \simeq \mathfrak{g}$. Choose a submanifold $S \subset M$ with $m \in S$ such that

$$T_m(G \cdot m) \oplus T_m S = T_m M.$$

We claim that the evaluation map $\mathrm{ev} : G \times M \to M$ restricts to a smooth map $\mathrm{ev} : G \times S \to M$ whose tangent map at the point (e, m) is

$$(d\mathrm{ev})_{e,m}(X, Y) = (d\mathrm{ev}_m)_e X_e + Y_m$$

which is invertible. In fact, to check the above formula, we take any $f \in C^\infty(M)$,

$$(d\mathrm{ev})_{e,m}(X, Y)(f) = (X, Y)_{e,m}(f \circ \mathrm{ev}) = X(f \circ \mathrm{ev}_m) + Y(f \circ \tau_e) = (d\mathrm{ev}_m)_e X(f) + Y(f).$$

The invertibility follows from the uniqueness of the direct sum decomposition above together with the invertibility of the map $(d\mathrm{ev}_m)_e : \mathfrak{g} \to T_m(G \cdot m)$.

By continuity, the tangent map $d\mathrm{ev}$ is still invertible at (e, m') for $m' \in S$ close to m. By choosing S small we may assume this is the case for all points in S. It follows from the commutative diagram (equivariance!)

$$\begin{array}{ccc} G \times S & \xrightarrow{\mathrm{ev}} & M \\ \big\downarrow{\scriptstyle L_{g_1} \times \mathrm{Id}} & & \big\downarrow{\scriptstyle \tau_{g_1}} \\ G \times S & \xrightarrow{\mathrm{ev}} & M \end{array}$$

that $\mathrm{ev} : G \times S \to M$ has invertible tangent map everywhere, and thus is a local diffeomorphism onto its image.

We claim that this is in fact a diffeomorphism onto its image if we choose S small enough. If not, we can choose two sequences (g_1^k, m_1^k) and (g_2^k, m_2^k) such that $m_1^k \to m$,

$m_2^k \to m$ and $g_1^k \cdot m_1^k = g_2^k \cdot m_2^k$. Note that if we denote $g^k = (g_1^k)^{-1} g_2^k$, then $g^k \cdot m_2^k = m_1^k$. Since ev is bijective near (e, m), there is a neighborhood U of e in G so that $g^k \notin U$ for all k. Now the set $\{(g^k \cdot m_2^k, m_2^k)\} \cup \{(m, m)\}$ is compact in $M \times M$. By properness, the sequence $\{(g^k, m_2^k)\}$ is contained in a compact set, thus has a converging subsequence. In other words, the sequence $\{g^k\}$ has a convergent subsequence. However, the only limit point of the set $\{g^k\}$ has to be e since the limit point must take m to m, and the action is free. This contradicts with the fact that no g^k lies in U.

Now denote by V the image of $G \times S$ under ev in M. Then the diffeomorphism $\mathrm{ev} : G \times S \to V \subset M$ identifies V/G with S, hence gives a manifold structure on M/G near the orbit $G \cdot m$. Obviously the quotient map $\pi : M \to M/G$ is a submersion. $\square$

Notice that if H is a closed Lie subgroup in G, then one can prove that the H-action on G defined by

$$\tau_h : G \to G, \quad g \mapsto gh^{-1}$$

is proper and free. So we have

Corollary 2.3. *Let G be a Lie group and H a closed subgroup. Then the quotient G/H is a manifold, with the quotient map $\pi : G \to G/H$ a smooth submersion.*

Proposition 2.4. *Suppose G acts on M smoothly. Then the map*

$$\Phi : G/G_m \to G \cdot m, \quad G_m \cdot g \mapsto g \cdot m$$

is a diffeomorphism between the quotient G/G_m and the orbit $G \cdot m$.

Proof. Φ is obviously surjective. It is injective since if $G_m \cdot g_1 \neq G_m \cdot g_2$, then $g_1 g_2^{-1} \notin G_m$, so $g_1 \cdot m \neq g_2 \cdot m$. So it is a bijective map from G/G_m to its image, $G \cdot m$.

Note that the tangent space of G/G_m at $G_m \cdot e$ is $\mathfrak{g}/\mathfrak{g}_m$. According to $\mathrm{ev}_m = \Phi \circ \pi$, we have $(d\mathrm{ev}_m)_e = (d\Phi)_{G_m \cdot e} \circ (d\pi)_e$. It follows $(d\Phi)_{G_m \cdot e}(X + \mathfrak{g}_m) = X_M(m)$. In other words, $d\Phi$ is bijective at $G_m \cdot e$. By equivariance,

$$\begin{array}{ccc} G/G_m & \xrightarrow{L_g} & G/G_m \\ \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \Phi} \\ G \cdot m & \xrightarrow{\tau_g} & G \cdot m \end{array}$$

$d\Phi$ is bijective everywhere. If follows that Φ is locally diffeomorphism everywhere. Since it is bijective, it is globally diffeomorphism. $\square$

3. Homogeneous Spaces

Let Lie group G acts smoothly on M.

Definition 3.1. The G-action is *transitive* if there is only one orbit, i.e. $M = G \cdot m$ for any $m \in M$.

Definition 3.2. Let G be a Lie group. A *homogeneous G-space*, or a *homogeneous space*, is a manifold M endowed with a transitive smooth G-action.

Example. The natural action of $O(n)$ on $S^{n-1} \subset \mathbb{R}^n$ is transitive. (Gram-Schmidt procedure.) So S^{n-1} is a homogeneous space.

Example. Let $H \subset G$ be a closed subgroup. Then the left G-action on the quotient manifold G/H is transitive. It follows that G/H is a homogeneous G-space.

It turns out that the last example above is the *universal* example. In fact, according to proposition 2.4,

Theorem 3.3. *Let M be a homogeneous G-space, $m \in M$. Then the map*

$$F_m : G/G_m \to M, \quad G_m \cdot g \mapsto g \cdot m$$

is an diffeomorphism.

It follows that to study homogeneous spaces of a Lie group G, it suffices to study its closed subgroups. One can take G/H as definitions of homogeneous spaces.

Example. Consider again the natural $O(n)$ action on S^{n-1}. If we choose the base point in S^{n-1} to be the north pole $N = (0, \cdots, 0, 1)$, it is easy to see that the isotropy group $G_N = O(n-1)$. We conclude

$$S^{n-1} \simeq O(n)/O(n-1).$$

Example. Let $k < n$. The manifold of all k-dimensional linear subspaces in $\mathbb{R}^n$ is the *Grassmann manifold* $\mathrm{Gr}_k(\mathbb{R}^n)$. Obviously $O(n)$ acts transitively on $\mathrm{Gr}_k(\mathbb{R}^n)$, and the isotropy group of $\hat{\mathbb{R}}^k$ is $O(k) \times O(n-k)$. It follows that

$$\mathrm{Gr}_k(\mathbb{R}^n) \simeq O(n)/(O(k) \times O(n-k)).$$

Note that in the special case $k = 1$, $\mathrm{Gr}_1(\mathbb{R}^n) = \mathbb{RP}^{n-1}$ is the real projective space. Similar the complex Grassmann manifolds and the complex projective spaces are homogeneous $U(n)$-spaces.