LECTURE 5: THE METHOD OF STATIONARY PHASE

1. A crash course on Fourier transform

¶ Some notions.
- For \( j = 1, \cdots, n \),
  - \( \partial_j = \frac{\partial}{\partial x_j} \).
  - \( D_j = \frac{1}{i} \partial_j \).
- For any multi-index \( \alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n \),
  - \( |\alpha| = \alpha_1 + \cdots + \alpha_n \).
  - \( \alpha! = \alpha_1! \cdots \alpha_n! \).
  - \( x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \).
  - \( \partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} \).
  - \( D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n} \).
- For any \( \varphi \in C^\infty(\mathbb{R}^n) \),
  - Gradient vector \( \nabla \varphi = (\partial_1 \varphi, \cdots, \partial_n \varphi) \).
  - Hessian matrix \( d^2 \varphi = [\partial_i \partial_j \varphi]_{n \times n} \).

¶ Schwartz functions.

Definition 1.1. A function \( \varphi \in C^\infty(\mathbb{R}^n) \) is called a Schwartz function (or a rapidly decreasing function) if

(1) \( \sup_{\mathbb{R}^n} |x^\alpha \partial^\beta \varphi| < \infty \)

for all multi-indices \( \alpha \) and \( \beta \).

We will denote the set of all Schwartz functions by \( \mathcal{S}(\mathbb{R}^n) \), or by \( \mathcal{S} \) for simplicity. Obviously \( \mathcal{S} \) is an infinitely dimensional vector space. It is easy to see

- \( C_0^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n) \) for any \( 1 \leq p \leq \infty \).
- For any \( \lambda \in \mathbb{C} \) with \( \text{Re}(\lambda) > 0 \), \( \varphi(x) = e^{-\lambda |x|^2} \in \mathcal{S}(\mathbb{R}^n) \).
- If \( \varphi \) is Schwartz, so are \( x^\alpha D^\beta \varphi, D^\alpha x^\beta \varphi \), where \( \alpha, \beta \) are any multi-indices.

Recall that a semi-norm on a vector space \( V \) is function \( \cdot |: \cdot |: V \rightarrow \mathbb{R} \) so that

(1) (Absolute homogeneity) \( |\lambda v| = |\lambda| |v| \).
(2) (Triangle inequality) \( |u + v| \leq |u| + |v| \).

On \( \mathcal{S} \) one can define, for each pair of multi-indices \( \alpha, \beta \) a semi-norm

(2) \( |\varphi|_{\alpha, \beta} = \sup_{\mathbb{R}^n} |x^\alpha \partial^\beta \varphi| \).
We say that $\varphi_j \to \varphi$ in $\mathcal{S}$ if

$$|\varphi_j - \varphi|_{\alpha, \beta} \to 0$$

as $j \to \infty$ for all $\alpha, \beta$. The topology induced by these semi-norms turns $\mathcal{S}$ into a Fréchet space.

Equivalently, one can define a metric $d$ on $\mathcal{S}$ via

$$(3) \quad d(\varphi, \psi) = \sum_{k=0}^{\infty} 2^{-k} \frac{\|\varphi - \psi\|_k}{1 + \|\varphi - \psi\|_k},$$

where for each $k \geq 0$,

$$\|\varphi\|_k = \max_{|\alpha| + |\beta| \leq k} \sup_{x \in \mathbb{R}^n} |x^\alpha D^\beta \varphi|$$

is a norm on $\mathcal{S}$. Then we have a metric topology on $\mathcal{S}$ which coincides with the one described above.

**Exercise 1.** Under the topology described above, $C_0^\infty(\mathbb{R}^n)$ is dense in $\mathcal{S}$.

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### Fourier transform on $\mathcal{S}$

Let $\varphi \in \mathcal{S}$ be a Schwartz function. By definition its Fourier transform $\mathcal{F}\varphi$ is

$$(4) \quad \mathcal{F}\varphi(\xi) = \hat{\varphi}(\xi) = \int_{\mathbb{R}^n} e^{-ix \cdot \xi} \varphi(x) dx.$$  

For example, one can prove

$$(5) \quad \mathcal{F}(e^{-\frac{|x|^2}{2}}) = (2\pi)^{\frac{n}{2}} e^{-\frac{|\xi|^2}{2}}.$$  

By definition we have the obvious inequality

$$\|\mathcal{F}\varphi\|_{L^\infty} \leq \|\varphi\|_{L^1}.$$  

It is easy to check that for any $\varphi \in \mathcal{S}$ and any multi-index $\alpha$,

$$(6) \quad \mathcal{F}(x^\alpha \varphi) = (-1)^{|\alpha|} D_\xi^\alpha (\mathcal{F}(\varphi))$$  

and

$$(7) \quad \mathcal{F}(D_\xi^\alpha \varphi) = \xi^\alpha \mathcal{F}(\varphi).$$

As a consequence, we see

$$\varphi \in \mathcal{S} \implies \mathcal{F}\varphi \in \mathcal{S}.$$  

From definition it is also easy to check

$$(8) \quad \int_{\mathbb{R}^n} \mathcal{F}\varphi(\xi) \psi(\xi) d\xi = \int_{\mathbb{R}^n} \varphi(x) \mathcal{F}\psi(x) dx.$$  

**Lemma 1.2.** Suppose $\varphi \in \mathcal{S}$.

1. For any $\lambda \in \mathbb{R}_+$ , if we let $\varphi_\lambda = \varphi(\lambda x)$, then $\varphi_\lambda \in \mathcal{S}$ and

$$\mathcal{F}\varphi_\lambda(\xi) = \frac{1}{\lambda^n} \mathcal{F}\varphi\left(\frac{\xi}{\lambda}\right).$$
(2) For any $a \in \mathbb{R}^n$, if we let $T_a \varphi(x) = \varphi(x + a)$, then $T_a \varphi \in \mathcal{S}$ and

$$ (\mathcal{F}T_a \varphi)(\xi) = e^{ia \cdot \xi} \mathcal{F} \varphi(\xi). $$

Now we can prove

**Theorem 1.3.** $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ is an isomorphism with inverse

$$ (\mathcal{F}^{-1} \psi)(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \psi(\xi) d\xi. $$

**Proof.** We have

$$ \int \mathcal{F} \varphi(\xi) \psi(\lambda \xi) = \frac{1}{\lambda^n} \int \varphi(x) \mathcal{F} \psi\left(\frac{x}{\lambda}\right)(x) dx = \int \varphi(\lambda x) \mathcal{F} \psi(x) dx. $$

Letting $\lambda \to 0$, we get

$$ \psi(0) \int \mathcal{F} \varphi(\xi) d\xi = \varphi(0) \int \mathcal{F} \psi(x) dx. $$

In particular, if we take $\psi(\xi) = e^{-\frac{|\xi|^2}{2}}$, then we get

$$ \varphi(0) = \frac{1}{(2\pi)^n} \int \mathcal{F} \varphi(\xi) d\xi. $$

Finally we replace $\varphi$ by $T_a \varphi$ in the above formula, then

$$ \varphi(a) = T_a \varphi(0) = \frac{1}{(2\pi)^n} \int e^{ia \cdot \xi} \mathcal{F} \varphi(\xi) d\xi. $$

As a consequence, we get

$$ \|\varphi\|^2_{L^2} = \frac{1}{(2\pi)^n} \|\hat{\varphi}\|^2_{L^2}. $$

**¶ The Fourier transform of the Gaussian $e^{-\frac{1}{2}x^T Q x}$.**

**Theorem 1.4.** Let $Q$ be any real, symmetric, positive definite $n \times n$ matrix, then

$$ \mathcal{F}(e^{-\frac{1}{2}x^T Q x})(\xi) = \frac{(2\pi)^{n/2}}{(\det Q)^{1/2}} e^{-\frac{1}{2} \xi^T Q^{-1} \xi}. $$

**Proof.** Let’s first assume $n = 1$, so that $Q = q$ is a positive number. By definition,

$$ \mathcal{F}(e^{-\frac{1}{2}x^T q x})(\xi) = \int_{\mathbb{R}} e^{-\frac{1}{2}qx^2 - i\xi x} dx. $$
Taking $\xi$ derivative, we get
\[
\frac{d}{d\xi} F(e^{-\frac{1}{2}x^T q x})(\xi) = -ix \int_\mathbb{R} e^{-\frac{1}{2}q x^2 - i x \xi} d\xi = \frac{i}{q} \int_\mathbb{R} \frac{d}{dx}(e^{-\frac{1}{2}q x^2})e^{-i x \xi} dx
\]
\[
= -\frac{i}{q} \int_\mathbb{R} e^{-\frac{1}{2}q x^2} \frac{d}{dx}(e^{-i x \xi}) d\xi = -\frac{\xi}{q} F(e^{-\frac{1}{2}x^T q x})(\xi).
\]
It follows
\[
F(e^{-\frac{1}{2}x^T q x})(\xi) = Ce^{-\frac{\xi^2}{2q}},
\]
where
\[
C = F(e^{-\frac{1}{2}x^T q x})(0) = \int_\mathbb{R} e^{-\frac{1}{2}q x^2} dx = \sqrt{\frac{2\pi}{q}}.
\]

Next suppose $Q$ is a diagonal matrix. Then the left hand side of (13) is a product of $n$ one-dimensional integrals, and the result follows from the one-dimensional case.

For the general case, one only need to choose an orthogonal matrix $O$ so that
\[
O^T Q O = D = \text{diag}(\lambda_1, \ldots, \lambda_n),
\]
where $\lambda_i$’s are the eigenvalues of $Q$, then if we set $y = O^{-1} x,$
\[
F(e^{-\frac{1}{2}x^T Q x})(\xi) = \int_{\mathbb{R}^n} e^{-\frac{1}{2}x^T Q x - i x \xi} dx = \int_{\mathbb{R}^n} e^{-\frac{1}{2}y^T D y - i y^T O^T \xi} dy
\]
\[
= \frac{(2\pi)^{n/2}}{(\det D)^{1/2}} e^{-\frac{1}{2}\xi^T D^{-1} O^T \xi} = \frac{(2\pi)^{n/2}}{(\det Q)^{1/2}} e^{-\frac{1}{2}\xi^T Q^{-1} \xi}.
\]

Now suppose $Q$ is an $n \times n$ complex matrix such that $Q = Q^T$ and $\text{Re}(Q)$ is positive definite. We observe that the integral on the left hand side of (13) is an analytic function of the entries $Q_{ij} = Q_{ji}$ of $Q$ in the region $\text{Re} Q > 0$. The same is true for the right hand side of (13), as long as we choose $\det^{1/2} Q$ to be the branch such that $\det^{1/2} Q > 0$ for real positive definite $Q$. So we end up with

**Theorem 1.5.** Let $Q$ be any $n \times n$ complex matrix such that $Q = Q^T$ and $\text{Re}(Q)$ is positive definite, then
\[
F(e^{-\frac{1}{2}x^T Q x})(\xi) = \frac{(2\pi)^{n/2}}{(\det Q)^{1/2}} e^{-\frac{1}{2}\xi^T Q^{-1} \xi}.
\]

Another way to describe this choice of $\det^{1/2}$: Since $Q = Q^T$, for any complex vector $w \in \mathbb{C}^n$ we have
\[
\text{Re}(\bar{w}^T Q w) = \bar{w}^T (\text{Re} Q) w.
\]
It follows that all eigenvalues $\lambda_1, \ldots, \lambda_n$ of $Q$ has positive real part. Then
\[
(\det Q)^{1/2} = \lambda_1^{1/2} \cdots \lambda_n^{1/2},
\]
where $\lambda_j^{1/2}$ is the square root of $\lambda_j$ that has positive real part.
The Fourier transform on $\mathcal{S}′$.

**Definition 1.6.** Any linear continuous map $u : \mathcal{S} \to \mathbb{C}$ is called a *tempered distribution*. The space of tempered distributions is denoted by $\mathcal{S}'$.

So $\mathcal{S}'$ is the dual of $\mathcal{S}$. The topology of the space $\mathcal{S}'$ is defined to be the weak-* topology, so that $u_j \to u$ in $\mathcal{S}'$ if

$$u_j(\varphi) \to u(\varphi), \quad \forall \varphi \in \mathcal{S}.$$  

For example,

- For any $a \in \mathbb{R}^n$, the *Dirac distribution* $\delta_a : \mathcal{S} \to \mathbb{C}$ defined by

  $$\delta_a(\varphi) = \varphi(a)$$

  is a tempered distribution.

- More generally, for any $a \in \mathbb{C}^n$ and any multi-index $\alpha$, the map $u : \mathcal{S} \to \mathbb{C}$ defined by

  $$u(\varphi) = D^\alpha \varphi(a)$$

  is a tempered distribution.

- For any bounded continuous function $\psi$ on $\mathbb{R}^n$ defines a tempered distribution by

  $$u(\varphi) = \int_{\mathbb{R}^n} \varphi(x)\psi(x)dx.$$  

Note that if $\psi \in \mathcal{S}$, then the distribution defined by the above formula is non-zero unless $\psi = 0$. It follows that $\mathcal{S} \subset \mathcal{S}'$.

**Definition 1.7.** Let $u \in \mathcal{S}'$. We define

1. $x^\alpha u \in \mathcal{S}'$ via $(x^\alpha u)(\varphi) = u(x^\alpha \varphi)$.
2. $D^\alpha u \in \mathcal{S}'$ via $(D^\alpha u)(\varphi) = (-1)^{|\alpha|}u(D^\alpha \varphi)$.
3. $\mathcal{F}u \in \mathcal{S}'$ via $(\mathcal{F}u)(\varphi) = u(\mathcal{F}\varphi)$.

Now let $Q$ be any real, symmetric, non-singular $n \times n$ matrix. Then the function $e^{\frac{i}{2}x^TQx}$ is not a Schwartz function, but we can think of it as a distribution. We would like to calculate its Fourier transform.

Recall that the *signature* of $Q$ is

$$(15) \quad \text{sgn}(Q) = N^+(Q) - N^-(Q),$$

where $N^\pm(Q) = \text{the number of positive/negative eigenvalues of } Q$.

**Theorem 1.8.** For any real, symmetric, non-singular $n \times n$ matrix $Q$,

$$(16) \quad \mathcal{F}(e^{\frac{i}{2}x^TQx}) = \frac{(2\pi)^{n/2}e^{\frac{i\pi}{4}\text{sgn}(Q)}}{|\det Q|^{1/2}} e^{-\frac{i}{2}\xi^TQ^{-1}\xi}.$$
Proof. Let’s first assume \( n = 1 \), so that \( Q = q \neq 0 \) is a non-zero real number. For any \( \varepsilon > 0 \) we let \( q_\varepsilon = q + i\varepsilon \). Applying theorem 1.5 we get

\[
\mathcal{F}(e^{\frac{i}{\hbar} q_\varepsilon x^2}) = \mathcal{F}(e^{-\frac{i}{2}(\varepsilon - iq)x^2}) = \frac{(2\pi)^{1/2}}{(\varepsilon - iq)^{1/2}} e^{-\frac{i}{2}(\varepsilon - iq)\xi^2},
\]

the square root is described after theorem 1.5. Note that when \( \varepsilon \to 0^+ \), we have

\[
(\varepsilon - iq)^{1/2} \to \begin{cases} 
\sqrt{q} e^{-i\pi/4} & q > 0 \\
-\sqrt{-q} e^{i\pi/4} & q < 0 
\end{cases} = \sqrt{|q|} e^{-i\pi/4} \text{sgn}(q),
\]

so the conclusion follows.

For general \( n \), one just proceed via the diagonalization method as before. \( \square \)

2. The method of stationary phase

\[ \texttt{Oscillatory integrals.} \]

The method of stationary phase is a method for estimating the small \( \hbar \) behavior of the oscillatory integrals

\[
I_\hbar = \int_{\mathbb{R}^n} e^{i\varphi(x)/\hbar} a(x)dx,
\]

where \( \varphi \in C^\infty(\mathbb{R}^n) \) is called the amplitude, and \( a \in C^\infty_c(\mathbb{R}^n) \) is called the phase. For simplicity we will assume \( \varphi \) is real-valued.

To illustrate, let’s start with two extremal cases:

- Suppose \( \varphi(x) = c \) is a constant. Then
  \[
  I_\hbar = e^{ic/\hbar} A,
  \]
  where \( A = \int_{\mathbb{R}^n} a(x)dx \) is a constant independent of \( \hbar \).

- Suppose \( n = 1 \) and \( \varphi(x) = x \). Then by definition \( I_\hbar = \mathcal{F}(a)(-\frac{1}{\hbar}) \). Since \( a \) is compactly supported, \( \mathcal{F}(a) \) is Schwartz. It follows
  \[
  I_\hbar = O(\hbar^N), \quad \forall N.
  \]

Intuitively, in the second case the exponential \( \exp(i\frac{x}{\hbar}) \) oscillates fast in any interval of \( x \) for small \( \hbar \), so that many cancellations take place and thus we get a function rapidly decreasing in \( \hbar \). This is in fact the case at any point \( x \) which is not a critical point of \( \varphi \). On the other hand, near a critical point of \( x \) the phase function \( \varphi \) doesn’t change much, i.e. “looks like” a constant, so that we are in case 1. According to these intuitive observation, we expect that the main contributions to \( I_\hbar \) should arise from the critical points of the phase function \( \varphi \).

We will use the following notions:

**Definition 2.1.** Let \( f = f(\hbar) \).
(1) We say \( f \sim \sum_{k=0}^{\infty} a_k \hbar^k \) if for each nonnegative integer \( N \),
\[
 f - \sum_{k=0}^{N} a_k \hbar^k = O(\hbar^{N+1}), \quad \hbar \to 0.
\]

(2) We say \( f = O(\hbar^{\infty}) \) if \( f \sim 0 \), i.e.
\[
 f = O(\hbar^N), \quad \hbar \to 0
\]
for all \( N \).

So in the second extremal case we discussed above, we can write \( I_\hbar = O(\hbar^{\infty}) \).

\textbf{Remark.} We don’t require the series \( \sum_{k=0}^{\infty} a_k \hbar^k \) to be convergent!

\textbf{¶ Non-stationary phase.}

\textbf{Proposition 2.2.} Suppose the phase function \( \varphi \) has no critical point in a neighborhood of the support of \( a \). Then
\[
 I_\hbar = O(\hbar^{\infty}).
\]

\textbf{Proof.} Let \( \chi \) be a smooth cut-off function which is identically one on the support of \( a \) so that \( \varphi \) has no critical point on the support of \( \chi \). Then \( a(x) = \chi(x)a(x) \), and \( \frac{\chi(x)}{|\nabla \varphi(x)|^2} \) is a smooth function. Define an operator \( L \) by
\[
 Lf(x) = \frac{\chi(x)}{|\nabla \varphi(x)|^2} \sum_j \partial_j \varphi(x) \partial_j f(x).
\]

Then it is easy to check
\[
 L(e^{i\frac{\varphi(x)}{\hbar}})(x) = \frac{\chi(x)}{|\nabla \varphi(x)|^2} \sum_j \partial_j \varphi(x) \frac{i \partial_j \varphi}{\hbar} e^{i\frac{\varphi(x)}{\hbar}} = \frac{i}{\hbar} \chi e^{i\frac{\varphi(x)}{\hbar}}.
\]

It follows
\[
 I_\hbar = \frac{\hbar}{i} \int_{\mathbb{R}^n} L(e^{i\frac{\varphi(x)}{\hbar}})a(x)dx = \frac{\hbar}{i} \int_{\mathbb{R}^n} e^{i\frac{\varphi(x)}{\hbar}}(L^*a)(x)dx.
\]
Repeating this \( N \) times, we get
\[
 I_\hbar = \left( \frac{\hbar \chi}{i} \right)^N \int_{\mathbb{R}^n} e^{i\frac{\varphi(x)}{\hbar}}(L^*)^N a(x)dx = O(\hbar^N).
\]
\( \square \)
Morse lemma.

Recall that a point \( p \) is called a critical point of a smooth function \( \varphi \) if
\[
\nabla \varphi(p) = 0.
\]
A critical point \( p \) of \( \varphi \) is called non-degenerate if the matrix \( d^2 \varphi(p) \) is non-degenerate:
\[
\det d^2 \varphi(p) = \det \left[ \frac{\partial^2 \varphi}{\partial x_i \partial x_j}(p) \right] \neq 0.
\]
A smooth function all of whose critical points are non-degenerate is called a Morse function.

**Theorem 2.3** (Morse lemma, form 1). Let \( \varphi \in C^\infty(\mathbb{R}^n) \). Suppose \( p \) is a non-degenerate critical point of \( \varphi \). Then there exists a neighborhood \( U \) of 0, a neighborhood \( V \) of \( p \) and a diffeomorphism \( f : U \to V \) so that
\[
f(0) = p \quad \text{and} \quad f^* \varphi(x) = \varphi(p) + \frac{1}{2} \left( x_1^2 + \cdots + x_r^2 - x_{r+1}^2 - \cdots - x_n^2 \right),
\]
where \( r \) is the number of positive eigenvalues of the Hessian matrix \( d^2 \varphi(p) \).

We shall prove the following form of the Morse lemma which is obviously equivalent to the above statement:

**Theorem 2.4** (Morse lemma, form 2). Let \( \varphi_0 \) and \( \varphi_1 \) be smooth functions on \( \mathbb{R}^n \) such that
\[
\begin{align*}
\bullet & \quad \varphi_i(0) = 0, \\
\bullet & \quad \nabla \varphi_0(0) = \nabla \varphi_1(0) = 0. \\
\bullet & \quad d^2 \varphi_0(0) = d^2 \varphi_1(0) \text{ is non-degenerate.}
\end{align*}
\]
Then there exist neighborhoods \( U_0 \) and \( U_1 \) of 0 and a diffeomorphism \( f : U_0 \to U_1 \) such that \( f(0) = 0 \) and \( f^* \varphi_1 = \varphi_0 \).

**Proof.** Let \( \varphi_t = (1 - t)\varphi_0 + t\varphi_1 \). Applying Moser’s trick, it is enough to find a time-dependent vector field \( X_t \) such that \( X_t(0) = 0 \) and
\[
\mathcal{L}_{X_t} \varphi_t = \varphi_0 - \varphi_1.
\]
Let \( X_t = \sum A_{ij}(x,t) \frac{\partial}{\partial x_j} \), where \( A_j(0,t) = 0 \). According to the second and the third conditions, \( \left[ \frac{\partial^2 \varphi_t}{\partial x_i \partial x_j}(0) \right] \) is non-degenerate. It follows that the system of functions
\[
\left\{ \frac{\partial \varphi_t}{\partial x_i} \mid i = 1, 2, \cdots, n \right\}
\]
form a system of coordinates near 0 with \( \frac{\partial \varphi_t}{\partial x_i}(0) = 0 \). According to the next lemma, one can find functions \( A_{ij}(x,t) \) with \( A_{ij}(0,t) = 0 \) so that
\[
A_j(x,t) = \sum A_{ij}(x,t) \frac{\partial \varphi_t}{\partial x_i}.
\]
Inserting this into the equation (24), we get
\begin{equation}
\sum A_{ij}(x,t) \frac{\partial \varphi_i}{\partial x_i} \frac{\partial \varphi_j}{\partial x_j} = \varphi_0 - \varphi_1.
\end{equation}

On the other hand, the three conditions implies that \( \varphi_0 - \varphi_1 \) vanishes to 2nd order at 0, so according the next lemma, one can find functions \( B_{ijk}(x,t) \) so that
\begin{equation}
\varphi_0 - \varphi_1 = \sum B_{ijk}(x,t) \frac{\partial \varphi_i}{\partial x_i} \frac{\partial \varphi_j}{\partial x_j} \frac{\partial \varphi_k}{\partial x_k}.
\end{equation}
In other words, the equation (25) is solvable with \( A_{ij}(0,t) = 0 \). This proves the existence of the diffeomorphism \( f \).

\textbf{Lemma 2.5.} Let \( \varphi \) be a smooth function with \( \varphi(0) = 0 \) and \( \partial^\alpha \varphi(0) = 0 \) for all \( |\alpha| < k \). Then there exists smooth functions \( \varphi_\alpha \) so that
\begin{equation}
\varphi(x) = \sum_{|\alpha|=k} x^\alpha \varphi_\alpha(x).
\end{equation}

\textit{Proof.} For \( k = 1 \), one just take
\[ \varphi_i(x) = \int_0^1 \frac{\partial \varphi}{\partial x_i}(tx) dt. \]
For larger \( k \), apply the above formula and induction. \qed

By exactly the same method, one can prove the following Morse lemma with parameters:

\textbf{Theorem 2.6} (Morse lemma with parameter). Let \( \varphi^*_s \in C^\infty(\mathbb{R}^n) \) be two family of smooth functions, depending smoothly on the parameter \( s \in \mathbb{R}^k \). Suppose
\begin{itemize}
  \item \( \varphi^*_0(0) = 0 \),
  \item \( \nabla \varphi^*_0(0) = \nabla \varphi^*_1(0) = 0 \),
  \item \( d^2 \varphi^*_0(0) = d^2 \varphi^*_1(0) \) are non-degenerate.
\end{itemize}
Then there exist an \( \varepsilon > 0 \), a neighborhood \( U \) of 0 and for all \( |s| < \varepsilon \) an open embedding \( f_s : U \to \mathbb{R}^n \), depending smoothly on \( s \), so that \( f_s(0) = 0 \) and \( f^*_s \varphi^*_1 = \varphi^*_0 \).

\textbf{The stationary phase formula for quadratic phase.}
Associated to any polynomial
\[ p(\xi) = \sum p_\alpha \xi^\alpha \]
we have a constant coefficient differential operator
\[ p(D) = \sum p_\alpha D^\alpha. \]
According to the Fourier transform formula that for any \( a \in \mathcal{S} \),
\[ \mathcal{F}(D^\alpha a) = \xi^\alpha \mathcal{F}a \]
we immediately get
Lemma 2.7. For any \( a \in S \),
\[
F(p(D)a)(\xi) = p(\xi)F a(\xi).
\]

In particular, associated to any \( n \times n \) symmetric matrix \( A = (a_{ij}) \) one has a quadratic function
\[
p_A(\xi) = \xi^T A \xi = \sum a_{ij} \xi_i \xi_j
\]
and thus a second order constant coefficient differential operator
\[
p_A(D) := \sum a_{kl} D_k D_l.
\]
It follows that
\[
F(p_A(D)a)(\xi) = (\xi^T A \xi) F a(\xi).
\]

More generally, for any nonnegative integer \( k \),
\[
F(p_A(D)^k a)(\xi) = (\xi^T A \xi)^k F a(\xi).
\]

Suppose the phase function is a quadratic function, i.e. \( \varphi(x) = \frac{1}{2} x^T Q x \) for some symmetric real \( n \times n \) matrix \( Q \). Then the only critical point of \( \varphi \) is the origin \( x = 0 \).

More over, the non-degeneracy condition implies that \( Q \) is non-singular.

Theorem 2.8. Let \( Q \) be a real, symmetric, non-singular \( n \times n \) matrix. For any \( a \in C_c^\infty(\mathbb{R}^n) \), one has
\[
\int_{\mathbb{R}^n} e^{\frac{-i}{2}\xi^T Q^{-1} \xi} \left( \sum_{k=0}^{N} \frac{(-i\frac{\hbar}{2})^k}{k!} (\xi^T Q^{-1} \xi)^k \right) a(\xi) d\xi.
\]

Proof. We start with the formula
\[
\int \mathcal{F} \varphi(\xi) \psi(\xi) d\xi = \int \varphi(x) \mathcal{F} \psi(x) dx.
\]
Recall from theorem 1.5 that if \( \varphi(x) = e^{\frac{i}{2} x^T Q x} \), then
\[
\mathcal{F} \varphi(\xi) = \frac{(2\pi)^n/2 e^{\frac{i}{2} \text{sgn}(Q)}}{|\det Q|^{1/2}} e^{-\frac{i}{2} \xi^T Q^{-1} \xi}.
\]
It follows
\[
\int_{\mathbb{R}^n} e^{\frac{i}{2} x^T Q x} a(x) dx = \frac{h^{n/2} e^{\frac{i}{2} \text{sgn}(Q)}}{(2\pi)^n/2 |\det Q|^{1/2}} \int_{\mathbb{R}^n} e^{-\frac{i}{2} \xi^T Q^{-1} \xi} \mathcal{F} a(\xi) d\xi.
\]
Using the Taylor’s expansion formula for the exponential function, we see that for any non-negative integer \( N \), the difference
\[
\int_{\mathbb{R}^n} e^{-\frac{i}{2} \xi^T Q^{-1} \xi} \mathcal{F} a(\xi) d\xi - \sum_{k=0}^{N} \frac{1}{k!} (-\frac{i\hbar}{2})^k \int_{\mathbb{R}^n} (\xi^T Q^{-1} \xi)^k \mathcal{F} a(\xi) d\xi
\]
is bounded by
\[
\hbar^{N+1} \frac{1}{2^{N+1} (N+1)!} \int_{\mathbb{R}^n} |(\xi^T Q^{-1} \xi)^k \mathcal{F} a(\xi)| d\xi.
\]
Note that
\[(\xi^T Q^{-1} \xi)^k \mathcal{F} a(\xi) = \mathcal{F}(p_{Q^{-1}}(D)^k a)(\xi),\]
so by the Fourier inversion formula,
\[p_{Q^{-1}}(D)^k a(0) = \frac{1}{(2\pi)^n} \int (\xi^T Q^{-1} \xi)^k \mathcal{F} a(\xi) d\xi.\]
The conclusion follows. \(\square\)

¶ The stationary phase formula for general phase.

As we have seen, only the critical points of the phase function \(\varphi\) give an essential contribution to the oscillatory integral
\[I_\hbar = \int_{\mathbb{R}^n} e^{i \varphi/\hbar} a(x) dx.\]
In what follows we will assume that \(\varphi \in C^\infty(\mathbb{R}^n)\) admits only non-degenerate critical points in a neighborhood of the support of \(a\). Since non-degenerate critical points must be discrete, \(\varphi\) has only finitely many critical points in the support of \(a\). Thus one can find a partition of unity \(\{U_i \mid 1 \leq i \leq N + 1\}\) of the support of \(a\) so that each \(U_i, 1 \leq i \leq N\), contains exactly one critical point \(p_i\) of \(\varphi\), and \(U_{N+1}\) contains no critical points of \(\varphi\). A partition of unity arguments converts the asymptotic behavior of \(I_\hbar\) to the sum of the asymptotic behavior of
\[I_{\hbar p_i} = \int_{U_i} e^{i \varphi_{p_i}/\hbar} \chi_i(x) a(x) dx,\]
where \(\chi_i\) is a function compactly supported in \(U_i\) and equals 1 identically near \(p_i\). According to the Morse lemma, for each \(U_i\) (one can always shrink \(U_i\) in the above integral if necessary) one can find a diffeomorphism \(f : V_i \to U_i\) so that \(f(0) = p_i\) and
\[f^* \varphi(x) = \varphi(p_i) + \frac{1}{2}(x_1^2 + \cdots + x_{r_i}^2 - x_{r_i+1}^2 - \cdots - x_n^2) =: \psi_{p_i}(x),\]
where \(r_i\) is the number of positive eigenvalues of the matrix \(d^2 \varphi(p_i)\). It follows from the change of variable formula that
\[I_{\hbar p_i} = \int_{V_i} e^{i \psi_{p_i}/\hbar} a(f(x)) \chi_i(f(x)) |\det df(x)| dx.\]
Moreover, since \(\psi_{p_i}(x)\) has a unique non-degenerate critical point at 0, modulo \(O(\hbar^\infty)\) one has
\[I_{\hbar p_i} = \int_{\mathbb{R}^n} e^{i \psi_{p_i}/\hbar} a(f(x)) |\det df(x)| dx + O(\hbar^\infty).\]
The asymptotic of this integral is basically given by theorem (2.8).

In particular, to get the leading term in the asymptotic expansion above, one only need to evaluate \(|\det df(0)|\), which follows from
Lemma 2.9. Let \( \psi, \varphi \) be smooth functions defined on \( V \) and \( U \) respectively, and suppose \( 0 \) is a non-degenerate critical point of both \( \psi \) and \( p \) a non-degenerate critical point of \( \varphi \). If \( f: V \to U \) a diffeomorphism so that \( f(0) = p \) and \( f^* \varphi = \psi \). Then

\[
df^T(0)d^2 \varphi(p)df(0) = d^2 \psi(0).
\]

Proof. We start with the equation \( \varphi \circ f = \psi \). Taking derivatives of both sides one gets

\[
\frac{\partial \varphi}{\partial y_k}(f(x)) \frac{\partial f_k}{\partial x_j} = \frac{\partial \psi}{\partial x_j}.
\]

Taking the second order derivatives at \( 0 \) of both sides, and using the conditions \( f(0) = p \) and \( \nabla \varphi(p) = 0 \), one gets

\[
\frac{\partial^2 \varphi}{\partial y_k \partial y_l}(p) \frac{\partial f_k}{\partial x_i}(0) \frac{\partial f_l}{\partial x_j}(0) = \frac{\partial^2 \psi}{\partial x_i \partial x_j}(0),
\]

in other words

\[
df^T(0)d^2 \varphi(p)df(0) = d^2 \psi(0).
\]

\[\square\]

It follows that in our case,

\[
|\det df(0)| = |\det d^2 \varphi(p_i)|^{-1/2}.
\]

In conclusion, we have

Theorem 2.10. As \( \hbar \to 0 \), the oscillatory integral has the asymptotic expansion

\[
I_\hbar \sim \sum_{d\varphi(p_i) = 0} I^\hbar_{p_i},
\]

where

\[
I^\hbar_{p_i} \sim (2\pi \hbar)^{n/2} e^{i\varphi(p_i)\hbar} e^{i\pi \text{sgn}(d^2 \varphi(p_i))} \frac{a(p_i)}{|\det d^2 \varphi(p_i)|^{1/2}} \sum_{j \geq 0} \hbar^j L_j(a)(p_i),
\]

where \( L_j = L_j(x, D) \) is a differential operator of order \( 2j \) (which depends on the phase function \( \varphi \)) with \( L_0 = 1 \). In particular, the leading term is

\[
I_\hbar = (2\pi \hbar)^{n/2} \sum_{d\varphi(p_i) = 0} e^{i\varphi(p_i)\hbar} e^{i\pi \text{sgn}(d^2 \varphi(p_i))} \frac{a(p_i)}{|\det d^2 \varphi(p_i)|^{1/2}} + O(\hbar^{n/2 + 1}).
\]

Remark. There are more complicated version of the stationary phase method where the phase function is allowed to have degenerate critical points.