Lecture 21: Schur Orthonormality

1. Schur's Lemma

Lemma 1.1 (Schur’s Lemma). Let $V$, $W$ be irreducible representations of $G$.

1. If $f : V \to W$ is a $G$-morphism, then either $f \equiv 0$, or $f$ is invertible.
2. If $f_1, f_2 : V \to W$ are two $G$-morphisms and $f_2 \neq 0$, then there exists $\lambda \in \mathbb{C}$ such that $f_1 = \lambda f_2$.

Proof. (1) Suppose $f$ is not identically zero. Since $\ker(f)$ is a $G$-invariant subset in $V$, it must be $\{0\}$. So $f$ is injective. In particular, $f(V)$ is a nonzero subspace of $W$. On the other hand, it is easy to check that $f(V)$ is a $G$-invariant subspace of $W$. It follows that $f(V) = W$, and thus $f$ is invertible.

(2) Since $f_2 \neq 0$, it is invertible. So $f = f_2^{-1} \circ f_1$ is a $G$-morphism from $V$ to $V$ itself. Let $\lambda$ be one of the eigenvalues of the linear map $f$. Then $f - \lambda \cdot \text{Id}$ is a $G$-morphism from $V$ to $V$ which is not invertible. It follows that $f - \lambda \cdot \text{Id} \equiv 0$, and thus $f_1 = \lambda f_2$. □

Note that in the proof we showed in particular

Corollary 1.2. Let $V$ be an irreducible representation of $G$, then $\text{Hom}_G(V, V) = \mathbb{C} \cdot \text{Id}$.

Conversely, we have

Lemma 1.3. If $(\pi, V)$ is a unitary representation of $G$, and $\text{Hom}_G(V, V) = \mathbb{C} \cdot \text{Id}$, then $(\pi, V)$ is an irreducible representation of $G$.

Proof. Let $0 \neq W \subset V$ be a $G$-invariant subspace. We need to show that $W = V$. Let $P : V \to W$ be the orthogonal projection (with respect to the given $G$-invariant inner product). Since both $W$ and $W^\perp$ are $G$-invariant, we have for any $g \in G$ and any $v = w + w^\perp \in V$,

$$P(g \cdot v) = P(g \cdot w + g \cdot w^\perp) = g \cdot w = g \cdot P(v),$$

i.e. $P : V \to W \subset V$ is a $G$-morphism. It follows that $P = \lambda \cdot \text{Id}$ for some $\lambda \in \mathbb{C}$. Now $P^2 = P$ implies $\lambda = 1$, and thus $W = V$. □

Recall that the center $Z(G)$ of a Lie group $G$ is

$$Z(G) = \{ h \in G : gh = hg, \forall g \in G \}.$$

Corollary 1.4. If $(\pi, V)$ is an irreducible representation of $G$, then for any $h \in Z(G)$, $\pi(h) = \lambda \cdot \text{Id}$ for some $\lambda \in \mathbb{C}$.
Proof. Suppose \( h \in Z(G) \), then for any \( g \in G \),
\[
\pi(h)\pi(g) = \pi(hg) = \pi(gh) = \pi(g)\pi(h).
\]
In other words, \( \pi(h) : V \to V \) is a \( G \)-morphism, and the conclusion follows. \( \square \)

**Corollary 1.5.** Any irreducible representation of an abelian Lie group is one dimensional.

*Proof.* Since \( G \) is abelian, \( Z(G) = G \). By the previous corollary, for any \( g \in G \), \( \pi(g) \) is a multiple of the identity map on \( V \). It follows that any subspace of \( V \) is \( G \)-invariant. So \( V \) has no nontrivial subspace, which is equivalent to \( \dim V = 1 \). \( \square \)

2. **Schur Orthogonality for Matrix Coefficients**

Let \((V, \pi)\) be a representation of a Lie group \( G \). If we choose a basis \( e_1, \ldots, e_n \) of \( V \), we can identify \( V \) with \( \mathbb{C}^n \), and represent any \( g \in G \) by a matrix:
\[
\pi(g)v = \begin{pmatrix}
\pi_{11}(g) & \cdots & \pi_{1n}(g) \\
\vdots & \ddots & \vdots \\
\pi_{n1}(g) & \cdots & \pi_{nn}(g)
\end{pmatrix}
\begin{pmatrix}
v_1 \\
\vdots \\
v_n
\end{pmatrix}
\]
for \( v = \sum v_ie_i \). So if we take \( L_j : V \to \mathbb{C} \) be the function
\[
L_j(\sum_i v_ie_i) = v_j,
\]
then \( \pi_{ij} \) is the function on \( G \) given by
\[
\pi_{ij}(g) = L_i(\pi(g)e_j).
\]

**Definition 2.1.** For any \( v \in V, L \in V^* \), the map
\[
\phi : G \to \mathbb{C}, \quad \phi(g) = L(\pi(g)v)
\]
is called a matrix coefficient of \( G \).

Obviously any matrix coefficients of \( G \) is a continuous function on \( G \). In fact, they form a subring of \( C(G) \):

**Proposition 2.2.** If \( \phi_1, \phi_2 \) are matrix coefficients for \( G \), so are \( \phi_1 + \phi_2 \) and \( \phi_1 \cdot \phi_2 \).

*Proof.* Let \((\pi_i, V_i)\) be representations of \( G \), \( v_i \in V_i, L_i \in V_i^* \) such that \( \phi_i(g) = L_i(\pi_i(g)v_i) \). Then \((\pi_1 \oplus \pi_2, V_1 \oplus V_2)\) is a representation of \( G \), \( L_1 \oplus L_2 \in V_1^* \oplus V_2^* = (V_1 \oplus V_2)^* \) and
\[
(L_1 \oplus L_2)((\pi_1 \oplus \pi_2)(g)(v_1, v_2)) = \phi_1(g) + \phi_2(g).
\]
Similarly we have a linear functional \( L_1 \otimes L_2 \) on \( V_1 \otimes V_2 \) satisfying \((L_1 \otimes L_2)(v_1 \otimes v_2) = L_1(v_1)L_2(v_2) \), and thus
\[
(L_1 \otimes L_2)((\pi_1 \otimes \pi_2)(g)(v_1 \otimes v_2)) = \phi_1(g)\phi_2(g).
\]
\( \square \)
Now suppose $G$ is a compact Lie group, and $dg$ the normalized Haar measure on $G$. Recall that $L^2(G)$, the space of square-integrable functions with respect to this Haar measure, is the completion of the space of continuous functions on $G$ with respect to the inner product
\[ \langle f_1, f_2 \rangle_{L^2} = \int_G f_1(g) \overline{f_2(g)} dg. \]

**Theorem 2.3** (Schur’s Orthogonality I). Let $(\pi_1, V_1)$ and $(\pi_2, V_2)$ are two non-isomorphic irreducible representations of a compact Lie group $G$. Then every matrix coefficient of $\pi_1$ is orthogonal in $L^2(G)$ to every matrix coefficient of $\pi_2$.

**Proof.** Fix $G$-invariant inner products on $V_1$ and $V_2$ respectively. Suppose
\[ \phi_i(g) = \langle \pi_i(g)v_i, w_i \rangle, \]
where $v_i, w_i \in V_i$. Fix a basis of $V_1$ such that $e_1 = v_1$. Define a linear map $f : V_1 \to V_2$ by $f(e_1) = v_2$ and $f(e_k) = 0$ for all $k \geq 2$. Consider the map
\[ F : V_1 \to V_2, \quad v \mapsto F(v) = \int_G \pi_2(g)f(\pi_1(g^{-1})v) dg. \]

$F$ is linear since $f$ is. It is also $G$-equivariant, since
\[ F(\pi_1(h^{-1})v) = \int_G \pi_2(g)f(\pi_1(hg)^{-1}v) dg = \pi_2(h^{-1}) \int_G \pi_2(hg)f(\pi_1(hg)^{-1}v) dg = \pi_2(h^{-1})F(v). \]

By Schur’s lemma, $F(v) = 0$ for any $v$, and in particular, $\langle F(v), w_2 \rangle = 0$. On the other hand, for any $j$,
\[ \pi_2(g)f(\pi_1(g^{-1})e_j) = \pi_2(g)f(\sum_k \pi_1(g^{-1})_{kj}e_k) = \pi_1(g^{-1})_{1j}\pi_2(g)(v_2), \]

where $\pi_1(g^{-1})_{kj} = \langle \pi_1(g^{-1})e_j, e_k \rangle$ is the matrix coefficients of $\pi_1$ with respect to the basis $\{e_1, \ldots, e_n\}$. It follows that
\[ \int_G \langle \pi_1(g^{-1})e_j, e_1 \rangle \langle \pi_2(g)v_2, w_2 \rangle dg = 0 \]
for any $j$. So by linearity,
\[ \int_G \langle \pi_1(g^{-1})w_1, v_1 \rangle \langle \pi_2(g)v_2, w_2 \rangle dg = 0. \]

Note that the inner product is $G$-invariant,
\[ \langle \pi_1(g^{-1})w_1, v_1 \rangle = \langle w_1, \pi_1(g)v_1 \rangle = \overline{\langle \pi_1(g)v_1, w_1 \rangle} = \overline{\phi_1(g)}. \]

So the theorem follows. \qed

**Theorem 2.4** (Schur’s Orthogonality II). Let $(\pi, V)$ be an irreducible representation of a compact Lie group $G$, with $G$-invariant inner product $\langle \cdot, \cdot \rangle$. Then
\[ \int_G \langle \pi(g)w_1, v_1 \rangle \overline{\langle \pi(g)w_2, v_2 \rangle} dg = \frac{1}{\dim V} \langle w_1, w_2 \rangle \langle v_1, v_2 \rangle. \]
Proof. Define the linear maps $f, F : V \to V$ as above. Then $F$ is $G$-equivariant, and thus $F = \lambda \cdot \text{Id}$ for some $\lambda = \lambda(v_1, v_2) \in \mathbb{C}$. On the other hand, when we take $\pi_1 = \pi_2 = \pi$, the computation in the previous proof shows

$$\lambda(v_1, v_2)\langle w_1, w_2 \rangle = \langle F(w_1), w_2 \rangle = \int_G \langle \pi(g^{-1})w_1, v_1 \rangle \langle \pi(g)v_2, w_2 \rangle dg.$$ 

Since the Haar measure is invariant under the inversion map $g \to g^{-1}$, the right hand side is invariant if we exchange $w_1$ with $v_2$, and exchange $w_2$ with $v_1$. It follows that

$$\lambda(v_1, v_2) = C \langle v_2, v_1 \rangle = C \langle v_1, v_2 \rangle$$

for some constant $C$. Finally if we take $v_1 = v_2 = e_1$ a unit vector, then

$$\text{Tr}(F) = \text{Tr} \int_G \pi(g) \circ f \circ \pi(g^{-1})dg = \int_G \text{Tr}(\pi(g) \circ f \circ \pi(g^{-1}))dg = \int_G \text{Tr}(f)dg = \int_G 1dg = 1.$$ 

On the other hand, in this case $F = C \cdot \text{Id}$. It follows from $\text{Tr}(F) = 1$ that $C = \frac{1}{\dim V}$.  \(\square\)