LECTURE 25-26: CARTAN'S THEOREM OF MAXIMAL TORI

1. Maximal Tori

By a *torus* we mean a compact connected abelian Lie group, so a torus is a Lie group that is isomorphic to $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$.

Definition 1.1. Let G be a compact connected Lie group. A subgroup $T \subset G$ is a *maximal torus* if T is a torus and there is no other torus T' with $T \subsetneqq T' \subset G$.

Remark. Let G be a compact Lie group, and $T \subset G$ a torus in G. Consider the conjugation action of G on itself,

$$c(g) : G \to G, \quad h \mapsto ghg^{-1}.$$

Obviously $c(g)(T) = gTg^{-1}$ is again a torus in G for any $g \in G$. Moreover, this conjugation action preserves the inclusion relation. It follows that if T is a maximal torus, so is gTg^{-1}.

Example. Consider $G = \mathrm{U}(n)$. Then the subgroup T of all diagonal matrices in $\mathrm{U}(n)$ is clearly isomorphic to $\mathbb{T}^n$. It is actually a maximal torus, for if there is a strictly larger one, then one can find some element g of $\mathrm{U}(n)$ that commutes with all elements of T. But T contains diagonal matrices with n distinct eigenvalues, and any matrix commutes with such matrices must be diagonal, so we get a contradiction. Note that any unitary matrix is unitarily diagonalizable, i.e. any $g \in \mathrm{U}(n)$ is conjugate to some element in this maximal torus by a unitary matrix. In other words, we have

$$\mathrm{U}(n) = \bigcup_{g \in \mathrm{U}(n)} gTg^{-1}.$$

It turns out that the same decomposition holds for any compact Lie group.

Theorem 1.2 (Cartan's Theorem)**.** *Let G be a compact connected Lie group, T a maximal torus of G. Then $G = \bigcup_{g \in G} gTg^{-1}$.*

Proof. Let $\mathfrak{t}$ be the Lie algebra of T. We will prove

- If G is compact, then $\exp : \mathfrak{g} \to G$ is surjective.
- We have the decomposition in the level of Lie algebra: $\mathfrak{g} = \bigcup_{g \in G} \mathrm{Ad}_g \mathfrak{t}$.

It follows

$$G = \exp(\mathfrak{g}) = \bigcup_{g \in G} \exp(\mathrm{Ad}_g \mathfrak{t}) = \bigcup_{g \in G} c(g)(\exp(\mathfrak{t})) = \bigcup_{g \in G} gTg^{-1}.$$

$\square$

Remarks. (1) We are proving the Cartan's theorem using the fact that exp is surjective for compact Lie groups. Conversely, if Cartan's theorem holds, i.e. every $g \in G$ is contained in some maximal torus, then exp is surjective on G since the exponential map is surjective on tori. So Cartan's theorem is equivalent to the fact that exp is surjective.

(2) We sketch two other proofs of the Cartan's theorem here. For more details, c.f. D.Bump, *Lie Groups*, Chapter 16-17.
   - Geometric proof: On any compact Lie group G there exists a *bi-invariant* Riemannian metric, under which the geodesics are translations of the one-parameter subgroups $t \mapsto \exp(tX)$. Now the theorem follows from the fact that any compact connected Riemannian manifold is *geodesically complete*.
   - Topologically proof: The map $G/T \times T \to G, (g,t) \mapsto gtg^{-1}$ has mapping degree $|W|$, where $W = N(T)/T$ is the Weyl group of G with respect to T (We will study this later). In particular, the map above is surjective.

(3) In general for noncompact Lie groups exp is not surjective and thus Cartan's theorem does not hold. For example, consider $G = SL(2,\mathbb{R})$. Then

$$A = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a > 0 \right\}$$

and

$$N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in \mathbb{R} \right\}$$

are both maximal connected abelian subgroups of G. Note that any element of A has trace $a + a^{-1}$ while any element in N has trace 2. So unless $a = 1$, an element of A cannot be conjugate to an element in N.

Corollary 1.3. *Let G be a compact connected Lie group and T a maximal torus of G. Then any $g \in G$ is conjugate to some $t \in T$.*

Since any representation is uniquely determined by its character, which is a conjugate invariant function, we conclude

Corollary 1.4. *Let G be a compact connected Lie group, T a maximal torus of G, and $(\pi_1, V_1), (\pi_2, V_2)$ are two finite dimensional complex representations of G. Then $\pi_1 \simeq \pi_2$ if and only if $\pi_1|_T \simeq \pi_2|_T$.*

Remark. Recall that any representation of a torus T can be decomposed into irreducible representations, which are characterized by their weights. So any representation of G is determined by a subset of the weight lattice Z_T^*, with multiplicity. This is called the *weight system* of the representation. In particular, if we take the representation to be the (complexified) adjoint representation $\mathrm{Ad} \otimes \mathbb{C}$, then any nonzero vector in the weight system is called a *root*. The set of roots is called the *root system* of G, which plays an important role in the classification theory of compact Lie groups.

2. Cartan Subalgebras

Since we are going to prove Cartan's decomposition via the corresponding Lie algebra decomposition, it is natural to study

Definition 2.1. Let $\mathfrak{g}$ be the Lie algebra of a compact Lie group. Then any maximal abelian subalgebra of $\mathfrak{g}$ is called a *Cartan subalgebra* of $\mathfrak{g}$.

Remark. Obviously if $\mathfrak{t} \subset \mathfrak{g}$ is a (maximal) abelian Lie subalgebra, then for any $g \in G$, $\mathrm{Ad}_g(\mathfrak{t})$ is a (maximal) abelian Lie subalgebra.

Proposition 2.2. *Let G be a compact Lie group. Then there is a one-to-one correspondence between maximal tori in G and Cartan subalgebras in $\mathfrak{g}$.*

Proof. This follows from the following facts:

- There is a one-to-one correspondence between conected Lie subgroups of G and Lie subalgebras of $\mathfrak{g}$ (Lecture 10).
- A connected Lie group T is abelian if and only if its Lie algebra $\mathfrak{t}$ is abelian (Lecture 8).
- The closure of any connected abelian Lie subgroup is still connected and abelian $\Longrightarrow$ a maximal connected abelian Lie subgroup must be compact, and thus a maximal torus.

$\square$

Since any one dimensional subspace of $\mathfrak{g}$ is abelian, it is clear that Cartan subalgebras of $\mathfrak{g}$ exist. It follows that maximal tori always exist. In fact,

Corollary 2.3. *Let G be a compact Lie group. Then any torus in G is contained in a maximal torus.*

Proof. Let $T_1 \subset G$ be a torus. Then its Lie algebra $\mathfrak{t}_1 = \mathrm{Lie}(T_1)$ is an abelian Lie subalgebra of $\mathfrak{g}$. Since $\dim \mathfrak{g}$ is finite, one can always find a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ that contains $\mathfrak{t}_1$. The corresponding connected Lie subgroup T is obviously a maximal torus in G that contains T_1. $\square$

Remark. Any $g \in G$ close to e has the form $\exp(X)$ for some $X \in \mathfrak{g}$, thus sits in the one-dimensional connected abelian Lie subgroup $\{\exp(tX) \mid t \in \mathbb{R}\}$. So any $g \in G$ in a small neighborhood of e sits in some maximal torus. As we have seen earlier, Cartan's theorem asserts that this holds for all $g \in G$.

In the rest of this section, we are going to prove the linear version of Cartan's decomposition: $\mathfrak{g} = \bigcup_{g \in G} \mathrm{Ad}_g \mathfrak{t}$. We first study the structure of Cartan's subalgebras:

Lemma 2.4. *Let G be a compact Lie group and $\mathfrak{t}$ a Cartan subgroup of $\mathfrak{g}$. Then there exists $X \in \mathfrak{t}$ such that $\mathfrak{t} = \ker(\mathrm{ad}_X)$.*

Proof. Since $\mathfrak{t}$ is abelian, $\mathfrak{t} \subset \ker(\mathrm{ad}_X))$ for any $X \in \mathfrak{t}$. In particular, if $\{X_1, \cdots, X_n\}$ is a basis of $\mathfrak{t}$, then $\mathfrak{t} \subset \cap_i \ker(\mathrm{ad}_{X_i})$. In fact, in this case we must have

$$\mathfrak{t} = \cap_i \ker(\mathrm{ad}_{X_i}),$$

otherwise we can take a vector $\widetilde{X} \in \cap_i \ker(\mathrm{ad}_{X_i}) \setminus \mathfrak{t}$, then

$$\tilde{\mathfrak{t}} = \mathrm{span}\{X_1, \cdots, X_n, \tilde{X}\}$$

is an abelian subalgebra of $\mathfrak{g}$ which is strictly larger than $\mathfrak{t}$, a contradiction. In what follows we will show

$$\ker(\mathrm{ad}_{X_1}) \cap \ker(\mathrm{ad}_{X_2}) = \ker(\mathrm{ad}_{X_1 + tX_2})$$

for some $t \in \mathbb{R}$, and thus by induction, $\mathfrak{t} = \ker(\mathrm{ad}_{X_1 + t_1 X_2 + \cdots + t_{n-1} X_n})$ for some $t_i \in \mathbb{R}$, completing the proof.

Take an inner product on $\mathfrak{g}$ that is invariant under the adjoint G-action on $\mathfrak{g}$, i.e.

$$\langle \mathrm{Ad}_g Y_1, \mathrm{Ad}_g Y_2 \rangle = \langle Y_1, Y_2 \rangle$$

for any $g \in G$ and $Y_1, Y_2 \in \mathfrak{g}$. Taking derivative, it follows that for any $X \in \mathfrak{g}$,

$$\langle \mathrm{ad}_X Y_1, Y_2 \rangle = -\langle Y_1, \mathrm{ad}_X Y_2 \rangle,$$

i.e. ad_X is skew-symmetric. Let $\mathfrak{t}_X = \ker(\mathrm{ad}_X)$ and $\mathfrak{u}_X = \mathrm{image}(\mathrm{ad}_X)$. Then it follows that $\mathfrak{u}_X \subset (\mathfrak{t}_X)^\perp$, and by dimension counting, $\mathfrak{u}_X = (\mathfrak{t}_X)^\perp$. Note that $\mathfrak{u}_X$ is an ad_X-invariant subspace of $\mathfrak{g}$.

Now suppose $X_1, X_2 \in \mathfrak{t}$. Obviously $\mathfrak{t}_{X_1} \cap \mathfrak{t}_{X_2} \subset \mathfrak{t}_{X_1 + X_2}$. If $\mathfrak{u}_{X_1} \cap \mathfrak{u}_{X_2} = \{0\}$, then $\mathfrak{t}_{X_1} \cap \mathfrak{t}_{X_2} = \mathfrak{t}_{X_1 + X_2}$, otherwise there is a $Y \in \mathfrak{g}$ such that $\mathrm{ad}_{X_1 + X_2}(Y) = 0$ while $\mathrm{ad}_{X_1}(Y) = \mathrm{ad}_{X_2}(-Y) \neq 0$, a contradiction.

Finally we suppose $\mathfrak{u}_{X_1} \cap \mathfrak{u}_{X_2} \neq 0$. Since X_1, X_2 commute, the Jacobi identity implies that ad_{X_1} and ad_{X_2} commute. So ad_{X_2} preserves the ad_{X_1}-invariant subspaces $\mathfrak{u}_{X_1}$. As a consequence, the intersection $\mathfrak{u}_{X_1} \cap \mathfrak{u}_{X_2}$ is invariant under $\mathrm{ad}_{X_1 + tX_2}$ for any $t \in \mathbb{R}$. Let

$$p(t) = \det(\mathrm{ad}_{X_1 + tX_2}|_{\mathfrak{u}_{X_1} \cap \mathfrak{u}_{X_2}}).$$

Then $p(t)$ is a polynomial in t, and is not identically zero since $p(0) \neq 0$. It follows that there exists a $t_1 \neq 0$ such that $p(t_1) \neq 0$, i.e. $\mathrm{ad}_{X_1 + t_1 X_2}$ is invertible on $\mathfrak{u}_{X_1} \cap \mathfrak{u}_{X_2}$. This implies $t_{X_1} \cap \mathfrak{t}_{X_2} = \mathfrak{t}_{X_1 + t_1 X_2}$, for otherwise there exists $Y \in \mathfrak{g}$ such that

$$0 \neq [X_1, Y] = -t_1 [X_2, Y] \in \mathfrak{u}_{X_1} \cap \mathfrak{u}_{X_2},$$

and thus

$$\mathrm{ad}_{X_1 + t_1 X_2}([X_1, Y]) = [X_1 + t_1 X_2, [X_1, Y]] = -[X_1, [Y, X_1 + t_1 X_2]] = 0.$$

Contradiction with the fact that $\mathrm{ad}_{X_1 + t_1 X_2}$ is invertible on $\mathfrak{u}_{X_1} \cap \mathfrak{u}_{X_2}$. $\square$

Remark. So any Cartan subalgebra is of the form $\ker(\mathrm{ad}_X)$. Such elements $X \in \mathfrak{g}$ are called *regular elements*. Obviously if X is a regular element, so is tX for $t \neq 0$.

Theorem 2.5. *Let G be a compact Lie group, and $\mathfrak{t}$ a Cartan subalgebra of $\mathfrak{g}$. Then $\mathfrak{g} = \bigcup_{g \in G} \mathrm{Ad}_g \mathfrak{t}$, i.e. for any $X \in \mathfrak{g}$, there exists $g \in G$ such that $\mathrm{Ad}_g(X) \in \mathfrak{t}$.*

Proof. According to the previous lemma, there exits $Y \in \mathfrak{g}$ such that $\mathfrak{t} = \ker(\mathrm{ad}(Y))$. Fix an Ad-invariant inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$, and consider the continuous function on G defined by

$$f(g) = \langle Y, \mathrm{Ad}_g(X) \rangle,$$

Since G is compact and f is continuous, f attains its maximum at some $g_0 \in G$. It follows that for any $Z \in \mathfrak{g}$, the function

$$t \mapsto \langle Y, \mathrm{Ad}_{\exp(tZ)}(\mathrm{Ad}_{g_0} X) \rangle$$

has a maximum at $t = 0$. Taking derivative at $t = 0$, we get

$$\langle Y, \mathrm{ad}_Z(\mathrm{Ad}_{g_0} X) \rangle = 0,$$

i.e.

$$\langle Y, \mathrm{ad}_{\mathrm{Ad}_{g_0} X}(Z) \rangle = 0$$

for all $Z \in \mathfrak{g}$. Since ad_Z is skew-symmetric, we have

$$\langle \mathrm{ad}_{\mathrm{Ad}_{g_0} X}(Y), Z \rangle = 0$$

for all $Z \in \mathfrak{g}$. So $\mathrm{ad}_{\mathrm{Ad}_{g_0} X}(Y) = 0$, i.e. $\mathrm{Ad}_{g_0}(X) \in \ker(\mathrm{ad}_Y) = \mathfrak{t}$. □

Corollary 2.6. *Let G be a compact Lie group, and $\mathfrak{t} \subset \mathfrak{g}$ a Cartan subalgebra. Then any Cartan subalgebra $\mathfrak{t}'$ of $\mathfrak{g}$ is of the form $\mathrm{Ad}_g(\mathfrak{t})$ for some $g \in G$.*

Proof. Suppose $\mathfrak{t} = \ker(\mathrm{ad}_X)$ and $\mathfrak{t}' = \ker(\mathrm{ad}_{X'})$. Then there exists $g \in G$ such that $\mathrm{Ad}_g X' \in \mathfrak{t}$. It follows

$$\begin{aligned} \mathrm{Ad}_g(\mathfrak{t}') &= \{\mathrm{Ad}_g Y \mid [Y, X'] = 0\} \\ &= \{\mathrm{Ad}_g Y \mid [\mathrm{Ad}_g Y, \mathrm{Ad}_g X'] = 0\} = \ker(\mathrm{Ad}_g X'). \end{aligned}$$

Since $\mathrm{Ad}_g X' \in \mathfrak{t}$ and $\mathfrak{t}$ is abelian, we have $\mathrm{Ad}_g(\mathfrak{t}') \supset \mathfrak{t}$. But $\mathfrak{t}$ is maximal abelian, so $\mathrm{Ad}_g(\mathfrak{t}') = \mathfrak{t}$. □

Corollary 2.7. *Let G be a compact Lie group and $T \subset G$ a maximal torus. Then any maximal torus of G is of the form gTg^{-1}.*

Proof. Suppose T' is a maximal torus in G, then $\mathfrak{t}'$ is a Cartan subalgebra of $\mathfrak{g}$. By the previous lemma, $\mathfrak{t}' = \mathrm{Ad}_g(\mathfrak{t})$ for some $g \in G$. It follows

$$gTg^{-1} = c(g)(T) = c(g)\exp(\mathfrak{t}) = \exp(\mathrm{Ad}_g(\mathfrak{t})) = \exp(\mathfrak{t}') = T'.$$

□

Remark. In particular, we see that any two maximal tori in a compact Lie group G have the same dimension. It is called the *rank* of G.

3. Surjectivity of the exponential map

In this section we are going to prove

Theorem 3.1. *Let G be a compact connected Lie group. Then the exponential map* exp *is surjective.*

Proof. We prove this by induction on the dimension of G. If $\dim G = 1$, then we have seen that G must be abelian, and thus exp is surjective.

Now suppose $\dim G = n > 1$, and the theorem holds for any Lie group of dimension less than n. Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{g}$. Then

$$\exp \mathfrak{g} = \bigcup_{g\in G} \exp(\mathrm{Ad}_g \mathfrak{t}) = \bigcup_{g\in G} c(g) \exp \mathfrak{t} = \bigcup_{g\in G} c(g)T,$$

where T is the maximal torus with Lie algebra $\mathfrak{t}$. So $\exp \mathfrak{g}$, as the image of the compact set $G \times T$ under the continuous map

$$G \times T \to G, \quad (g,t) \mapsto gtg^{-1},$$

is compact and therefore closed in G. We will show that it is also open, and thus equals G by connectivity.

Fix any $0 \neq X_0 \in \mathfrak{g}$, and write $g_0 = \exp(X_0)$. We need to show that $\exp \mathfrak{g}$ contains a neighborhood of g_0. Let

$$A = Z_G(g_0)^0 = \{g \in G \mid g_0 g g_0^{-1} = g\}^0,$$

the connected component of the centralizer of g_0 in G. It is easy to see that A is a closed Lie subgroup of G whose Lie algebra is given by

$$\mathfrak{a} = Z_{\mathfrak{g}}(g_0) := \{Y \in \mathfrak{g} \mid \mathrm{Ad}_{g_0} Y = Y\}.$$

Note that $\exp(tX_0) \in A$ for any $t \in \mathbb{R}$. In particular, $g_0 \in A$.

<u>Case 1</u>: $\dim \mathfrak{a} = \dim \mathfrak{g}$. Then $\mathfrak{a} = \mathfrak{g}$, and $A = G$. So $g_0 \in Z(G)$. Let $\mathfrak{t}$ be any Cartan subalgebra containing X_0. Then for any $X \in \mathfrak{t}$ and $g \in G$,

$$g_0 \exp(\mathrm{Ad}_g X) = g_0 c(g)(\exp(X)) = c(g)(\exp(X_0)\exp(X)) = \exp(\mathrm{Ad}_g(X_0 + X)).$$

Since g and X are arbitrary, we get

$$g_0 \exp(\mathfrak{g}) \subset \exp(\mathfrak{g}).$$

But $\exp(\mathfrak{g})$ contains a neighborhood U of e. So $\exp(\mathfrak{g})$ must also contains $g_0 U$, a neighborhood of g_0.

<u>Case 2</u>: $\dim \mathfrak{a} < \dim \mathfrak{g}$. Since $X_0 \in \mathfrak{a}$, we have $1 \leq \dim \mathfrak{a} < n$. By induction assumption, $A = \exp(\mathfrak{a})$. So

$$\bigcup_{g\in G} g^{-1}Ag = \bigcup_{g\in G} \exp(\mathrm{Ad}_g \mathfrak{a}) \subset \exp(\mathfrak{g}).$$

We will show that $\bigcup_{g\in G} g^{-1}Ag$ contains a neighborhood of g_0. It follows that $\exp(\mathfrak{g})$ contains a neighborhood of g_0.

As before we will take an Ad-invariant inner product on $\mathfrak{g}$. Let $\mathfrak{b} = \mathfrak{a}^\perp$. Then both $\mathfrak{a}$ and $\mathfrak{b}$ are $\mathrm{Ad}_{g_0^{-1}}$ invariant. In particular, $\mathrm{Ad}_{g_0^{-1}} - \mathrm{Id}$ is an invertible endomorphism of $\mathfrak{b}$. Consider the smooth map $\varphi : \mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b} \to G$ given by

$$\varphi(X, Y) = g_0^{-1} \exp(Y) g_0 \exp(X) \exp(-Y).$$

Then

$$(d\varphi)_{(0,0)}(X, 0) = \left.\frac{d}{dt}\right|_{t=0} \varphi(tX, 0) = X$$

and

$$(d\varphi)_{(0,0)}(0, Y) = \left.\frac{d}{dt}\right|_{t=0} \varphi(0, tY) = \mathrm{Ad}_{g_0^{-1}} Y - Y.$$

It follows that $d\varphi$ is an isomorphism at $(0, 0) = 0 \in \mathfrak{g}$. Thus φ is a diffeomorphism from a neighborhood of $0 \in \mathfrak{g}$ to a neighborhood of $e \in G$. Since L_{g_0} is a diffeomorphism, the set

$$\{\exp(Y) g_0 \exp(X) \exp(-Y) \mid X \in \mathfrak{a}, Y \in \mathfrak{b}\}$$

contains a neighborhood of g_0 in G. Note that $\exp(Y) \in G$, $g_0 \in A$, and $\exp(X) \in A$ for any $X \in \mathfrak{a}$, we conclude that $\bigcup_{g \in G} g^{-1} A g$ contains a neighborhood of g_0 in G. □

4. Applications of Catan's theorem to Centralizers

We have seen in pset 2 that the centralizer

$$Z(G) = \{g \in G \mid gh = hg, \quad \forall h \in G\}$$

of G is a normal Lie subgroup of G. Now we give a characterization.

Corollary 4.1. *If G is a compact connected Lie group, then $Z(G)$ is the intersection of all maximal tori in G.*

Proof. Suppose $g \in Z(G)$, then for any maximal torus T, there is some $h \in G$ such that $hgh^{-1} \in T$. But $hgh^{-1} = g$, so $g \in T$ for any maximal torus T.

Conversely suppose g lies in any maximal torus. For any $h \in G$, there is a maximal torus T such that $h \in T$. Since T is abelian, $hg = gh$. So $g \in Z(G)$. □

In general for any subgroup $H \subset G$, the centralizer

$$Z_G(H) = \{g \in G \mid gh = hg, \quad \forall h \in H\}$$

is a Lie subgroup of G.

Corollary 4.2. *Supposer G is compact, and $A \subset G$ is a connected abelian Lie subgroup. Then $Z_G(A)$ is the union of all maximal tori in G that contains A. In particular, $Z_G(A)$ is connected.*

Proof. Note that $Z_G(A) = Z_G(\bar{A})$. So we may assume that A is a torus, otherwise we may replace A by $\bar{A}$.

Suppose T is a maximal torus containing A. Then by definition $T \subset Z_G(A)$. So $Z_G(A)$ contains the union of all maximal tori in G that contains A.

Conversely, let $g \in Z_G(A)$, or equivalently, $A \subset Z_G(g)$. Then $Z_G(g)^0$ is a compact connected Lie group, and $A \subset Z_G(g)^0$ since $e \in A$ and A is connected. Let T_1 be a maximal torus in $Z_G(g)^0$ that contains A. Since exp is surjective, there exists $X \in \mathfrak{g}$ such that $g = \exp(X) \in Z_G(g)^0$. So by definition, $g \in Z(Z_G(g)^0)$. By the previous corollary, $g \in T_1$. In other words, we find a torus $T_1 \subset Z_G(g)^0 \subset G$ that contains both A and g. In particular, if T is any maximal torus in G that contains T_1, then $g \in T$. This completes the proof. $\square$

As a consequence, we get the following characterizations of maximal tori:

Corollary 4.3. *Let G be a compact connected Lie group and T a torus in G. Then the following are equivallent:*

(1) *T is a maximal torus.*
(2) *$Z_G(T) = T$.*
(3) *$G = \bigcup_{g \in G} gTg^{-1}$.*

Proof. (1)$\Leftrightarrow$(2): This is an obvious consequence of corollary 4.2.

(1)$\Rightarrow$(3): This is part of Cartan's theorem.

(3)$\Rightarrow$(1): Suppose $G = \bigcup_{g \in G} gTg^{-1}$. Then the map

$$G/Z_G(T) \times T \to G, \quad (gZ_G(T), t) \mapsto gtg^{-1}$$

is surjective. In particular, $\dim G/Z_G(T) + \dim T \geq \dim G$. So $\dim Z_G(T) \leq \dim T$. This implies T is maximal, since if T is not maximal, then $Z(T)$ contains all maximal torus that contains T, and thus $\dim Z(T) > \dim T$. $\square$

5. The Maximal Tori of Classical Groups

5.1. **$G = \mathrm{U}(n)$ or $\mathrm{SU}(n)$.** Recall that

$$T = \left\{ \begin{pmatrix} e^{it_1} & & \\ & \ddots & \\ & & e^{it_n} \end{pmatrix} \ : \ t_i \in \mathbb{R} \right\}$$

is a maximal torus in $\mathrm{U}(n)$.

Similarly one can study $G = \mathrm{SU}(n) = \mathrm{U}(n) \cap \mathrm{SL}(n, \mathbb{C})$. Then a maximal torus is given by

$$\tilde{T} = \left\{ \begin{pmatrix} e^{it_1} & & \\ & \ddots & \\ & & e^{it_n} \end{pmatrix} \ : \ t_i \in \mathbb{R}, t_1 + \cdots + t_n = 0 \right\}.$$

5.2. $G = \mathrm{SO}(n)$.

Next let's study the special orthogonal group $\mathrm{SO}(n)$, which is the identity component of $\mathrm{O}(n)$. We first assume $n = 2l$ is even. Then

$$T = \mathrm{SO}(2) \times \cdots \times \mathrm{SO}(2) = \left\{ \begin{pmatrix} \cos t_1 & -\sin t_1 & & & \\ \sin t_1 & \cos t_1 & & & \\ & & \ddots & & \\ & & & \cos t_l & -\sin t_l \\ & & & \sin t_l & \cos t_l \end{pmatrix} : t_i \in \mathbb{R} \right\}$$

is a torus in $\mathrm{SO}(n)$. Moreover, from linear algebra we know that any orthogonal matrix is conjugate using orthogonal matrices to a matrix in T. In other words,

$$\mathrm{SO}(n) = \bigcup_{g \in \mathrm{SO}(n)} gTg^{-1}.$$

It follows that T is a maximal torus in $\mathrm{O}(n)$.

Similarly for $n = 2l + 1$ an odd number, a maximal torus of $\mathrm{SO}(n)$ is

$$T = \mathrm{SO}(2) \times \cdots \times \mathrm{SO}(2) \times \{1\} = \left\{ \begin{pmatrix} \cos t_1 & -\sin t_1 & & & & \\ \sin t_1 & \cos t_1 & & & & \\ & & \ddots & & & \\ & & & \cos t_l & -\sin t_l & \\ & & & \sin t_l & \cos t_l & \\ & & & & & 1 \end{pmatrix} : t_i \in \mathbb{R} \right\}$$

5.3. $G = \mathrm{Sp}(n)$.

Now consider the compact symplectic group $\mathrm{Sp}(n) = \mathrm{Sp}(2n, \mathbb{C}) \cap \mathrm{U}(2n)$. $\mathrm{Sp}(n)$ consists of unitary matrices of the form $\begin{pmatrix} A & -\bar{B} \\ B & \bar{A} \end{pmatrix}$. There is a canonical inclusion

$$\mathrm{U}(n) \to \mathrm{Sp}(n), \quad A \mapsto \begin{pmatrix} A & 0 \\ 0 & \bar{A} \end{pmatrix}.$$

We denote by T the image of the maximal torus of $U(n)$ described above under this inclusion, i.e.

$$T = \left\{ \begin{pmatrix} e^{it_1} & & & & & \\ & \ddots & & & & \\ & & e^{it_n} & & & \\ & & & e^{-it_1} & & \\ & & & & \ddots & \\ & & & & & e^{-it_n} \end{pmatrix} \ : \ t_i \in \mathbb{R} \right\}.$$

It is easy to see that T is a torus in $\mathrm{Sp}(n)$ and is maximal.