LECTURE 19: HAAR MEASURE

1. Haar Measure

Recall that to integrate a function on a manifold, one could start with a fixed volume form, which requires the manifold to be orientable. (Well, if the manifold is not orientable, one could also develop a theory of integration using a fixed *density* instead of a volume forms. We will not discuss that theory here.) Now suppose G is a Lie group. Since any Lie group is orientable (because the tangent bundle $TG \simeq G \times \mathfrak{g}$ is trivial), volume forms always exist on G. Of course we would like to choose a volume form that behaves well under the group operations.

Definition 1.1. A volume form ω on a Lie group G is called *left invariant* if $L_g^*\omega = \omega$ for all $g \in G$.

Theorem 1.2. *Left invariant volume form exists on any Lie group G, and is unique up to a multiplicative constant.*

Proof. Take any basis of T_e^*G to form an nonzero element $\omega_e \in \Lambda^n T_e^*G$. Then define an n-form ω on G by letting $\omega_g = L_{g^{-1}}^*\omega_e$. This is left-invariant since

$$(L_g^*\omega)_h = L_g^*\omega_{gh} = L_g^*L_{h^{-1}g^{-1}}^*\omega_e = (L_{h^{-1}g^{-1}} \circ L_g)^*\omega_e = L_{h^{-1}}^*\omega_e = \omega_h.$$

Moreover, suppose ω' is any left invariant volume form on G. Since $\dim \Lambda^n T_e G = 1$, there exists some non-zero constant C so that $\omega'_e = C\omega_e$. It follows from left-invariance that for any g,

$$\omega'_g = L_{g^{-1}}^*\omega'_e = CL_{g^{-1}}^*\omega_e = C\omega_g.$$

So the left invariant volume form is unique up to a multiplicative constant. □

Now suppose ω is a left invariant volume form on G. Replacing ω by $-\omega$ if necessary, we may assume ω is positive with respect to the orientation of G. This gives us a measure on G via

$$f \in C_c(G) \mapsto I(f) = \int_G f(g)\omega(g).$$

In what follows we will not distinguish this measure and the corresponding positive volume form. This measure is *left invariant* in the sense that for any $h \in G$,

$$I(f) = \int_G f\omega = \int_G L_h^*(f\omega) = \int_G (L_h^*f)\omega = I(L_h^*f).$$

In particular, for any Borel set $E \subset G$, $m(E) = m(L_h E)$.

Definition 1.3. We will call such a left invariant measure a *left Haar measure.*

So as we just proved, left Haar measure always exists on any Lie group, and is unique up to a *positive* constant. In the case G is compact, a Haar measure ω is called *normalized* if

$$\mathrm{Vol}(G) = \int_G \omega = 1.$$

In this case we will denote $\omega = dg$. Note that the left invariance means

$$d(hg) = dg,$$

or equivalently,

$$\int_G f(hg)dg = \int_G f(g)dg$$

for any fixed $h \in G$. Since the volume of a compact Lie group is always finite, and any two volume forms differ by a multiplicative constant, we immediately get

Corollary 1.4. *There exists a unique normalized left Haar measure on any compact Lie group.*

2. Modular Function

Similarly one can define the *right invariant volume forms* and *right Haar measures* on a Lie group, and prove their existence and uniqueness (up to a constant). In general a left Haar measure need not be a right Haar measure.

Example. Consider

$$G = \left\{ \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \mid x, y \in \mathbb{R}, y > 0 \right\},$$

then one can check that up to a multiplicative constant,

$$\omega_L = y^{-2} dx dy$$

is the left Haar measure on G, and

$$\omega_R = y^{-1} dx dy$$

is the right Haar measure on G.

One can check that ω_L and ω_R in previous example satisfies $\iota^* \omega_L = \omega_R$, where $\iota : G \to G$ is the inversion operation. In general,

Lemma 2.1. *Let G be a Lie group and ω a left Haar measure on G. Then $\iota^*\omega$ is a right invariant Haar measure on G.*

Proof. Using the relation $\iota \circ R_h = L_{h^{-1}} \circ \iota$, we get

$$R_h^* \iota^* \omega = (\iota \circ R_h)^* \omega = (L_{h^{-1}} \circ \iota)^* \omega = \iota^* L_{h^{-1}}^* \omega = \iota^* \omega.$$

$\square$

Left invariant Haar measures also behaves nice under right multiplications.

Lemma 2.2. *For any $g \in G$ and any left Haar measure ω, $R_g^*\omega$ is also left invariant.*

Proof. This follows from the fact that any left multiplication commutes with any right multiplication:

$$L_h^*(R_g^*\omega) = (R_g \circ L_h)^*\omega = (L_h \circ R_g)^*\omega = R_g^* L_h^*\omega = R_g^*\omega.$$

□

It follows that there exists a positive constant, $\Delta(g)$, such that

$$\omega = \Delta(g) R_g^*\omega.$$

Note that the number $\Delta(g)$ is independent of the choices of a left Haar measure ω, since any two left Haar measure differ only by a constant.

Definition 2.3. The function $\Delta : G \to \mathbb{R}^+$ is called the *modular function* of G.

Proposition 2.4. *The modular function $\Delta : G \to \mathbb{R}^+$ is a Lie group homomorphism.*

Proof. Obviously Δ is continuous. Moreover, by definition

$$\omega = \Delta(g_1 g_2) R_{g_1 g_2}^*\omega = \Delta(g_1 g_2)(R_{g_2} R_{g_1})^*\omega = \Delta(g_1 g_2) R_{g_1}^* R_{g_2}^*\omega.$$

On the other hand, we have

$$R_{g_2}^*\omega = \Delta(g_1) R_{g_1}^* R_{g_2}^*\omega,$$

and thus

$$\omega = \Delta(g_2) R_{g_2}^*\omega = \Delta(g_2)\Delta(g_1) R_{g_1}^* R_{g_2}^*\omega.$$

It follows $\Delta(g_1 g_2) = \Delta(g_1)\Delta(g_2)$. □

Example. In the previous example, one can check that $\Delta\left(\begin{pmatrix} b & a \\ 0 & 1 \end{pmatrix}\right) = b$. Note that this implies $\Delta(g)\omega_L(g) = \omega_R(g) = (\iota^*\omega_L)(g)$. This is actually true for any Lie group:

Lemma 2.5. *For any left invariant Haar measure ω on G, $\iota^*\omega = \Delta(g)\omega(g)$.*

Proof. We first prove $\Delta(g)\omega(g)$ is right invariant:

$$R_h^*(\Delta(g)\omega(g)) = \Delta(gh)(R_h^*\omega)(g) = \Delta(g)\Delta(h)(R_h^*\omega)(g) = \Delta(g)\omega(g).$$

It follows that there exists a positive constant C so that

$$\Delta(g)\omega(g) = C(\iota^*\omega)(g).$$

It remains to show $C = 1$. This follows from the fact

$$\omega(g) = \Delta(g^{-1})(C(\iota^*\omega)(g)) = C\iota^*(\Delta\omega)(g) = C^2(\iota^*\iota^*\omega)(g) = C^2\omega(g).$$

□

As a consequence, we see that for any $f \in C_c(G)$ and any left Haar measure,

$$\int_G f(g^{-1})\omega(g) = \int_G f(g)\Delta(g)\omega(g).$$

We are interested in those Lie groups whose left Haar measure are also right invariant.

Definition 2.6. G is called *unimodular* if $\Delta(g) \equiv 1$ for any $g \in G$.

Note that by definition, a Lie group is unimodular if and only if every left Haar measure is also a right Haar measure. So we can speak of "Haar measure" on unimodular Lie groups, without indicating left or right.

Example. Any commutative Lie group is unimodular.

Theorem 2.7. *Any compact Lie group is unimodular.*

Proof. If G is compact, the image $\Delta(G)$ of G is a compact subgroup of $\mathbb{R}^+$. However, the only compact subgroup of $\mathbb{R}^+$ is $\{1\}$. The theorem follows. $\square$

In particular, we see

Corollary 2.8. *The normalized Haar measure dg on a compact Lie group is left invariant, right invariant and invariant under inversion, i.e.*

$$\int_G f(hg)dg = \int_G f(gh)dg = \int_G f(g^{-1})dg = \int_G f(g)dg.$$