LECTURE 28: THE STRUCTURE OF COMPACT LIE GROUPS

1. Structure of Compact Lie Algebras

Let $\mathfrak{g}$ be a Lie algebra. Recall that a subset $\mathfrak{h} \subset \mathfrak{g}$ is an *ideal* if $[\mathfrak{g}, \mathfrak{h}] \subset \mathfrak{h}$. In particular any ideal is a Lie subalgebra.

Example. The center $Z(\mathfrak{g})$ is an ideal because $[\mathfrak{g}, Z(\mathfrak{g})] = 0$.

Example. The *derived Lie subalgebra* $\mathfrak{g}' = \mathrm{span}\{[g,h] \mid g, h \in \mathfrak{g}\}$ is an ideal of $\mathfrak{g}$.

Definition 1.1. Let $\mathfrak{g}$ be a Lie algebra.

(1) $\mathfrak{g}$ is *simple* if it has no nonzero proper ideals and if $\dim \mathfrak{g} > 1$.
(2) $\mathfrak{g}$ is *semisimple* if it is a direct sum of simple Lie algebras.
(3) $\mathfrak{g}$ is *reductive* if it is a direct sum of a semisimple Lie algebra and an abelian Lie algebra.

Our first theorem shows that the Lie algebra of a compact Lie group is reductive.

Theorem 1.2. *Let G be a compact Lie group, then $\mathfrak{g}$ is reductive. More explicitly,*

$$\mathfrak{g} = \mathfrak{g}' \oplus Z(\mathfrak{g}),$$

where $Z(\mathfrak{g})$ is abelian, and $\mathfrak{g}'$ is semisimple.

Proof. Choose any adjoint invariant inner product on $\mathfrak{g}$. Then we have seen earlier, ad_X is skew-symmetric for any $X \in \mathfrak{g}$. It follows that if $\mathfrak{a}$ is an ideal of $\mathfrak{g}$, so is $\mathfrak{a}^\perp$. As a consequence, $\mathfrak{g}$ can be decomposed into a direct sum of minimal ideals

$$\mathfrak{g} = \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_k \oplus \mathfrak{z}_1 \oplus \cdots \oplus \mathfrak{z}_l,$$

where $\dim \mathfrak{s}_j > 1$ and $\dim \mathfrak{z}_j = 1$. Obviously

$$\mathfrak{s} = \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_k$$

is semisimple, and

$$\mathfrak{z} = \mathfrak{z}_1 \oplus \cdots \oplus \mathfrak{z}_l$$

is abelian. It remains to show $\mathfrak{s} = \mathfrak{g}'$ and $\mathfrak{z} = Z(\mathfrak{g})$.

First notice that $[\mathfrak{s}_i, \mathfrak{z}_j] \subset \mathfrak{s}_i \cap \mathfrak{z}_j = \{0\}$ for any i, j. So $[\mathfrak{s}_i, \mathfrak{z}_j] = 0$ for any i, j. Similarly $[\mathfrak{z}_i, \mathfrak{z}_j] = 0$ for all $i \neq j$. Moreover, $[\mathfrak{z}_i, \mathfrak{z}_i] = 0$ for any i since $\dim \mathfrak{z}_i = 1$. So $[\mathfrak{g}, \mathfrak{z}_i] = 0$ for all i. In other words, $\mathfrak{z} = \mathfrak{z}_1 \oplus \cdots \oplus \mathfrak{z}_l \subset Z(\mathfrak{g})$. Conversely, suppose $Z \in Z(\mathfrak{g})$. Decompose Z as

$$Z = S_1 + \cdots + S_k + Z_1 + \cdots + Z_l,$$

where $S_i \in \mathfrak{s}_i$, and $Z_i \in \mathfrak{z}_i$. Then $0 = [Z, \mathfrak{s}_i] = [S_i, \mathfrak{s}_i]$, i.e., $S_i \in Z(\mathfrak{s}_i)$. Since $\mathfrak{s}_i$ is a minimal ideal of dimension $\dim \mathfrak{s}_i > 1$, we must have $S_i = 0$. So $Z \in \mathfrak{z}$. This proves $\mathfrak{z} = Z(\mathfrak{g})$.

Similarly for $i \neq j$, $[\mathfrak{s}_i, \mathfrak{s}_j] = 0$. We claim that for any i, $\mathfrak{s}_i' = \mathrm{span}[\mathfrak{s}_i, \mathfrak{s}_i] = \mathfrak{s}_i$. In fact, since $\dim \mathfrak{s}_i > 1$ and $\mathfrak{s}_i \cap Z(\mathfrak{g}) = \{0\}$, $\dim \mathfrak{s}_i' \geq 1$. So $\dim \mathfrak{s}_i' = \dim \mathfrak{s}_i$, otherwise it is a nonzero proper ideal. Now it follows that

$$\mathrm{span}[\mathfrak{g}, \mathfrak{g}] = \mathrm{span}[\mathfrak{s}_1, \mathfrak{s}_1] \oplus \cdots \oplus \mathrm{span}[\mathfrak{s}_k, \mathfrak{s}_k] = \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_k,$$

or in other words, $\mathfrak{s} = \mathfrak{g}'$. □

2. The Commutator Subgroup

Recall that for any group G, its *commutator subgroup* G' is the normal subgroup generated by elements of the form $g_1 g_2 g_1^{-1} g_2^{-1}$.

Theorem 2.1. *Let G be a compact connected Lie group. Then G' is a connected closed normal Lie subgroup of G with Lie algebra $\mathfrak{g}'$.*

Proof. Since G is compact, it is a closed Lie subgroup of $\mathrm{U}(N)$ for some N. Decompose the standard representation of G on $\mathbb{C}^N$ into irreducible ones,

$$\mathbb{C}^N = \mathbb{C}^{n_1} \oplus \cdots \oplus \mathbb{C}^{n_k},$$

where $n_1 + \cdots + n_k = N$, and denote by π_i the irreducible representation of G on $\mathbb{C}^{n_i}$.

Consider the map

$$\varphi : G \to (S^1)^k, \quad g \mapsto (\det \pi_1(g), \cdots, \det \pi_k(g)).$$

Then this is a Lie group homomorphism. It follows that $H = \ker(\varphi)$ is a closed Lie subgroup of G with Lie algebra $\mathfrak{h} = \ker(d\varphi)$. We will show that $\mathfrak{h} = \mathfrak{g}'$, and $H^0 = G'$, which will finish the proof.

We first show $\mathfrak{h} = \mathfrak{g}'$. Suppose $Z \in Z(\mathfrak{g})$, then $e^{tZ} \in Z(G)$. By Schur's lemma, $\pi_i(e^{tZ}) = c_i(t)\mathrm{Id}$ for some scalar $c_i(t)$ with $c_i(0) = 1$. So if we think of G as a closed subgroup of $U(n_1) \times \cdots \times U(n_k)$, then Z is given by the diagonal matrix

$$\left.\frac{d}{dt}\right|_{t=0} (\pi_1 \oplus \cdots \oplus \pi_k)(e^{tZ}) = \mathrm{diag}(c_1'(0), \cdots, c_1'(0), \cdots, c_k'(0), \cdots, c_k'(0)).$$

It follows that

$$d\varphi(Z) = \left(d\det \left.\frac{d}{dt}\right|_{t=0} \pi_1(e^{tZ}), \cdots, d\det \left.\frac{d}{dt}\right|_{t=0} \pi_k(e^{tZ}) \right) = (n_1 c_1'(0), \cdots, n_k c_k'(0)).$$

As a consequence, $\ker(d\varphi) \cap Z(\mathfrak{g}) = \{0\}$. On the other hand, since $\mathrm{tr}(AB) = \mathrm{tr}(BA)$, for any $X \in \mathfrak{g}'$ we must have $\mathrm{tr}(d\pi_i(X)) = 0$. So $\det \pi_i(e^{tX}) \equiv 1$. We thus get $d\varphi(X) = 0$ for all $X \in \mathfrak{g}'$, i.e. $\mathfrak{g}' \subset \ker(d\varphi)$. Combine this with $\ker(d\varphi) \cap Z(\mathfrak{g}) = \{0\}$, we conclude $\ker(d\varphi) = \mathfrak{g}'$.

Finally we show $H^0 = G'$. Obviously $G' \subset H$ since determinant is multiplicative. G' is connected since $G' = \cup_j U^j$, where $U = \{g_1 g_2 g_1^{-1} g_2^{-1} \mid g_1, g_2 \in G\}$ is connected, and $e \in U^j$ for all j. So $G' \subset H^0$. It remains to show $H^0 \subset G'$, or equivalently, to show G' contains a neighborhood of e in H. According to theorem 1.2 in lecture 8, for any $X, Y \in \mathfrak{h}$, $[X, Y]$ is the tangent vector of the curve

$$t \mapsto c_{X,Y}(t) = \exp(\sqrt{t}X)\exp(\sqrt{t}Y)\exp(-\sqrt{t}X)\exp(-\sqrt{t}Y)$$

at $t = 0$. Now let $\{[X_1, Y_1], \cdots, [X_m, Y_m]\}$ be a basis of $\mathfrak{g}'$. Consider the map

$$c : \mathbb{R}^m \to H, \quad (t_1, \cdots, t_m) \mapsto c_{X_1,Y_1}(t_1) \cdots c_{X_m,Y_m}(t_m).$$

Then dc_0 is an isomorphism onto $\mathfrak{h}$. So c is locally a diffeomorphism near 0. Thus the image of c contains a neighborhood of e in H. This completes the proof since the image of c is contained in G'. □

Corollary 2.2. *$G' \cap Z(G)$ is a finite group.*

Proof. We have just seen in the proof that for any $g \in Z(G)$, $\pi(g)$ must be a diagonal matrix of the form $\mathrm{diag}(c_1, \cdots, c_1, \cdots, c_k, \cdots, c_k)$. If we also have $g \in G' = (\ker(\varphi))^0$, then $c_1^{n_1} = \cdots = c_k^{n_k} = 1$. So c_i is an n_i^{th}-root of unity. It follows that $G' \cap Z(G)$ is a finite group. □

Proposition 2.3. *Let $\mathfrak{g}' = \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_k$ be the decomposition of $\mathfrak{g}'$ into simple ideals and let $S_i = \exp \mathfrak{s}_i$. Then*

1. *S_i is a connected closed Lie subgroup of G' with Lie algebra $\mathfrak{s}_i$.*
2. *Any proper closed normal Lie subgroup of S_i are finite and lies in the center of S_i.*

Proof. (1) Let $K_i = \{g \in G' \mid \mathrm{Ad}_g|_{\mathfrak{s}_j} = \mathrm{Id}, \forall j \neq i\}^0$. Then K_i is a connected closed Lie subgroup of G'. We will show $K_i = S_i$. In fact, if we denote the Lie algebra of K_i by $\mathfrak{k}_i$, then

$$\begin{aligned} X \in \mathfrak{k}_i &\iff \exp(tX) \in K_i, \forall t \in \mathbb{R} \\ &\iff \exp(t\,\mathrm{ad}_X)|_{s_j} = \mathrm{Ad}_{\exp(tX)}|_{\mathfrak{s}_j} = \mathrm{Id}, \forall j \neq i \\ &\iff \mathrm{ad}_X(\mathfrak{s}_j) = 0, \forall j \neq i \\ &\iff [X, \mathfrak{s}_j] = 0, \forall j \neq i. \end{aligned}$$

So $\mathfrak{s}_i \subset \mathfrak{k}_i$. Conversely, if $X \in \mathfrak{k}_i$, then projection of X onto $\mathfrak{s}_j$, $j \neq i$, must be zero. So $\mathfrak{k}_i = \mathfrak{s}_i$. It follows $K_i = \exp(\mathfrak{k}_i) = \exp(\mathfrak{s}_i) = S_i$.

(2) Suppose N is a proper closed normal Lie subgroup of S_i, i.e. $sNs^{-1} = N$ for all $s \in S_i$. Taking derivative, we get $\mathrm{Ad}_s\mathfrak{n} = \mathfrak{n}$ for any $s \in S_i$, where $\mathfrak{n}$ is the Lie algebra of N. Taking derivative again, we have $\mathrm{ad}_X\mathfrak{n} \subset \mathfrak{n}$ for any $X \in \mathfrak{s}_i$. So $\mathfrak{n}$ is an ideal of $\mathfrak{s}_i$. It follows that $\mathfrak{n} = \{0\}$, i.e. N is discrete. Since N is closed, it is also compact. So N is finite.

To show N lies in the center of S_i, for each $n \in N$ we consider

$$C_n = \{sns^{-1} \mid s \in S_i\}.$$

It is connected since S_i is connected. Since N is normal, $C_n \subset N$. So C_n contains only one element since N is discrete. It follows that $C_n = \{n\}$, and thus n lies in the center of S_i. □

3. The Structure of Compact Lie Groups

Theorem 3.1. *Let G be a compact connected Lie group. Then*

(1) *G is the product of a semisimple Lie group with an abelian Lie group,*
$$G = G' Z(G)^0.$$
(2) *Let $F = \{(g, g^{-1}) \mid g \in G' \cap Z(G)^0\}$, then F is finite and*
$$G \cong (G' \times Z(G)^0)/F.$$
(3) *There is a finite abelian subgroup F' of $S_1 \times \cdots \times S_k$ such that*
$$G' \cong (S_1 \times \cdots \times S_k)/F'.$$

Proof. (1) Since G' is closed, it is compact. So $G' = \exp \mathfrak{g}'$, and
$$G = \exp \mathfrak{g} = \exp(\mathfrak{g}' \oplus Z(\mathfrak{g})) = G' Z(G)^0.$$

(2) F is finite since $G' \cap Z(G)^0 \subset G' \cap Z(G)$ and the later is finite. From (1), the Lie group homomorphism
$$G' \times Z(G)^0 \to G, \quad (g, z) \mapsto gz$$
is surjective. The kernel of this map is F. So $G \cong (G' \times Z(G)^0)/F$.

(3) Since $[\mathfrak{s}_i, \mathfrak{s}_j] = 0$ for $i \neq j$,
$$G' = \exp \mathfrak{g}' = \exp(\mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_k) = S_1 \cdots S_k.$$
So the Lie group homomorphism
$$S_1 \times \cdots \times S_k \to G', \quad (s_1, \cdots, s_k) \mapsto s_1 \cdots s_k$$
is surjective. Moreover, its differential at e is the identity map. So the kernel of this map is a discrete normal closed subgroup, which has to be finite and lies in the center. □