LECTURE 23-24: PETER-WEYL THEOREM AND ITS APPLICATIONS

1. Some Functional Analysis

Let H be a (complex) Hilbert space, i.e. a (finite or infinite dimensional) vector space with an inner product, such that H is *complete* with respect to the induced metric $|v| = \langle v, v\rangle^{1/2}$. A linear operator $T : H \to H$ is said to be *bounded* if there exists $C > 0$ such that

$$|Tv| \le C|v|, \quad \forall v \in H.$$

Definition 1.1. Let H be a Hilbert space, and $T : H \to H$ a bounded operator.

(1) T is *self-adjoint* if for any $v, w \in H$, $\langle Tv, w\rangle = \langle v, Tw\rangle$.
(2) T is *compact* if for any bounded sequence $v_1, v_2, \cdots$ in H, the sequence $Tv_1, Tv_2, \cdots$ has a convergent subsequence.

We say that $\lambda \in \mathbb{C}$ is an eigenvalue of an operator T on H, with eigenvector $0 \ne v \in H$, if $Tv = \lambda v$. We will denote the set of all eigenvalues of T by $\mathrm{Spec}(T)$, and the eigenspace for $\lambda \in \mathrm{Spec}(T)$ by H_λ. Self-adjoint operators on Hilbert spaces are the (infinite dimensional) generalization of $n \times n$ Hermitian matrices acting on $\mathbb{C}^n$. For example, we have

Lemma 1.2. *If T is self-adjoint, then*

(1) $\mathrm{Spec}(T) \subset \mathbb{R}$,
(2) *for any* $\lambda \ne \mu \in \mathrm{Spec}(T)$, $H_\lambda \perp H_\mu$.

Proof. If $\lambda \in \mathrm{Spec}(T)$ with eigenvector v, and T is self-adjoint, then

$$\lambda\langle v, v\rangle = \langle Tv, v\rangle = \langle v, Tv\rangle = \langle v, \lambda v\rangle = \bar{\lambda}\langle v, v\rangle.$$

So $\lambda = \bar{\lambda}$, i.e. $\lambda \in \mathbb{R}$.

Similarly if $\lambda \ne \mu \in \mathrm{Spec}(T)$, $0 \ne v \in H_\lambda$ and $0 \ne w \in H_\mu$, then

$$\lambda\langle v, w\rangle = \langle Tv, w\rangle = \langle v, Tw\rangle = \langle v, \mu w\rangle = \mu\langle v, w\rangle.$$

So $\langle v, w\rangle = 0$, i.e. $v \perp w$. □

The following spectral theorem for compact self-adjoint operators in Hilbert space will play a crucial rule in the proof of Peter-Weyl theorem.

Theorem 1.3 (The Spectral Theorem). *Let T be a compact self-adjoint operator on a Hilbert space H, and denote by $N(T) = \mathrm{Ker}(T)$ the null space of T. Then*

(1) *The set of nonzero eigenvalues is a countable set (and thus discrete).*

(2) *If the set* $\mathrm{Spec}(T) = \{\lambda_1, \lambda_2, \cdots\}$ *is not a finite set, then* $\lim_{k\to\infty} \lambda_k = 0$.
(3) *For each eigenvalue* $0 \neq \lambda \in \mathrm{Spec}(T)$, $\dim H_\lambda < \infty$.
(4) *Denote by* $v_1^{(k)}, \cdots, v_{n(k)}^{(k)}$ *an orthonormal basis of* H_{λ_k}, *where* $n(k) = \dim H_{\lambda_k}$. *Then*

$$\{v_j^{(k)} \mid 1 \leq j \leq n(k), k = 1, 2, \cdots\}$$

form an orthonormal basis of $N(T)^\perp$.

2. Convolution on Compact Lie Groups

Let G be a compact Lie group with the normalized Haar measure dg, and $C(G)$ the ring of continuous functions on G. Since G has total volume 1, we have inequalities

$$\|f\|_1 \leq \|f\|_2 \leq \|f\|_\infty$$

for any $f \in C(G)$, where $\|\cdot\|_p$ is the is the L^p norm, i.e.

$$\|f\|_p = \left(\int_G |f(g)|^p dg\right)^{1/p},$$

and

$$\|f\|_\infty = \sup_{g\in G} |f(g)|.$$

It follows that $C(G) \subset L^\infty(G) \subset L^2(G) \subset L^1(G)$. Recall that $L^2(G)$ is a Hilbert space, with inner product

$$\langle f_1, f_2\rangle = \int_G f_1(g)\overline{f_2(g)}dg.$$

Definition 2.1. For any $f_1 \in C(G)$ and $f_2 \in L^1(G)$, we can define the *convolution* by

$$(f_1 * f_2)(g) := \int_G f_1(gh^{-1})f_2(h)dh.$$

Remark. Use the change of variable $h \mapsto h^{-1}g$ one can

$$(f_1 * f_2)(g) = \int_G f_1(h)f_2(h^{-1}g)dh.$$

Now for any $\phi \in C(G)$ and any $f \in L^1(G)$, we define

$$T_\phi(f) = \phi * f.$$

Proposition 2.2. *For any* $\phi \in C(G)$ *and* $f \in L^1(G)$, $T_\phi(f) \in C(G)$, *and*

$$\|T_\phi(f)\|_\infty \leq \|\phi\|_\infty \cdot \|f\|_1.$$

In particular, the operator T_ϕ *is a bounded operator on* $L^2(G)$.

Proof. Since ϕ is uniformly continuous, there exits a neighborhood U of e such that $|\phi(g) - \phi(kg)| < \varepsilon$ for all $g \in G$ and $k \in U$. It follows that for any $g, g' \in G$ with $g'g^{-1} \in U$,

$$\begin{aligned}|T_\phi f(g) - T_\phi f(g')| &= \left| \int_G (\phi(gh^{-1}) - \phi(g'h^{-1}))f(h)dh \right| \\ &\le \int_G \left|\phi(gh^{-1}) - \phi(g'g^{-1}gh^{-1})\right| |f(h)dh| \le \varepsilon \|f\|_1.\end{aligned}$$

So $T_\phi f$ is a continuous function. Moreover,

$$\|T_\phi(f)\|_\infty = \sup_{g\in G} \left| \int_G \phi(gh^{-1})f(h)dh \right| \le \|\phi\|_\infty \int_G |f(h)|dh = \|\phi\|_\infty \|f\|_1.$$

□

The main result in this section is

Proposition 2.3. *Let $\phi \in C(G)$. Then*

1. *The operator T_ϕ is a compact operator on $L^2(G)$.*
2. *If $\phi(g^{-1}) = \overline{\phi(g)}$, then T_ϕ is self-adjoint on $L^2(G)$.*

Before proving this let's first remind you the classical Ascoli-Arzela theorem. Recall that a subset $U \subset C(X)$ is *equicontinuous* if for any $x \in X$ and $\varepsilon > 0$, there exists a neighborhood N of x such that $|f(x) - f(y)| < \varepsilon$ holds for all $y \in N$ and all $f \in U$.

Theorem 2.4 (Ascolli-Arzela)**.** *Let X be compact, $U \subset C(X)$ a bounded and equicontinuous subset. Then every sequence in U has a uniformly convergent subsequence.*

Proof of Proposition 2.3. (1) It suffices to show that any sequence in

$$\mathcal{B} = \{T_\phi(f) \mid f \in L^2(G), \|f\|_2 \le 1\}$$

has a convergent subsequence. According to the fact $\|f\|_2 \le \|f\|_\infty$ and the Arzela-Ascoli theorem, it suffices to show that $\mathcal{B}$ is bounded and equicontinuous in $(C(G), \|\cdot\|_\infty)$. The boundedness follows from the previous proposition. Now let's show the equicontinuity: Since G is compact, ϕ is uniformly continuous, i.e. $\forall \varepsilon > 0$, there exists a neighborhood U of $e \in G$ such that $|\phi(kg) - \phi(g)| < \varepsilon$ for all $g \in G$ and $k \in U$. So if $\|f\|_2 \le 1$,

$$\begin{aligned}|(\phi * f)(kg) - (\phi * f)(g)| &= \left| \int_G (\phi(kgh^{-1}) - \phi(gh^{-1}))f(h)dh \right| \\ &\le \int_G \left|\phi(kgh^{-1}) - \phi(gh^{-1})\right| \ |f(h)|dh \\ &\le \varepsilon \|f\|_1 \le \varepsilon\end{aligned}$$

holds for all $k \in N$ and all $f \in L^2(G)$ with $\|f\|_2 \le 1$.

(2) We have by definition

$$\langle T_\phi(f_1), f_2\rangle = \int_G\int_G \phi(gh^{-1})f_1(h)\overline{f_2(g)}dgdh$$

while

$$\langle f_1, T_\phi(f_2)\rangle = \int_G\int_G \overline{\phi(hg^{-1})}f_1(h)\overline{f_2(g)}dgdh.$$

So if $\phi(g^{-1}) = \overline{\phi(g)}$, T_ϕ is self-adjoint. □

3. The Peter-Weyl Theorem

The right translation $R(g)$ of G induces a linear G-action on $C(G)$, given by

$$(R(g)f)(x) = f(xg).$$

Lemma 3.1. *Suppose $\phi \in C(G)$, and $\lambda \in \mathrm{Spec}(T_\phi)$. Then the λ-eigenspace H_λ is G-invariant.*

Proof. For any $f \in H_\lambda$ and any $g \in G$,

$$(T_\phi R(g)f)(x) = \int_G \phi(xh^{-1})f(hg)dh = \int_G \phi(xgh^{-1})f(h)dh = R(g)(T_\phi f)(x) = \lambda R(g)f(x).$$

□

Recall that a matrix coefficient of G associated to a representation (V,π) is a function on G of the form $L(\pi(g)v)$ for some $v \in V$ and $L \in V^*$.

Lemma 3.2. *A function $f \in C(G)$ is a matrix coefficient on G if and only if the functions $R(g)f$ span a finite dimensional vector space.*

Proof. It is easy to check that if $f \in C(G)_\pi$ for some finite dimensional representation π, then $R(g)f \in C(G)_\pi$ for any g. It follows that $R(g)f$'s span a vector space of dimension no more than n^2, where $n = \dim\pi$.

Conversely, suppose the functions $R(g)f$ span a finite dimensional vector space V, then (R,V) is a finite dimensional representation of G, and if we define a functional $L : V \to \mathbb{C}$ by $L(\phi) = \phi(e)$, it is clear that $f \in V$ and

$$L(R(g)f) = f(g),$$

so as a function f is a matrix coefficient of G associated to this representation. □

Definition 3.3. Given representation (V,π), we will denote by $R(\pi)$ the linear span of of all matrix coefficients in $C(G)_\pi$.

Now we are ready to prove

Theorem 3.4 (Peter-Weyl Theorem)**.** *Let G be a compact Lie group. Then the matrix coefficients of G are dense in $C(G)$.*

Proof. Let $f \in C(G)$ be a continuous function on G. Since G is compact, f is uniformly continuous, i.e. there exists a neighborhood U of e in G such that for any $g \in U$ and $h \in G$,

$$|f(g^{-1}h) - f(h)| < \varepsilon/2.$$

Now let ϕ be a nonnegative (real-valued) function supported in U such that $\int_G \phi(g)dg = 1$ and $\phi(g) = \phi(g^{-1})$. It follows that T_ϕ is a self-adjoint compact operator on $L^2(G)$, and for any $h \in G$,

$$|(T_\phi f)(h) - f(h)| = \left| \int_G (\phi(g)f(g^{-1}h) - \phi(g)f(h))dg \right| \le \int_U \phi(g)|f(g^{-1}h) - f(h)|dg \le \frac{\varepsilon}{2}.$$

It follows by the spectral theorem that for all $\lambda \in \mathrm{Spec}(T_\phi)$, H_λ are finite dimensional for $\lambda \neq 0$, mutually orthogonal, and span $L^2(G)$. So if we write $f = \sum_\lambda f_\lambda$, where $f_\lambda \in H_\lambda$, then $\sum_\lambda \|f_\lambda\|_2^2 = \|f\|_2^2 < \infty$. In particular, one can find δ such that

$$\sqrt{\sum_{0<|\lambda|<\delta} \|f_\lambda\|_2^2} < \frac{\varepsilon}{2\|\phi\|_\infty}.$$

Now let $\tilde{f} = T_\phi(\sum_{|\lambda|\ge\delta} f_\lambda)$. Then $\tilde{f} \in \bigoplus_{|\lambda|\ge\delta} H_\lambda$, and moreover for any $g \in G$, $R(g)\tilde{f} \in \bigoplus_{|\lambda|\ge\delta} H_\lambda$. Since $\bigoplus_{|\lambda|\ge\delta} H_\lambda$ is a finite dimensional vector space, we conclude that $\tilde{f}$ is a matrix coefficient of G. Note that

$$T_\phi(f) - \tilde{f} = T_\phi(f_0 + \sum_{0<|\lambda|<\delta} f_\lambda) = T_\phi(\sum_{0<|\lambda|<\delta} f_\lambda),$$

so

$$\|T_\phi(f) - \tilde{f}\|_\infty \le \|\phi\|_\infty \cdot \|\sum_{0<|\lambda|<\delta} f_\lambda\|_2 < \frac{\varepsilon}{2}.$$

It follows

$$\|f - \tilde{f}\|_\infty \le \|f - T_\phi f\|_\infty + \|T_\phi f - \tilde{f}\|_\infty < \varepsilon.$$

□

Since $C(G)$ is dense in $L^2(G)$, we immediately get

Corollary 3.5. *Let G be a compact Lie group. Then the matrix coefficients of G are dense in $L^2(G)$.*

Since any representation of G is completely reducible, and the matrix coefficients for distinct irreducible representations are orthogonal, we can restate Peter-Weyl theorem as

Theorem 3.6 (Peter-Weyl)**.** *Let G be a compact Lie group and $\widehat{G}$ be the set of equivalence classes of irreducible representations of G. Then*

$$L^2(G) = \widehat{\bigoplus_{\rho\in\hat{G}}} R(\rho),$$

where the right hand side denote the closure in L^2 of $\oplus_\rho R(\rho)$.

Last time we have seen through examples that irreducible characters generate a dense subspace of class functions. Now we can prove this for all compact groups:

Theorem 3.7 (Peter-Weyl theorem for class functions)**.** *Suppose G is compact. Then the irreducible characters of G generate a dense subspace of the space of continuous class functions on G.*

Proof. Suppose ϕ is a class function. For any $\varepsilon > 0$, by the Peter-Weyl theorem, one can find a representation (V, π) of G and a matrix coefficient $f \in C(G)_\pi$ such that $\|\phi - f\|_\infty < \varepsilon$. Consider the function defined on G by

$$\psi(x) = \int_G f(gxg^{-1})dg.$$

Obviously ψ is a class function, and $\|\phi - \psi\|_\infty < \varepsilon$. We claim that ψ is a linear combination of irreducible characters, thus prove the theorem.

In fact, since G is compact, $(V, \pi) = \bigoplus_i (V_i, \pi_i)$ is a direct sum of finitely many irreducible representations. Since $f \in C(G)_\pi = \bigoplus C(G)_{\pi_i}$, we can write

$$f(g) = \sum_i L_i(\pi_i(g)v_i)$$

for some $v_i \in V_i$ and $L_i \in V_i^*$. So

$$\psi(x) = \sum_i L_i \left(\int_G \pi_i(g)\pi_i(x)\pi_i(g^{-1})v_i \; dg \right).$$

We have already shown in the proof of Schur's orthogonality that the map

$$v_i \mapsto \int_G \pi_i(g)\pi_i(x)\pi_i(g^{-1})v_i \; dg,$$

as the "average" of $\pi_i(x)$, is a linear equivariant map on V_i. So

$$\int_G \pi_i(g)\pi_i(x)\pi_i(g^{-1})v_i \; dg = \lambda(x)v_i$$

for some $\lambda(x) \in \mathbb{C}$. Computing the trace as we did before, we conclude

$$\lambda(x) = \frac{1}{\dim V_i}\mathrm{Tr}(\pi_i(x)) = \frac{1}{\dim V_i}\chi_{\pi_i}(x).$$

It follows that

$$\psi(x) = \sum \frac{1}{\dim V_i} L_i(v_i)\chi_{\pi_i}(x)$$

is a linear combination of irreducible characters. □

Corollary 3.8. *For any class function f in $L^2(G)$,*

$$f = \sum_{\pi \in \hat{G}} \langle f, \chi_\pi \rangle \chi_\pi$$

as an L^2 function with respect to L^2-convergence, and

$$\|f\|_2^2 = \sum_{\pi \in \hat{G}} \langle f, \chi_\pi \rangle^2.$$

4. Faithful Representations

Definition 4.1. A representation (V, π) of G is *faithful* if as a linear action, π is effective, i.e. the group homomorphism $\pi : G \to \mathrm{GL}(V)$ is injective.

Lemma 4.2. *Let G be a compact Lie group. Then for any $g \neq e$, there exists a representation (V, π) of G such that $\pi(g) \neq \mathrm{Id}$.*

Proof. Take any function $f \in C(G)$ such that $f(e) = 0$ and $f(g) = 1$. Then there is a representation (V, π) of G and a matrix coefficient $\phi \in C(G)_\pi$ such that $\|f - \phi\|_\infty < \frac{1}{3}$. So in particular $\phi(e) \neq \phi(g)$. Since $\phi(x) = L(\pi(x)v)$ for some fixed $v \in V$ and $L \in V^*$, we must have $\pi(g) \neq \pi(e) = \mathrm{Id}$. □

Theorem 4.3. *Any compact Lie group possesses a faithful representation.*

Proof. According to the previous lemma, for any $g_1 \in G^0$ (the connected component of G containing e), $g_1 \neq e$, there exists a representation (V_1, π_1) of G such that $\pi_1(g_1) \neq \mathrm{Id}$. The kernel $\mathrm{Ker}(\pi_1)$ is a closed subgroup of G, thus a Lie group by itself. Moreover, since $G^0 \not\subseteq \mathrm{Ker}(\pi_1)$, we must have $\dim \mathrm{Ker}(\pi_1) < \dim G$. If $\dim \mathrm{Ker}(\pi_1) \neq 0$, we do this procedure again, i.e. take an element $g_2 \in \mathrm{Ker}(\pi_1)^0$ and a representation (V_2, π_2) such that $\pi_2(g_2) \neq \mathrm{Id}$. It follows that $\mathrm{Ker}(\pi_1 \oplus \pi_2)$ is a compact Lie subgroup in G with $\dim \mathrm{Ker}(\pi_1 \oplus \pi_2) < \dim \mathrm{Ker}(\pi_1)$. Continuing this procedure, we will get a sequence of representations $(V_i, \pi_i), 1 \leq i \leq N$, of G, such that $\dim \mathrm{Ker}(\pi_1 \oplus \cdots \oplus \pi_N) = 0$. Since G is compact, $\mathrm{Ker}(\pi_1 \oplus \cdots \oplus \pi_N) = \{h_1, \cdots, h_M\}$ is a finite set. Now we choose representations $(W_i, \rho_i), 1 \leq i \leq M$ such that $\rho_i(h_i) \neq \mathrm{Id}$. It follows that the representations $\pi_1 \oplus \cdots \oplus \pi_N \oplus \rho_1 \oplus \cdots \oplus \rho_M$ is a faithful representation of G. □

As a corollary, we will prove that any compact Lie group is a linear Lie group:

Corollary 4.4. *Any compact Lie group is isomorphic to a closed subgroup of $\mathrm{U}(N)$ for N large.*

Proof. Take a faithful representation (V, π) of G. Since G is compact, there exists a G-invariant inner product on V. It follows $\pi : G \to \mathrm{GL}(V)$ maps G into $\mathrm{U}(V) \simeq \mathrm{U}(N)$. The injectivity of π implies that G is isomorphic to its image, a closed subgroup of $\mathrm{U}(N)$. □

Remark. There exists noncompact Lie groups that does not admit any finite dimensional faithful representation, and thus are not linear Lie groups. One example of such Lie groups is the *metaplectic group* Mp_{2n}, the double cover of the symplectic group Sp_{2n}. Although it doesn't have any finite dimensional faithful representation, it does have faithful infinite dimensional representations, such as the *Weil representation*.