LECTURE 20: BASIC REPRESENTATION THEORY

1. Representations of Lie Groups and Lie Algebras

We have studied the Lie group actions on smooth manifolds. A very important special case is that the manifolds are linear vector spaces, and the Lie group actions are also linear:

Definition 1.1. A *representation* of a Lie group G is a pair (V, π), where V is a vector space, and $\pi : G \to \mathrm{GL}(V)$ is a linear G-action on V.

Remarks. (1) Although a representation contains a pair (V, π), we will always use V or π to denote the representation. Again we will write $g \cdot v$ instead of $\pi(g)v$ for simplicity.

(2) In this definition we did not specify whether G and V are real or complex. Usually we will take V to be a complex vector space, even if G is real Lie group. (In general complex representations are much simpler than real ones, since complex matrices theory are much easier than real matrices theory.)

(3) In this definition V might be infinite dimensional, e.g. a Hilbert space. But we will mainly concentrate on finite dimensional representations in this course. The dimension of V is called the *dimension* of this representation.

Example. The following are some examples of representations.

- The trivial representation: G is arbitrary, $V = \mathbb{C}$, and $\pi(g) = \mathrm{Id}$ for any $g \in G$.
- The standard representation for linear Lie groups: G is a linear Lie group, and π is the inclusion map $\pi : G \hookrightarrow \mathrm{GL}(n, \mathbb{C})$.
- The adjoint representation: G is arbitrary, $V = \mathfrak{g}$, and $\pi(g) = \mathrm{Ad}_g$.
- The coadjoint representation: G is arbitrary, $V = \mathfrak{g}^*$, and $\pi(g) = \mathrm{Ad}_g^*$.
- Let G acts on a smooth manifold M, and $F(M)$ be the space of complex-valued functions on M. Then the action induces a representation of G on $F(M)$ by
$$g \cdot \varphi(x) = \varphi(g^{-1}x).$$
In particular if $M = G$ and the G-action on G is the left action, we call this *left regular representation* of G.

Definition 1.2. Let V, W be two representations of G. A *G-morphism* (or an *intertwining map*) between them is a linear G-equivariant map $f : V \to W$, i.e. f is linear and
$$f(g \cdot v) = g \cdot f(v)$$
for all $g \in G$ and $v \in V$. The set of all such morphisms is denoted by $\mathrm{Hom}_G(V, W)$. As usual, an invertible morphism is called an *isomorphism*, and isomorphic representations are always called *equivalent* representations.

One of the central problems in algebra is to classify all the objects with the same algebraic structure. In representation theory we have

Basic problem: Given G, classify all its representations up to isomorphism.

The Lie algebra representations are defined by the same way:

Definition 1.3. A *representation* of a Lie algebra $\mathfrak{g}$ is a pair (V, ρ), where V is a vector space and $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ a Lie algebra morphism. (In algebra a $\mathfrak{g}$-representation V is also called a $\mathfrak{g}$-*module*.)

Example. The differential of the adjoint representation G on $\mathfrak{g}$ gives the adjoint representation of $\mathfrak{g}$ on $\mathfrak{g}$: for any $X \in \mathfrak{g}$, $\mathrm{ad}(X) \in \mathrm{End}(\mathfrak{g})$ maps Y to $[X, Y]$.

Similarly one can define $\mathfrak{g}$-morphisms of representations of Lie algebra. The set of $\mathfrak{g}$-morphisms between V and W is denoted by $\mathrm{Hom}_{\mathfrak{g}}(V, W)$.

According to Lie group-Lie algebra correspondence, we have

Proposition 1.4. *Let G be a Lie group with Lie algebra $\mathfrak{g}$.*

(1) *Every representation $\pi : G \to \mathrm{GL}(V)$ of G defines a representation $\rho = d\pi : \mathfrak{g} \to \mathfrak{gl}(V)$ of $\mathfrak{g}$. Moreover, any morphism between representations of G is automatically a morphism between the corresponding representations of $\mathfrak{g}$.*
(2) *If G is connected and simply connected, then any representation of $\mathfrak{g}$ can be uniquely lifted to a representation of G. Moreover,* $\mathrm{Hom}_G(V, W) = \mathrm{Hom}_{\mathfrak{g}}(V, W)$.

2. Operations on Representations

It is well known that given several vector spaces, one can produce many other vector spaces using algebraic operations such as direct sum. Similarly given several representations, one can produce new representations using these algebraic operations.

♠ Subrepresentations and Quotient representations

Definition 2.1. Let V be a representation of G. A *subrepresentation* of V is a G-invariant linear subspace $W \subset V$ together with the restriction of π to W.

Suppose W is a subrepresentation of V. Then since W is a linear subspace of V, one can form the quotient space V/W. It follows from the G-invariance of W that the G-action on V descends to a G-action on V/W by

$$g \cdot (v + W) = g \cdot v + W.$$

This gives a representation of G on the quotient V/W, and is called the *quotient representation* of V under W.

Similarly one can define the subrepresentations and quotient representations of a given $\mathfrak{g}$-representation. It is easy to check that if G is a connected Lie group, and V a representation of G, then $W \subset V$ is a subrepresentation of V if and only if it is a subrepresentation of $\mathfrak{g}$.

♣ Direct Sum, Tensor Product and Dual

Recall that if V, W are vector spaces of dimension n and m respectively, with basis $\{v_1, \cdots, v_n\}$ and $\{w_1, \cdots, w_m\}$, then

- $V \oplus W$ is a vector space of dimension $n+m$. A basis of $V \oplus W$ is given by $\{v_1, \cdots, v_n, w_1, \cdots, w_n\}$.
- $V \otimes W$ is a vector space of dimension nm. A basis of $V \otimes W$ is given by $\{v_i \otimes w_j : 1 \le i \le n, 1 \le j \le m\}$. In general,
$$(\sum a_i v_i) \otimes (\sum b_j w_j) = \sum a_i b_j v_i \otimes w_j.$$
- The dual V^* is a vector space of dimension n. A basis of V^* is given by the dual basis v_i^*, where $v_i^*(v_j) = \delta_j^i$.
- The set of linear maps $\mathrm{Hom}(V, W)$ is a vector space of dimension mn. In fact, $\mathrm{Hom}(V, W) = W \otimes V^*$.

Now we will extend these operations to representations. Suppose V and W be two representations of G or $\mathfrak{g}$, then

- The *direct sum* $V \oplus W$ is a representation of G or $\mathfrak{g}$: The G-action on $V \oplus W$ is given by
$$g \cdot (v, w) := (g \cdot v, g \cdot w).$$
Similarly one can define the $\mathfrak{g}$-action on $V \oplus W$.
- The *tensor product* $V \otimes W$ is a representation of G if we define
$$g \cdot (v \otimes w) := g \cdot v \otimes g \cdot w.$$
However, the $\mathfrak{g}$-action on $V \otimes W$ is tricker. For example, one cannot define the $\mathfrak{g}$-action to be $X \cdot (v \otimes w) = X \cdot v \otimes X \cdot w$ as before, since this is not even linear in X. Note that the $\mathfrak{g}$-action is the differential of the G-action, we have
$$X(v \otimes w) = \frac{d}{dt}\Big|_{t=0} (e^{tX} v \otimes e^{tX} w) = (X \cdot v) \otimes w + v \otimes (X \cdot w).$$
- The *dual representation* V^* of G is defined such that for any $v^* \in V^*$ and any $v' \in V$, (note that this is a special case of G-representation on $F(V)$ that we just discussed)
$$(g \cdot v^*)(v') = v^*(g^{-1} \cdot v').$$
If we denote the natural pairing between V and V^* by $\langle \cdot, \cdot \rangle$, then for any $v' \in V$ and $v^* \in V^*$, we have
$$\langle g \cdot v', g \cdot v^* \rangle = \langle v', v^* \rangle.$$
Similarly the $\mathfrak{g}$-action on V^* is defined by
$$(X \cdot v^*)(v') = v^*(-X \cdot v'),$$
and by using the symbol $\langle \cdot, \cdot \rangle$,
$$\langle X \cdot v', v^* \rangle + \langle v', X \cdot v^* \rangle = 0.$$

- It follows that the space of linear maps $\mathrm{Hom}(V, W) = W \otimes V^*$ admits a natural structure as a representation of G and $\mathfrak{g}$. More explicitly, the G-action on $\mathrm{Hom}(V, W)$ is given by
$$(g \cdot f)(v) = g \cdot (f(g^{-1} \cdot v)),$$
and its differential is
$$(X \cdot f)(v) = X \cdot f(v) - f(X \cdot v).$$

3. Irreducible Representations

To classify all the representations of a given Lie group G, it is natural to study the simplest possible representations.

Definition 3.1. Let (V, π) be a representation of Lie group G. V is *irreducible* (or *simple*) if it has no subrepresentations other than 0 and V.

Example. Any one dimensional representation is irreducible.

Example. The standard representation of $\mathrm{SO}(n)$ on $\mathbb{R}^n$ is irreducible.

If a representation is not irreducible, then it has a nontrivial subrepresentation W, which give us a short exact sequence
$$0 \to W \to V \to V/W \to 0$$
of representations. A natural question is: whether this exact sequence splits, i.e. whether $V = W \oplus V/W$. If this is the case, then W and V/W are built-up blocks of the representation V, which are simpler then V since the dimensions are lower.

Definition 3.2. A finite dimensional representation V of G is called *completely reducible* (or *semi-simple*) if it is isomorphic to a direct sum of irreducible representations.

Modified Basic Problems:

(1) Classify all the irreducible representations of a given Lie group G.
(2) Decompose any completely reducible representation into a direct sum of irreducible representations.
(3) Determine for which G all representations are completely irreducible.

Definition 3.3. A representation (V, π) of G is *unitary* if V admits a G-invariant positive-definite Hermitian inner product, i.e. $\pi(g)$ is unitary for any $g \in G$.

Proposition 3.4. *Any unitary representation is completely reducible.*

Proof. Let V be any reducible representation of G, and $W \subset V$ a subrepresentation. Pick an invariant inner product on V, then $W^\perp$ is also invariant under the G-action. It follows that $W^\perp$ is also a subrepresentation, and $V = W \oplus W^\perp$. We can do this procedure again and again until each component is irreducible. $\square$

A natural question is which representation of G admits an invariant inner product, and thus unitary?

Theorem 3.5. *Any representation of a compact Lie group admits an invariant inner product.*

Proof. Let V be a representation of G. Start with any inner product on V, we can define a new inner product via

$$\langle v, w\rangle^{new} := \int_G \langle g\cdot v, g\cdot w\rangle dg,$$

where dg is the Haar measure on G. It is easy to check that $\langle\cdot,\cdot\rangle^{new}$ is an inner product on V and is invariant under the G-action:

$$\langle g\cdot v, g\cdot w\rangle^{new} = \langle v, w\rangle^{new}.$$

□

It follows that any representation of a compact Lie group is unitary.

Corollary 3.6. *Any representation of a compact Lie group is completely reducible.*

Since any finite group is a 0-dimensional Lie group, we get

Corollary 3.7. *Any representation of a finite group is completely reducible.*