LECTURE 17-18: THE TUBE THEOREM AND ITS APPLICTIONS

1. Associated Bundles

Let G be a Lie group. Recall that a right G-action on a manifold P is a Lie group anti-homomorphism $\hat{\tau} : G \to \mathrm{Diff}(M)$, i.e. $\hat{\tau}_{gh} = \hat{\tau}_h \circ \hat{\tau}_g$. For example, let U be any manifold, then

$$h \cdot (u, g) := (u, gh)$$

defines a right G-action on $U \times G$.

Definition 1.1. A *principal G-bundle* over a manifold M is a manifold P together with a free right G-action on P and a fibration map $\pi : P \to M$ such that for every $m \in M$,

(1) each fiber $\pi^{-1}(m)$ is a G-orbit,
(2) there exists a neighborhood U of m in M and a diffeomorphism $\phi : \pi^{-1}(U) \to U \times G$ that sends $\pi^{-1}(m)$ to the fiber $\{m\} \times G$,
(3) ϕ is equivariant with respect to the right G-action on $\pi^{-1}(U)$ and the right G-action on $U \times G$ described above: $\phi(p \cdot g) = \phi(p) \cdot g$.

Example. Let $E \to M$ be any vector bundle of rank k. Its *frame bundle* is the principal $\mathrm{GL}(k)$-bundle whose fiber over $m \in M$ is the set of linear isomorphisms $g : \mathbb{R}^k \to E_m$, i.e. the set of basis of E_m, with $A \in \mathrm{GL}(k)$ acting by $g \mapsto g \circ A$.

Example. Suppose G acts on M properly and freely, then $\pi : M \to M/G$ makes M into a principal G-bundle over M/G. Here the right G-action on M is

$$\hat{\tau} : G \to \mathrm{Diff}(M), \quad \hat{\tau}(g)(m) := g^{-1} \cdot m.$$

Now suppose Lie group G acts properly and freely on P and makes P a principal G-bundle over M. Moreover, suppose G also acts linearly on a vector space W. Then G acts on the product $P \times W$ by

$$g \cdot (p, w) := (p \cdot g^{-1}, g \cdot w).$$

This action is obviously a free and proper (left) G-action on $P \times W$.

Definition 1.2. The *associated bundle* (with respect to previous data) is

$$P \times_G W := (P \times W)/G.$$

We will denote by $[p, w]$ the equivalence class of $(p, w) \in P \times W$ in $P \times_G W$. The projection map $\pi : P \to M$ induces a map $P \times_G W \to M$ which sends $[p, w]$ to $\pi(p)$.

Proposition 1.3. *The associated bundle $P \times_G W$ is a vector bundle over M.*

Proof. Let $U \subset M$ be an open set whose preimage in P under the map $\pi : P \to M$ is $U \times G$. Then the preimage of U in $P \times_G W$ under the map $P \times_G W \to M$ is $(U \times G) \times_G W = U \times W$. $\square$

Remark. Every vector bundle can be obtained as an associated bundle. In fact, if $E \to M$ is a rank k vector bundle, then $E = P \times_{\mathrm{GL}(k)} \mathbb{R}^k$, where P is the frame bundle defined above.

2. The Tube Theorem

Let G be a Lie group acts properly on a manifold M. Recall that for any $m \in M$, the stabilizer of m is

$$G_m = \{g \in G \mid g \cdot m = m\}.$$

Taking the differential of τ_g at m, we get the *isotropy action* of G_m on T_mM via

$$g \cdot v := d\tau_g(v).$$

(Note: $d\tau_g$ is the true differential of the smooth map $\tau_g : M \to M$ which sends vectors in T_mM to $T_{g\cdot m}M = T_mM$, not the "formal differential" that we used to define the infinitesimal action.)

Consider the orbit $G \cdot m$. We have seen that this is an embedded submanifold in M, and the map

$$F : G/G_m \to G \cdot m, \quad gG_m \mapsto g \cdot m$$

is a diffeomorphism between the quotient G/G_m and the orbit $G \cdot m$.

Obviously the tangent space $T_m(G \cdot m)$ is a subspace of T_mM which is invariant under the isotropic G_m-action. Since the G-action on M is proper, G_m is compact. Hence there exists a G_m invariant decomposition,

$$T_mM = T_m(G \cdot m) \oplus W,$$

where W is orthogonal to the orbit. (For example, we can fix a G_m-invariant inner product on T_mM and take W to be the orthogonal complement of $T_m(G \cdot m)$ in T_mM. Such an inner product exists since G_m is compact: one can take an arbitrary inner product and then average it over G_m using Haar measure.)

Since G_m is a closed subgroup in G, G is a principal G_m-bundle over G/G_m. Since G_m acts on W, we can take D be a small disc in W around the origin with respect to some G_m-invariant metric so that G_m also acts on D. From this we can form the associated *disc* bundle $G \times_{G_m} D$ over G/G_m. (So locally for small open set $U \subset G/G_m$ the bundle looks like $U \times D$.) Obviously the left G-action on G give rise to a G-action on $G \times_{G_m} D$.

Theorem 2.1 (The Tube Theorem)**.** *Let G be a Lie group acts properly on a manifold M, $m \in M$. Then there exists a G-equivariant diffeomorphism from the disc bundle $G \times_{G_m} D$ onto a G-invariant neighborhood of the orbit $G \cdot m$ in M, whose restriction to the zero section $G \times_{G_m} \{0\} = G/G_m$ is the diffeomorphism $F : G/G_m \to G \cdot m$ described above.*

We note that for the extremal case that the action is also free (so that $G_m = \{e\}$), the theorem is already proven in previous lecture. Before we prove the tube theorem, we will first prove another extremal case where $G_m = G$ (so in particular G is compact), i.e. m is a fixed point of the G-action.

Theorem 2.2 (The Local Linearization Theorem). *Let G be a compact Lie group acting on a manifold M and let $m \in M^G$ be a fixed point. Then there exists a G-equivariant diffeomorphism from a neighborhood of the origin in T_mM onto a neighborhood of m in M.*

Proof. Let U be an invariant neighborhood of m in M, and let $f : U \to T_mM$ be any smooth map whose differential at m is the identity map on T_mM. Consider the average

$$F : U \to T_mM, \quad u \mapsto F(u) = \int_G (d\tau_g)_m(f(g^{-1} \cdot u))dg,$$

where dg is the haar measure on G. (We will study the details of Haar measure later.) Then for any $g_1 \in G$, since $d(g_1g) = dg$,

$$F(g_1 \cdot u) = \int_G (d\tau_g)_m f(g^{-1}g_1 \cdot u)dg = \int_G d\tau_{g_1g} f(g^{-1} \cdot u)d(g_1g) = d\tau_{g_1}F(u).$$

In other words, F is equivariant with respect to the isotropy G-action on T_mM and the given G action on U. Moreover, since

$$d((d\tau_g)_m \circ f \circ \tau_g^{-1})_m = (d\tau_g)_m \circ df_m \circ (d\tau_g^{-1})_m = \mathrm{Id}$$

for all $g \in G$, we claim that dF_m is the identity map. So by inverse function theorem, F is a diffeomorphism near m. □

Proof of the tube theorem: Since G acts on M properly, the stabilizer G_m is compact, acting on M smoothly, and has m as a fixed point. By the local linearization theorem above, there exists a G_m-equivariant diffeomorphism φ from a neighborhood of 0 in T_mM to a neighborhood of m in M such that $\varphi(0) = m$. Moreover, according to the proof above, one can take φ so that $d\varphi_0 = \mathrm{Id}$. Take a small disc D with respect to some G_m-invariant inner product on W as described above. Consider the map

$$\psi : G \times_{G_m} D \to M, \quad [g, v] \mapsto g \cdot \varphi(v).$$

This is well-defined for D small enough contained in the domain of φ, since if $(g_1, v_1) \sim (g_2, v_2)$, then there is some $g \in G_m$ such that $g_2 = g_1g^{-1}$ and $v_2 = g \cdot v_1$. So

$$g_2 \cdot \varphi(v_2) = g_1g^{-1} \cdot \varphi(g \cdot v_1) = g_1 \cdot \varphi(v_1).$$

Obviously this map is G-invariant. It remains to prove that ψ is a local diffeomorphism onto its image for small D. At $[e, 0]$, if we take a small neighborhood $U \subset G/G_m$ of $G_m \cdot e$, identify a small neighborhood of $[e, 0]$ in $G \times_{G_m} D$ with $U \times D$, and identify $T_{G_m \cdot e}U$ with $T_m(G \cdot m)$, then we get identification $T_{[e,0]}(G \times_{G_m} D) = T_{[e,0]}(U \times D) = T_m(G \cdot m) \oplus D$, under which the differential of ψ at $[e, 0]$ is

$$d\psi_{[e,0]}(X, Y) = X + d\varphi_0(Y) = X + Y.$$

Since the decomposition $T_mM = T_m(G \cdot m) \oplus W$ is a direct sum decomposition, $d\psi_{[e,0]}$ is bijective, and thus ψ is a local diffeomorphism at $[e, 0]$. By G-equivariance, ψ is a local diffeomorphism at all points of the form $[g, 0]$. It remains to show that ψ is bijective onto its image for D small enough.

Assume to the contrary that there exists $u_n, v_n \to 0$ in W and $g_n, h_n \in G$ such that $[g_n, u_n] \neq [h_n, v_n]$ while $g_n \cdot \varphi(u_n) = h_n \cdot \varphi(v_n)$. Without loss of generality, we may assume $h_n = e$. Then $g_n \cdot \varphi(u_n) = \varphi(v_n) \to m$. Since the action is proper, and under the action map $G \times M \to M \times M$ the sequence $(g_n, \varphi(u_n))$ is mapped to the convergent sequence $(\varphi(v_n), \varphi(u_n))$, there is a converging subsequence $g_{n_i} \to g_\infty$. Obviously the limit $g_\infty \in G_m$, so that $[g_n, u_n]$ is close to $[g_\infty, 0] = [e, 0]$ for n large. Also $[e, v_n]$ is close to $[e, 0]$ for n large, but

$$\psi([g_n, u_n]) = g_n \cdot \varphi(u_n) = e \cdot \varphi(v_n) = \psi([e, v_n]),$$

contradicts with the fact that ψ is a local diffeomorphism near $[e, 0]$. □

3. Applications

As an application of the local linearization theorem, we have

Proposition 3.1. *Suppose G acts on M properly. Then for any subgroup $H \subset G$, the fixed point set*

$$M^H = \{m \in M \mid g \cdot m = m \text{ for all } g \in H\}$$

is a disjoint union of closed submanifolds of M.

Proof. Obviously M^H is closed in M for any H. Observe that the fixed point set of H coincides with the fixed point set of its closure $\bar{H}$, and moreover, for any $m \in M^H$, $\bar{H} \subset G_m$. So without loss of generality, we may assume that H is a compact Lie subgroup of G.

Let F be a connected component of M^H, and $m \in F$ be a point. By the local linearization theorem, there exists a neighborhood U of m in M and an H-equivariant diffeomorphism of U with an open subset V of the vector space $W = T_mM$. This diffeomorphism carries $U \cap F$ to $V \cap W^H$, a linear subspace consisting of those vectors that are fixed by H. It follows that F is a submanifold. □

In particular,

Corollary 3.2. *For any vector $X \in \mathfrak{g}$, the zero set*

$$M^X = \{m \in M \mid X_M(m) = 0\}$$

is a disjoint union of closed submanifolds of M.

Proof. Let $H = \{\exp(tX) \mid t \in \mathbb{R}\}$. Then $M^X = M^H$. □

In what follows we will give more applications in geometry. We have already seen how to apply averaging trick with respect to a compact group action. To apply the same method to proper actions of non-compact groups, we need to use the following invariant partition of unity theorem. Recall that a partition of unity subordinate to an open covering $\{U_\alpha\}$ of a manifold M is a collection $\{\rho_\alpha\}$ of non-negative smooth functions such that

- $\mathrm{supp}(\rho_\alpha) \subset U_\alpha$.
- Each $p \in M$ has a neighborhood that intersects with only finitely many $\mathrm{supp}(\rho_\alpha)$.
- $\sum \rho_\alpha = 1$.

Now suppose G acts on M smoothly and each U_α is a G-invariant subset of M. A natural question is: can we choose ρ_α carefully so that each ρ_α is a G-invariant function? The answer is yes, provided the action is proper.

Theorem 3.3 (Invariant partition of unity)**.** *Suppose G acts on M properly. For every covering of M by G-invariant open sets, there exists a G-invariant partition of unity subordinate to the covering.*

Sketch of proof. First take open subsets $W_n'' \subset\subset W_n' \subset\subset W_n$ such that each W_n is contained in a tube and in some element of the given covering, and such that W_n'' also cover M. Let $V_n = (G \cdot W_n) \setminus \cup_{k<n}(G \cdot \overline{W_k''})$ and $C_n = (G \cdot \overline{W_n'}) \setminus \cup_{k<n}(G \cdot W_k')$. Then V_n is a locally finite refinement of the given covering and each V_n is G-invariant and is isomorphic to $G \times_H D$ for some compact subgroup H of G and some H-invariant open subset D of a vector space. Moreover, C_n is sitll a covering of M and the isomorphism described just now takes C_n to a set of the form $G \times_H K$ for some compact set $K \subset D$. Finally we take any smooth function on D which is positive on K and whose support is a compact subset of D, and average this function with respect to the H-action (which is possible since H is compact). This gives us a function ρ_n' that is supported on V_n and is strictly positive on C_n. Then the functions $\rho_n = \rho_n'/\sum_k \rho_k'$ form an invariant partition of unity on M, subordinate to the given covering. □

As a consequence, now we can "average" with respect to proper actions of non-compact groups.

Corollary 3.4. *Suppose G acts on M properly. Then there exists a G-invariant Riemannian metric on M.*

Proof. According to the tube theorem, each point has a G-invariant neighborhood U and a G-equivariant diffeomorphism $\psi : G \times_H D \to U$, where H is a compact subgroup of G acting on a vector space W and $D \subset W$ an invariant open subset. On G we can pick a G-invariant Riemannian metric since $TG \simeq G \times \mathfrak{g}$. On D we can pick an H-invariant Riemannian metric since H is compact (so that one can average an arbitrary initial metric). It is not hard to verify that the resulting Riemannian metric on $G \times_H D$ is G-invariant. In other words, near each orbit we can construct a G-invariant Riemannian metric. Using invariant partition of unity, one can glue them into a G-invariant inner product over all of M. $\square$