LECTURE 23-24: PETER-WEYL THEOREM AND ITS APPLICATIONS

1. SOME FUNCTIONAL ANALYSIS

Let $H$ be a (complex) Hilbert space, i.e. a (finite or infinite dimensional) vector space with an inner product, such that $H$ is complete with respect to the induced metric $|v| = \langle v, v \rangle^{1/2}$. A linear operator $T : H \rightarrow H$ is said to be bounded if there exists $C > 0$ such that $|Tv| \leq C|v|$, $\forall v \in H$.

**Definition 1.1.** Let $H$ be a Hilbert space, and $T : H \rightarrow H$ a bounded operator.

1. $T$ is self-adjoint if for any $v, w \in H$, $\langle Tv, w \rangle = \langle v, Tw \rangle$.
2. $T$ is compact if for any bounded sequence $v_1, v_2, \cdots$ in $H$, the sequence $Tv_1, Tv_2, \cdots$ has a convergent subsequence.

We say that $\lambda \in \mathbb{C}$ is an eigenvalue of an operator $T$ on $H$, with eigenvector $0 \neq v \in H$, if $Tv = \lambda v$. We will denote the set of all eigenvalues of $T$ by $\text{Spec}(T)$, and the eigenspace for $\lambda \in \text{Spec}(T)$ by $H_\lambda$. Self-adjoint operators on Hilbert spaces are the (infinite dimensional) generalization of $n \times n$ Hermitian matrices acting on $\mathbb{C}^n$. For example, we have

**Lemma 1.2.** If $T$ is self-adjoint, then

1. $\text{Spec}(T) \subset \mathbb{R}$,
2. for any $\lambda \neq \mu \in \text{Spec}(T)$, $H_\lambda \perp H_\mu$.

**Proof.** If $\lambda \in \text{Spec}(T)$ with eigenvector $v$, and $T$ is self-adjoint, then $\lambda \langle v, v \rangle = \langle Tv, v \rangle = \langle v, Tv \rangle = \langle v, \lambda v \rangle = \bar{\lambda} \langle v, v \rangle$. So $\lambda = \bar{\lambda}$, i.e. $\lambda \in \mathbb{R}$.

Similarly if $\lambda \neq \mu \in \text{Spec}(T)$, $0 \neq v \in H_\lambda$ and $0 \neq w \in H_\mu$, then $\lambda \langle v, w \rangle = \langle Tv, w \rangle = \langle v, Tw \rangle = \langle v, \mu w \rangle = \mu \langle v, w \rangle$. So $\langle v, w \rangle = 0$, i.e. $v \perp w$. \hfill \Box

The following spectral theorem for compact self-adjoint operators in Hilbert space will play a crucial rule in the proof of Peter-Weyl theorem.

**Theorem 1.3 (The Spectral Theorem).** Let $T$ be a compact self-adjoint operator on a Hilbert space $H$, and denote by $N(T) = \text{Ker}(T)$ the null space of $T$. Then

1. The set of nonzero eigenvalues is a countable set (and thus discrete).
(2) If the set $\text{Spec}(T) = \{\lambda_1, \lambda_2, \cdots\}$ is not a finite set, then $\lim_{k \to \infty} \lambda_k = 0$.

(3) For each eigenvalue $0 \neq \lambda \in \text{Spec}(T)$, dim $H_\lambda < \infty$.

(4) Denote by $v_1^{(k)}, \cdots, v_{n(k)}^{(k)}$ an orthonormal basis of $H_{\lambda_k}$, where $n(k) = \text{dim } H_{\lambda_k}$.

Then

$$\{v_j^{(k)} \mid 1 \leq j \leq n(k), k = 1, 2, \cdots\}$$

form an orthonormal basis of $N(T)^\perp$.

2. Convolutions on Compact Lie Groups

Let $G$ be a compact Lie group with the normalized Haar measure $dg$, and $C(G)$ the ring of continuous functions on $G$. Since $G$ has total volume 1, we have inequalities

$$\|f\|_1 \leq \|f\|_2 \leq \|f\|_\infty$$

for any $f \in C(G)$, where $\|\cdot\|_p$ is the $L^p$ norm, i.e.

$$\|f\|_p = \left( \int_G |f(g)|^p \, dg \right)^{1/p},$$

and

$$\|f\|_\infty = \sup_{g \in G} |f(g)|.$$

It follows that $C(G) \subset L^\infty(G) \subset L^2(G) \subset L^1(G)$. Recall that $L^2(G)$ is a Hilbert space, with inner product

$$\langle f_1, f_2 \rangle = \int_G f_1(g) \overline{f_2(g)} \, dg.$$

**Definition 2.1.** For any $f_1 \in C(G)$ and $f_2 \in L^1(G)$, we can define the convolution by

$$(f_1 * f_2)(g) := \int_G f_1(gh^{-1}) f_2(h) \, dh.$$  

**Remark.** Use the change of variable $h \mapsto h^{-1}g$ one can

$$(f_1 * f_2)(g) = \int_G f_1(h) f_2(h^{-1}g) \, dh.$$  

Now for any $\phi \in C(G)$ and any $f \in L^1(G)$, we define

$$T_\phi(f) = \phi * f.$$  

**Proposition 2.2.** For any $\phi \in C(G)$ and $f \in L^1(G)$, $T_\phi(f) \in C(G)$, and

$$\|T_\phi(f)\|_\infty \leq \|\phi\|_\infty \cdot \|f\|_1.$$  

In particular, the operator $T_\phi$ is a bounded operator on $L^2(G)$. 
Proof. Since $\phi$ is uniformly continuous, there exists a neighborhood $U$ of $e$ such that $|\phi(g) - \phi(kg)| < \varepsilon$ for all $g \in G$ and $k \in U$. It follows that for any $g, g' \in G$ with $g^{-1}g' \in U$,

$$|T_\phi f(g) - T_\phi f(g')| = \left| \int_G (\phi(gh^{-1}) - \phi(g'h^{-1}))f(h)dh \right|$$

$$\leq \int_G |\phi(gh^{-1}) - \phi(g'h^{-1})| |f(h)|dh \leq \varepsilon\|f\|_1.$$ 

So $T_\phi f$ is a continuous function. Moreover,

$$\|T_\phi(f)\|_\infty = \sup_{g \in G} \left| \int_G \phi(g) f(h)dh \right| \leq \|\phi\|_\infty \int_G |f(h)|dh = \|\phi\|_\infty \|f\|_1.$$

The main result in this section is

**Proposition 2.3.** Let $\phi \in C(G)$. Then

1. The operator $T_\phi$ is a compact operator on $L^2(G)$.
2. If $\phi(g^{-1}) = \overline{\phi}(g)$, then $T_\phi$ is self-adjoint on $L^2(G)$.

Before proving this let’s first remind you the classical Ascoli-Arzela theorem. Recall that a subset $U \subset C(X)$ is equicontinuous if for any $x \in X$ and $\varepsilon > 0$, there exists a neighborhood $N$ of $x$ such that $|f(x) - f(y)| < \varepsilon$ holds for all $y \in N$ and all $f \in U$.

**Theorem 2.4** (Ascoli-Arzela). Let $X$ be compact, $U \subset C(X)$ a bounded and equicontinuous subset. Then every sequence in $U$ has a uniformly convergent subsequence.

**Proof of Proposition 2.3.** (1) It suffices to show that any sequence in

$$\mathcal{B} = \{T_\phi(f) \mid f \in L^2(G), \|f\|_2 \leq 1\}$$

has a convergent subsequence. According to the fact $\|f\|_2 \leq \|f\|_\infty$ and the Arzela-Ascoli theorem, it suffices to show that $\mathcal{B}$ is bounded and equicontinuous in $(C(G), \|\cdot\|_\infty)$. The boundedness follows from the previous proposition. Now let’s show the equicontinuity: Since $G$ is compact, $\phi$ is uniformly continuous, i.e. $\forall \varepsilon > 0$, there exists a neighborhood $U$ of $e \in G$ such that $|\phi(kg) - \phi(g)| < \varepsilon$ for all $g \in G$ and $k \in U$. So if $\|f\|_2 \leq 1$,

$$|(\phi * f)(kg) - (\phi * f)(g)| = \left| \int_G (\phi(kgh^{-1}) - \phi(gh^{-1}))f(h)dh \right|$$

$$\leq \int_G |\phi(kgh^{-1}) - \phi(gh^{-1})| |f(h)|dh$$

$$\leq \varepsilon\|f\|_1 \leq \varepsilon$$

holds for all $k \in N$ and all $f \in L^2(G)$ with $\|f\|_2 \leq 1$. 

(2) We have by definition
\[ \langle T \phi(f_1), f_2 \rangle = \int_G \int_G \phi(gh^{-1}) f_1(h) \overline{f_2(g)} dg dh \]
while
\[ \langle f_1, T \phi(f_2) \rangle = \int_G \int_G \overline{\phi(hg^{-1})} f_1(h) f_2(g) dg dh. \]
So if \( \phi(g^{-1}) = \overline{\phi(g)} \), \( T \phi \) is self-adjoint. \( \square \)

3. The Peter-Weyl Theorem

The right translation \( R(g) \) of \( G \) induces a linear \( G \)-action on \( C(G) \), given by
\[ (R(g)f)(x) = f(xg). \]

**Lemma 3.1.** Suppose \( \phi \in C(G) \), and \( \lambda \in \text{Spec}(T \phi) \). Then the \( \lambda \)-eigenspace \( H_\lambda \) is \( G \)-invariant.

**Proof.** For any \( f \in H_\lambda \) and any \( g \in G \),
\[ (T \phi R(g)f)(x) = \int_G \phi(xh^{-1}) f(hg) dh = \int_G \phi(xgh^{-1}) f(h) dh = R(g)(T \phi f)(x) = \lambda R(g)f(x). \] \( \square \)

Recall that a matrix coefficient of \( G \) associated to a representation \((V, \pi)\) is a function on \( G \) of the form \( L(\pi(g)v) \) for some \( v \in V \) and \( L \in V^* \).

**Lemma 3.2.** A function \( f \in C(G) \) is a matrix coefficient on \( G \) if and only if the functions \( R(g)f \) span a finite dimensional vector space.

**Proof.** It is easy to check that if \( f \in C(G)_\pi \) for some finite dimensional representation \( \pi \), then \( R(g)f \in C(G)_\pi \) for any \( g \). It follows that \( R(g)f \)'s span a vector space of dimension no more than \( n^2 \), where \( n = \dim \pi \).

Conversely, suppose the functions \( R(g)f \) span a finite dimensional vector space \( V \), then \( (R,V) \) is a finite dimensional representation of \( G \), and if we define a functional \( L : V \to \mathbb{C} \) by \( L(\phi) = \phi(e) \), it is clear that \( f \in V \) and
\[ L(R(g)f) = f(g), \]
so as a function \( f \) is a matrix coefficient of \( G \) associated to this representation. \( \square \)

**Definition 3.3.** Given representation \((V, \pi)\), we will denote by \( R(\pi) \) the linear span of all matrix coefficients in \( C(G)_\pi \).

Now we are ready to prove

**Theorem 3.4** (Peter-Weyl Theorem). Let \( G \) be a compact Lie group. Then the matrix coefficients of \( G \) are dense in \( C(G) \).
Proof. Let $f \in C(G)$ be a continuous function on $G$. Since $G$ is compact, $f$ is uniformly continuous, i.e. there exists a neighborhood $U$ of $e$ in $G$ such that for any $g \in U$ and $h \in G$,

$$|f(g^{-1}h) - f(h)| < \varepsilon/2.$$

Now let $\phi$ be a nonnegative (real-valued) function supported in $U$ such that $\int_{G} \phi(g)dg = 1$ and $\phi(g) = \phi(g^{-1})$. It follows that $T_\phi$ is a self-adjoint compact operator on $L^2(G)$, and for any $h \in G$,

$$|(T_\phi f)(h) - f(h)| = \left| \int_{G} (\phi(g)f(g^{-1}h) - \phi(g)f(h))dg \right| \leq \int_{U} |\phi(g)||f(g^{-1}h) - f(h)|dg \leq \frac{\varepsilon}{2}.$$

It follows by the spectral theorem that for all $\lambda \in \text{Spec}(T_\phi)$, $H_\lambda$ are finite dimensional for $\lambda \neq 0$, mutually orthogonal, and span $L^2(G)$. So if we write $f = \sum_{\lambda} f_\lambda$, where $f_\lambda \in H_\lambda$, then $\sum_{\lambda} \|f_\lambda\|_2^2 = \|f\|_2^2 < \infty$. In particular, one can find $\delta$ such that

$$\sqrt{\sum_{0<|\lambda|<\delta} \|f_\lambda\|_2^2} < \frac{\varepsilon}{2\|\phi\|_\infty}.$$

Now let $\tilde{f} = T_\phi(\sum_{|\lambda|\geq\delta} f_\lambda)$. Then $\tilde{f} \in \bigoplus_{|\lambda|\geq\delta} H_\lambda$, and moreover for any $g \in G$, $R(g)\tilde{f} \in \bigoplus_{|\lambda|\geq\delta} H_\lambda$. Since $\bigoplus_{|\lambda|\geq\delta} H_\lambda$ is a finite dimensional vector space, we conclude that $\tilde{f}$ is a matrix coefficient of $G$. Note that

$$T_\phi(f) - \tilde{f} = T_\phi(f_0 + \sum_{0<|\lambda|<\delta} f_\lambda) = T_\phi(\sum_{0<|\lambda|<\delta} f_\lambda),$$

so

$$\|T_\phi(f) - \tilde{f}\|_\infty \leq \|\phi\|_\infty \cdot \| \sum_{0<|\lambda|<\delta} f_\lambda\|_2 \leq \frac{\varepsilon}{2}.$$

It follows

$$\|f - \tilde{f}\|_\infty \leq \|f - T_\phi f\|_\infty + \|T_\phi f - \tilde{f}\|_\infty < \varepsilon. \quad \square$$

Since $C(G)$ is dense in $L^2(G)$, we immediately get

**Corollary 3.5.** Let $G$ be a compact Lie group. Then the matrix coefficients of $G$ are dense in $L^2(G)$.

Since any representation of $G$ is completely reducible, and the matrix coefficients for distinct irreducible representations are orthogonal, we can restate Peter-Weyl theorem as

**Theorem 3.6 (Peter-Weyl).** Let $G$ be a compact Lie group and $\hat{G}$ be the set of equivalence classes of irreducible representations of $G$. Then

$$L^2(G) = \bigoplus_{\rho \in \hat{G}} R(\rho),$$

where the right hand side denote the closure in $L^2$ of $\bigoplus_{\rho} R(\rho)$. 

Last time we have seen through examples that irreducible characters generate a dense subspace of class functions. Now we can prove this for all compact groups:

**Theorem 3.7** (Peter-Weyl theorem for class functions). *Suppose $G$ is compact. Then the irreducible characters of $G$ generate a dense subspace of the space of continuous class functions on $G$.***

**Proof.** Suppose $\phi$ is a class function. For any $\varepsilon > 0$, by the Peter-Weyl theorem, one can find a representation $(V, \pi)$ of $G$ and a matrix coefficient $f \in C(G)_{\pi}$ such that $\|\phi - f\|_{\infty} < \varepsilon$. Consider the function defined on $G$ by

$$\psi(x) = \int_{G} f(gxg^{-1})dg.$$  

Obviously $\psi$ is a class function, and $\|\phi - \psi\|_{\infty} < \varepsilon$. We claim that $\psi$ is a linear combination of irreducible characters, thus prove the theorem.

In fact, since $G$ is compact, $(V, \pi) = \bigoplus_{i} (V_{i}, \pi_{i})$ is a direct sum of finitely many irreducible representations. Since $f \in C(G)_{\pi} = \bigoplus_{i} C(G)_{\pi_{i}}$, we can write

$$f(g) = \sum_{i} L_{i}(\pi_{i}(g)v_{i})$$

for some $v_{i} \in V_{i}$ and $L_{i} \in V_{i}^{\ast}$. So

$$\psi(x) = \sum_{i} L_{i} \left( \int_{G} \pi_{i}(g)\pi_{i}(x)\pi_{i}(g^{-1})v_{i} \, dg \right).$$

We have already shown in the proof of Schur’s orthogonality that the map

$$v_{i} \mapsto \int_{G} \pi_{i}(g)\pi_{i}(x)\pi_{i}(g^{-1})v_{i} \, dg,$$

as the “average” of $\pi_{i}(x)$, is a linear equivariant map on $V_{i}$. So

$$\int_{G} \pi_{i}(g)\pi_{i}(x)\pi_{i}(g^{-1})v_{i} \, dg = \lambda(x)v_{i}$$

for some $\lambda(x) \in \mathbb{C}$. Computing the trace as we did before, we conclude

$$\lambda(x) = \frac{1}{\dim V_{i}} \text{Tr}(\pi_{i}(x)) = \frac{1}{\dim V_{i}} \chi_{\pi_{i}}(x).$$

It follows that

$$\psi(x) = \sum_{i} \frac{1}{\dim V_{i}} L_{i}(v_{i})\chi_{\pi_{i}}(x)$$

is a linear combination of irreducible characters. \qed

**Corollary 3.8.** *For any class function $f$ in $L^{2}(G)$,

$$f = \sum_{\pi \in \hat{G}} \langle f, \chi_{\pi} \rangle \chi_{\pi}$$*
as an $L^2$ function with respect to $L^2$-convergence, and
$$\|f\|^2_2 = \sum_{\pi \in \hat{G}} \langle f, \chi_{\pi} \rangle^2.$$ 

4. Faithful Representations

**Definition 4.1.** A representation $(V, \pi)$ of $G$ is **faithful** if as a linear action, $\pi$ is effective, i.e. the group homomorphism $\pi : G \to \text{GL}(V)$ is injective.

**Lemma 4.2.** Let $G$ be a compact Lie group. Then for any $g \neq e$, there exists a representation $(V, \pi)$ of $G$ such that $\pi(g) \neq \text{Id}$.

**Proof.** Take any function $f \in C(G)$ such that $f(e) = 0$ and $f(g) = 1$. Then there is a representation $(V, \pi)$ of $G$ and a matrix coefficient $\phi \in C(G)_r$ such that $\|f - \phi\|_{\infty} < \frac{1}{3}$. So in particular $\phi(e) \neq \phi(g)$. Since $\phi(x) = L(\pi(x)v)$ for some fixed $v \in V$ and $L \in V^*$, we must have $\pi(g) \neq \pi(e) = \text{Id}$. □

**Theorem 4.3.** Any compact Lie group possesses a faithful representation.

**Proof.** According to the previous lemma, for any $g_1 \in G^0$ (the connected component of $G$ containing $e$), $g_1 \neq e$, there exists a representation $(V_1, \pi_1)$ of $G$ such that $\pi_1(g_1) \neq \text{Id}$. The kernel $\text{Ker}(\pi_1)$ is a closed subgroup of $G$, thus a Lie group by itself. Moreover, since $G^0 \not\subseteq \text{Ker}(\pi_1)$, we must have $\dim \text{Ker}(\pi_1) < \dim G$. If $\dim \text{Ker}(\pi_1) \neq 0$, we do this procedure again, i.e. take an element $g_2 \in \text{Ker}(\pi_1)^0$ and a representation $(V_2, \pi_2)$ such that $\pi_2(g_2) \neq \text{Id}$. It follows that $\text{Ker}(\pi_1 \oplus \pi_2)$ is a compact Lie subgroup in $G$ with $\dim \text{Ker}(\pi_1 \oplus \pi_2) < \dim \text{Ker}(\pi_1)$. Continuing this procedure, we will get a sequence of representations $(V_i, \pi_i), 1 \leq i \leq N$, of $G$, such that $\dim \text{Ker}(\pi_1 \oplus \cdots \oplus \pi_N) = 0$. Since $G$ is compact, $\text{Ker}(\pi_1 \oplus \cdots \oplus \pi_N) = \{h_1, \cdots, h_M\}$ is a finite set. Now we choose representations $(W_i, \rho_i), 1 \leq i \leq M$ such that $\rho_i(h_i) \neq \text{Id}$. It follows that the representations $\pi_1 \oplus \cdots \oplus \pi_N \oplus \rho_1 \oplus \cdots \oplus \rho_M$ is a faithful representation of $G$. □

As a corollary, we will prove that any compact Lie group is a linear Lie group:

**Corollary 4.4.** Any compact Lie group is isomorphic to a closed subgroup of $U(N)$ for $N$ large.

**Proof.** Take a faithful representation $(V, \pi)$ of $G$. Since $G$ is compact, there exists a $G$-invariant inner product on $V$. It follows $\pi : G \to \text{GL}(V)$ maps $G$ into $U(V) \simeq U(N)$. The injectivity of $\pi$ implies that $G$ is isomorphic to its image, a closed subgroup of $U(N)$. □

**Remark.** There exists noncompact Lie groups that does not admit any finite dimensional faithful representation, and thus are not linear Lie groups. One example of such Lie groups is the **metaplectic group** $\text{Mp}_{2n}$, the double cover of the symplectic group $\text{Sp}_{2n}$. Although it doesn’t have any finite dimensional faithful representation, it does have faithful infinite dimensional representations, such as the **Weil representation**.