Recall: Generating functions in lecture 3.

Let $M = T^*X$ be the cotangent bundle of a smooth manifold $X$.

- A horizontal submanifold (= the graph of a 1-form $\mu$)
  
  $$\Lambda_\mu = \{(x, \mu_x) \mid x \in X\}$$

  is a Lagrangian submanifold of $M$ if and only if $d\mu = 0$.
- If $\mu = d\varphi$ is exact, then we call $\varphi$ a generating function of $\Lambda_\mu$.
- A diffeomorphism $f : M_1 \to M_2$ is a symplectomorphism if and only if the twisted graph
  
  $$\Gamma_{\sigma_2 \circ f} = \{(x, \sigma_2 \circ f(x)) \mid x \in M_1\}$$

  is a Lagrangian submanifold of $M_1 \times M_2$.
- A generating function $\varphi \in C^\infty(X_1 \times X_2)$ of $\Gamma_{\sigma_2 \circ f}$ is also called a generating function of the symplectomorphism $f$.
- If $\det[\frac{\partial^2 \varphi}{\partial x_i \partial y_j}] \neq 0$, then locally $\varphi$ generates a symplectomorphism.

Example. The function

$$\varphi(x, y) = \sum x_i y_i$$

generates the symplectomorphism

$$(x, \xi) \mapsto (\xi, -x).$$
Unfortunately not all Lagrangian submanifolds are generated (even locally) by those kind of generating functions we introduced in lecture 3. In what follows we will extend the conception of generating functions by introducing “auxiliary variables” so that every Lagrangian submanifold of $T^*X$ is locally represented by a generating function (in the new sense).

¶ Recall: Symplectic “category” in lecture 4.

Let $M$ and $N$ be symplectic manifold. By a canonical relation “from $M$ to $N$” we mean a Lagrangian submanifold of the “twisted product” $M \times N^-$, i.e. the product manifold $M \times N$ endowed with the twisted symplectic form $\omega_1 \oplus (-\omega_2)$.

The symplectic “category” $\mathcal{S}$:

- $\text{Ob}(\mathcal{S})$ = symplectic manifolds
- $\text{Mor}(M, N)$ = canonical relations from $M$ to $N$.

This is not a true category because not all morphisms are composable. However, for a large class of morphisms one can compose them: If $\Gamma_i \subset M_i \times M_{i+1}$ are canonical relations so that the maps $
abla : \Gamma_1 \to M_2$, $(m_1, m_2) \mapsto m_2$
and
$\rho : \Gamma_2 \to M_2$, $(m_2, m_3) \mapsto m_2$
intersect cleanly, (i.e. if the fiber product $F = (\pi \times \rho)^{-1}(\Delta_{M_2})$ is a submanifold of $\Gamma_1 \times \Gamma_2$ whose tangent space at $(m_1, m_2, m_2, m_3)$ is $\{(v_1, v_2, v_2, v_3) \mid v_i \in T_{m_i}M_i, (v_i, v_{i+1}) \in T_{(m_i, m_{i+1})}\Gamma_i, i = 1, 2\}$), then $\Gamma_2 \circ \Gamma_1$ is an immersed canonical relation from $M_1$ to $M_3$. One can drop the word “immersed” by further assumptions, for example we may assume that the intersection is transversal.

We have already seen

(1) Any smooth map $f : X_1 \to X_2$ induces a canonical relation
$$\Gamma_f = \{(x_1, \xi_1, x_2, \xi_2) \mid x_2 = f(x_1), \xi_1 = df_{x_1}^*(\xi_2)\}$$
from $T^*X_1$ to $T^*X_2$.

(2) The graph of any symplectomorphism is a canonical relation which is composable with any canonical relation.

¶ Generating function with respect to a fibration.

Let $Z, X$ are smooth manifolds and $\pi : Z \to X$ a smooth fibration, then
$$\Gamma_\pi = \{(z, \xi, x, \eta) \mid x = \pi(z), \xi = (d\pi_z)^*\eta\}$$
is a canonical relation in $T^*Z \times (T^*X)^-$. Let $H^*Z$ be the horizontal subbundle of $T^*Z$ which is the image of $\Gamma_z$ under the projection $pr_1 : M_1 \times M_2 \to M_1$. In other words, the fiber of $H^*Z$ at $z$ is
$$H^*Z_z = \{(d\pi_z)^*\eta \mid \eta \in T_{\pi(z)}^*X\}.$$
Since $H^*Z$ is a vector sub-bundle of $T^*Z$, one has a short exact sequence of vector bundles

$$(1) \quad 0 \to H^*Z \to T^*Z \to V^*Z \to 0,$$

where $(V^*Z)_z = T^*_z Z/(H^*Z)_z = T^*_z \pi^{-1}(\pi(z)))$ is the cotangent space to the fiber through $z$.

Now let $\Lambda_\varphi$ be a horizontal Lagrangian submanifold of $T^*Z$ generated by a function $\varphi \in C^\infty(Z)$, i.e.

$$\Lambda_\varphi = \{ (z, d\varphi(z)) \mid z \in Z \}.$$ 

Then one can think of $\Lambda_\varphi$ as a morphism from $pt$ to $T^*Z$. So if $\Gamma_\pi$ and $\Lambda_\varphi$ are composable, then

$$(2) \quad \Lambda = \Gamma_\pi \circ \Lambda_\varphi$$

is a canonical relation from “pt” to $T^*X$, i.e. a Lagrangian submanifold of $T^*X$.

**Definition 1.1.** We call $\varphi$ a *generating function* of $\Lambda$ with respect to the fibration $(Z, \pi)$.

Next let’s look for conditions so that $\Gamma_\pi$ and $\Lambda_\varphi$ are transversally composable.

Assume $\Lambda_\varphi$ intersects $H^*Z$ transversally. By the exact sequence (1) above, any section $d\varphi$ of $T^*Z$ gives a section $d_{\text{vert}}\varphi$ of $V^*Z$, and $\Lambda_\varphi$ intersects $H^*Z$ transversally if and only if this section $d_{\text{vert}}\varphi$ intersects the zero section of $V^*Z$ transversally. It follows

$$(3) \quad C_\varphi := \{ z \in Z \mid (d_{\text{vert}}\varphi)_z = 0 \}$$

is a submanifold of $Z$ whose dimension is $\dim X$. Further, at any $z \in C_\varphi$,

$$d\varphi_z = (d\pi_z)^* \eta$$

for a unique $\eta \in T^*_\pi(z)X$, and $\Lambda = \Gamma_\pi \circ \Lambda_\varphi$ is the image of the map

$$C_\varphi \to T^*X, \quad z \mapsto (\pi(z), \eta).$$

We will denote this map by $p_\varphi$:

$$(4) \quad p_\varphi : C_\varphi \to \Lambda.$$

### The generating function in local coordinates.

Locally assume $X$ is an open subset of $\mathbb{R}^n$ and $Z = X \times \mathbb{R}^k$. Let $(x, s)$ be the coordinates on $Z$ so that $\varphi = \varphi(x, s)$. Then $C_\varphi \subset Z$ is defined by the $k$ equations

$$(5) \quad \frac{\partial \varphi}{\partial s_i} = 0, \quad i = 1, 2, \ldots, k,$$

and the transversality condition is that the differentials of these functions,

$$d\left( \frac{\partial \varphi}{\partial s_i} \right), \quad i = 1, 2, \ldots, k$$
are linearly independent. In this case, $\Lambda \subset T^*X$ is the image of the embedding
\[ C_\varphi \to T^*X, \quad (x, s) \mapsto (x, d_x \varphi(x, s)). \]

**Example.** Let $Y \subset X$ be a submanifold defined by $k$ equations
\[ f_1(x) = \cdots = f_k(x) = 0 \]
and assume that these equations are functionally independent, i.e. $df_1, \cdots, df_k$ are linearly independent. Let $\varphi : X \times \mathbb{R}^k \to \mathbb{R}$ be the function
\[ \varphi(x, s) = \sum f_i(x)s_i. \]
We claim that $\Lambda = \Gamma_{\pi_1 \circ \Lambda_{\varphi}}$ is the conormal bundle $N^*Y$ of $Y$. In fact, since $\frac{\partial \varphi}{\partial s_i} = f_i$ we see
\[ C_\varphi = Y \times \mathbb{R}^k, \]
and the map $C_\varphi \to T^*X$ is given by
\[ (x, s) \mapsto (x, \sum s_i d_x f_i(x)). \]
The conclusion follows from the fact that $d_x f_i$’s span the conormal bundle to $Y$ at each $x$.

**Example.** In particular, if we let $X = \mathbb{R}^n \times \mathbb{R}^n$ and let $Y$ be the diagonal
\[ Y = \text{diag}(X) = \{(x, x) \mid x \in X\}, \]
then $Y \subset X$ is defined by the equations
\[ x_i - y_i = 0, \quad i = 1, 2, \cdots, n. \]
So the function
\[ \varphi(x, y, s) = (x - y) \cdot s = \sum (x_i - y_i) s_i, \]
is the generating function of $N^*(\text{diag}(X))$. (Note: the canonical relation generated by the identity map is a twisting of this example, $\sigma_2(N^*(\text{diag}(X)))$.)

---

### Facts about the generating function.

In Guillemin-Sternberg chapter 5, many nice facts were proven for the generating functions (with respect to fibrations), e.g.

- If $\Gamma \in \text{Mor}(T^*X, T^*Y)$ is a canonical relation, $\pi : Z \to X \times Y$ a fibration, and $\varphi$ a generating function of $\Gamma$ related to this fibration. Locally $\varphi = \varphi(x, y, s)$. Then the function $\psi(y, x, s) = -\varphi(x, y, s)$ is a generating function for the transpose canonical relation $\Gamma^T \in \text{Mor}(T^*Y, T^*X)$.

- If $\Gamma_i \in \text{Mor}(T^*X_i, T^*X_{i+1})$, $i = 1, 2$ are canonical relations which are transversally composable, $\pi_i : Z_i \to X_i \times X_{i+1}$ are fibrations and $\varphi_i \in C^\infty(Z_i)$ are generating functions for $\Gamma_i$ with respect to $\pi_i$, then one can construct a fibration $\tilde{Z} \to X_1 \times X_3$ with
\[ Z = (\pi_1 \times \pi_2)^{-1}(X_1 \times \Delta_{Z_2} \times Z_3), \]
Let $\varphi$ be the restriction to $Z$ of the function

$$
(9) \quad (z_1, z_2) \mapsto \varphi_1(z_1) + \varphi_2(z_2),
$$

then $\varphi$ is a generating function for $\Gamma_2 \circ \Gamma_1$ with respect to the fibration $Z \to X_1 \times X_3$. Similar facts holds for clean compositions.

- Suppose that the fibration $\pi : Z \to X$ can be factored as a succession of fibrations $\pi = \pi_1 \circ \pi_0$, where $\pi_0 : Z \to Z_1$ and $\pi_1 : Z_1 \to X$ are fibrations. Moreover, suppose that the restriction of the generating function $\varphi$ to each fiber $\pi_0^{-1}(z_1)$ has a unique non-degenerate critical point $\gamma(z_1)$, so that we get a section $\gamma : Z_1 \to Z$. Then the function $\phi_1 = \gamma_1^* \phi$ is a generating function of $\Lambda$ with respect to $\pi_1$.

Of course the most important thing is the existence and “uniqueness” of generating functions. We state the results here. Details is left as a possible topic for the final term paper. c.f. Guillemin-Sternberg sections 5.9 and 5.11

**Theorem 1.2** (Existence). Every Lagrangian submanifold of $T^*X$ can be locally represented by a generating function relative to a fibration.

Of course the generating function is not unique. However,

**Theorem 1.3.** Suppose $\varphi_i$, $i = 1, 2$, are generating functions for the same Lagrangian submanifold $\Lambda$ with respect to fibrations $\pi_i : Z_i \to X$. Then locally one can obtain one description from the other by applying a sequence of “moves” of the following three types:

- Adding a constant: replace $\varphi$ by $\varphi + c$.
- Equivalence: For a diffeomorphism $g : Z \to \tilde{Z}$, replace $\pi$ by $g^* \pi$ and $\varphi$ by $g^* \varphi$.
- Increasing the number of fiber variables: replace $Z$ by $Z = Z \times \mathbb{R}^d$ and $\varphi$ by $\varphi(z) + \frac{1}{2} \langle Az, z \rangle$, where $A$ is a non-degenerate $d \times d$ matrix.

The proof to the above two theorems is left as an term paper topic.

### 2. The Calculus of Densities

**Densities on vector space.**

Let $V$ be a vector space of dimension $n$. We denote by $\mathcal{F}(V)$ the set of all bases of $V$. Then for any two bases $\{e_i\}$ and $\{f_j\}$ of $V$, there exists a unique $A \in GL(n, \mathbb{R})$ that maps $\{e_i\}$ to $\{f_j\}$.

**Definition 2.1.** Let $\alpha \in \mathbb{C}$ be a complex number. A $\alpha$-density on $V$ is a map $\mu : \mathcal{F}(V) \to \mathbb{C}$ satisfying

$$
(10) \quad \mu(Av_1, \ldots, Av_n) = |\det A|^\alpha \mu(v_1, \ldots, v_n)
$$

for any $v_i \in V$ and $A \in End(V)$. We will denote the space of $\alpha$-densities on $V$ by $|V|^\alpha$. 
**Remark.** An \( n \)-form on \( V \) is a map \( \omega : V^n \to \mathbb{C} \) such that
\[
\omega(Av_1, \ldots, Av_n) = (\det A) \omega(v_1, \ldots, v_n).
\]
So if \( \omega \in \Lambda^n(V) \) is a \( n \)-form, \( |\omega| \) is a 1-density. As a consequence, \( |\omega|^{\alpha} \) is an \( \alpha \)-density.

So \( \alpha \)-densities exists. Obviously \( |V|^\alpha \) is a linear space. The following theorem is a direct consequence of the transitivity of the action \( GL(n, \mathbb{R}^n) \) on \( \mathcal{F}(V) \):

**Proposition 2.2.** The set of \( \alpha \)-densities on \( V \), \( |V|^\alpha \), is a one dimensional vector space over \( \mathbb{C} \).

By definition it is easy to check

- If \( \rho \in |V|^\alpha \) and \( \tau \in |V|^\beta \), then \( \rho \cdot \tau \in |V|^{\alpha+\beta} \). In other words, we have an isomorphism \( |V|^\alpha \otimes |V|^\beta \cong |V|^{\alpha+\beta} \).
- If \( \rho \in |V|^\alpha \), then \( \bar{\rho} \in |V|^\bar{\alpha} \). This gives us an anti-linear isomorphism \( |V|^\alpha \cong |V|^\bar{\alpha} \).
- Combining the two facts, we get a sesquilinear map \( |V|^\alpha \otimes |V|^\beta \to |V|^{\alpha+\beta} \). This is specially useful if we take \( \alpha = \beta = \frac{1}{2} + is \), in which case we get a sesquilinear map
  \[
  |V|^\frac{1}{2} + is \otimes |V|^\frac{1}{2} + is \cong |V|.
  \]
- \( |V|^\alpha \cong |V^*|^{-\alpha} \).
- Natural isomorphism \( |V|^\alpha \cong |V'|^\alpha \otimes |V''|^\alpha \) if there is a short exact sequence \( 0 \to V' \to V \to V'' \to 0 \). In fact, given any basis \( (e_1, \ldots, e_k) \) of \( V' \), one can extend it to a basis \( (e_1, \ldots, e_k, e_{k+1}, \ldots, e_n) \) of \( V \). And the images of \( e_{k+1}, \ldots, e_n \) under the map \( V \to V'' \) gives a basis of \( V'' \). Any two basis of \( V \) of this type is related by a matrix \( A \in Gl(n) \) of the form
  \[
  A = \begin{pmatrix} A' & * \\ 0 & A'' \end{pmatrix}.
  \]
  Since \( \det A = \det A' \det A'' \), we get a canonical isomorphism \( |V|^\alpha \cong |V'|^\alpha \otimes |V''|^\alpha \). Similar results hold for long exact sequences.
- If \( L : V \to W \) is a linear isomorphism, then one has pull-back \( L^* : |W|^\alpha \to |V|^\alpha \) and push-forward \( L_* = (L^{-1})^* : |W|^\alpha \to |V|^\alpha \).

### Densities on smooth manifolds.

For any real vector bundle \( E \to X \), where \( X \) is a smooth manifold, one can consider the complex line bundle
\[
|E|^\alpha \to X
\]
whose fiber at \( x \) is \( |E_x|^\alpha \).

**Definition 2.3.** A smooth section of \( |TX|^\alpha \) is called an \( \alpha \)-density on \( X \).
The operations like multiplication, complex conjugation etc in the linear theory extends easily to this setting. Moreover, since an $\alpha$-density is a section of the $\alpha$-density bundle, it makes sense to say that a density is zero not zero at a point, and thus the support of an $\alpha$-density is easily defined.

\section*{Integrating 1-Densities on smooth manifolds.}

Suppose $(U, x_1, \cdots, x_n)$ is a coordinate patch near $x \in X$, then we can write any 1-density on $U$ as

$$\mu(x) = f(x)|dx_1 \wedge \cdots \wedge dx_n|$$

for some smooth function $f$ on $U$.

We can pull back densities as follows: If $f : M \to N$ is a diffeomorphism, and $\mu$ is a density on $N$, then $f^* \mu$, the pull-back of $\mu$, is a density on $M$ defined by

$$(f^* \mu)_m(X_1, \cdots, X_n) = \mu_{f(m)}(df_m(X_1), \cdots, df_m(X_n)).$$

Check: $f^* \mu$ is a density on $X$.

Now we define the integration of compactly supported continuous densities on $X$.

\textbf{Step 1.} First suppose $\mu$ is a compactly supported continuous density on $\mathbb{R}^n$. Then we can write $\mu = f|dx_1 \wedge \cdots \wedge dx_n|$ for some continuous function $f$ support on a compact set $D \subset \mathbb{R}^n$. Define

$$\int_{\mathbb{R}^n} \mu := \int_{\mathbb{R}^n} f(x)dx_1 \cdots dx_n = \int_D f(x)dx_1 \cdots dx_n.$$

To define the integration of densities on manifolds, we need the following

\begin{lemma}
Suppose $U, V$ are open sets in $\mathbb{R}^n$, and $\varphi : U \to V$ is a diffeomorphism, $\mu$ is a density on $V$, then

$$\int_V \mu = \int_U \varphi^* \mu.$$

\end{lemma}

\textbf{Proof.} Denote $\mu = f|dx_1 \wedge \cdots \wedge dx_n|$, then

$$\varphi^* \mu = f(\varphi(x))|\det d\varphi| |dx_1 \wedge \cdots \wedge dx_n|,$$

and the lemma follows from the change of variable formula. \hfill \Box

\textbf{Step 2.} Secondly suppose $\mu$ is a 1-density on $M$ supported on a coordinate chart $(\varphi, U, V)$, we define

$$\int_U \mu := \int_V (\varphi^{-1})^* \mu.$$

This is well-defined, since if $(\tilde{\varphi}, \tilde{U}, \tilde{V})$ is another coordinate chart and $\mu$ is also supported in $\tilde{U}$, then

$$\int_{\tilde{V}} (\tilde{\varphi}^{-1})^* \mu = \int_{\tilde{V}} (\tilde{\varphi} \circ \varphi^{-1})^*(\varphi^{-1})^* \mu = \int_{\tilde{V}} (\varphi^{-1})^* \mu,$$
where we used the fact that \( \tilde{\varphi} \circ \varphi^{-1} \) is a diffeomorphism from \( \varphi(U \cap \tilde{U}) \) to \( \tilde{\varphi}(U \cap \tilde{U}) \), and that \( (\tilde{\varphi} \circ \varphi^{-1})^* = (\varphi^{-1})^* \circ \tilde{\varphi}^* \).

**Step 3.** Finally suppose \( \mu \) is any compactly supported continuous density on \( M \). Take a finite open cover \( \{U_i\} \) of support of \( \mu \) by coordinate charts, then \( \{U_i, U_0 = M - \cup U_i\} \) is a finite cover of \( M \). The partition of unity theorem claims that there exists smooth functions \( \psi_i \) supported in \( U_i \) satisfying \( 0 \leq \psi_i \leq 1 \) and \( \sum \psi_i \equiv 1 \). Now we can define

\[
\int_M \mu = \sum \int_{U_i} \psi_i \mu.
\]

It is not hard to check that this is independent of choices of open cover, and choices of partition of unity, so the integration of compactly supported densities are well defined.

Moreover, the integration satisfies the following propositions:

**Proposition 2.5.** Let \( \mu, \nu \) be compactly supported densities on \( M \).

1. (Linearity) \( \int_M (a\mu + b\nu) = a \int_M \mu + b \int_M \nu \).
2. (Positivity) If \( \mu \) is a positive density, \( \int_M \mu > 0 \).
3. (Diffeomorphism Invariance) If \( \varphi : N \to M \) is a diffeomorphism, then

\[
\int_M \mu = \int_N (\varphi^* \mu).
\]

Note that if \( X \) is compact and \( \alpha = \frac{1}{2} + is \), then the set of \( \alpha \)-densities form a pre-Hilbert space:

\[
\langle \rho, \tau \rangle = \int_X \rho \bar{\tau}.
\]

<table>
<thead>
<tr>
<th>Push-forward under a fibration.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Now suppose ( \pi : Z \to X ) is a fibration with compact fibers. Denote by ( F_x = \pi^{-1}(x) ) the fiber over ( x ). Then for any ( z \in F_x ), we have an exact sequence of vector spaces</td>
</tr>
</tbody>
</table>
| \[
0 \to T_z F_x \to T_z X \xrightarrow{d\pi_z} T_x X \to 0
\] |
| which gives an isomorphism |
| (12) \( |T_z F_x| \otimes |T_x X| \simeq |T_z Z| \). |
| Now let \( \mu \) be a one density on \( Z \). We first fix a one density \( \nu \) on \( X \). Then according to the isomorphism above we get a one density \( \sigma \) on \( F_x \) so that \( \sigma \otimes \nu = \mu \). We define the push-forward of \( \mu \) under the fibration \( \pi \) to be the one density |
| \[
(\int_{F_x} \sigma) \nu.
\] |
| Note that if we replace \( \nu \) by \( c\nu \), then \( \sigma \) is replaced by \( \frac{1}{c} \sigma \), so the push-forward is well defined. |
3. The Enhanced symplectic “category”

Now we would like enhance the “category” \( S \) by adding half densities as a piece of data on canonical relations, so that integrals are intrinsically defined. More precisely, we would like to define a “category” \( ES \) with

- Objects = symplectic manifolds
- \( Mor(\mathcal{M}_1, \mathcal{M}_2) = \text{pairs } (\Gamma, \sigma) \), where \( \Gamma \subset \mathcal{M}_1 \times \mathcal{M}_2^- \) is a Lagrangian submanifold, and \( \sigma \) is a half density on \( \Gamma \).

Still, the question is: Given two morphisms \((\Gamma_i, \sigma_i) \in Mor(\mathcal{M}_i, \mathcal{M}_{i+1})\), how do we compose? We have seen how to compose canonical relations modulo clean intersection conditions. So the remaining question is: Given half densities \( \sigma_i \) on \( \Gamma_i \), how to form a new half density \( \sigma_2 \circ \sigma_1 \) on \( \Gamma_2 \circ \Gamma_1 \)?

Locally if \((x_1, \ldots, x_n, s_1, \ldots, s_d)\) are coordinates on \( Z \), with \((x_1, \ldots, x_n)\) coordinates on \( X \), and

\[
\mu = u(x_1, \ldots, x_n, s_1, \ldots, s_d) dx_1 \cdots dx_n ds_1 \cdots ds_d
\]

is compactly supported in one chart, then

\[
\pi_*\mu = \left( \int u(x_1, \ldots, x_n, s_1, \ldots, s_d) ds_1 \cdots ds_d \right) dx_1 \cdots dx_n.
\]

\[\text{¶ Some linear theory.}\]

Let \( V_1, V_2 \) and \( V_3 \) be symplectic vector spaces, and \( \Gamma_1 \subset V_1 \times V_2^- \), \( \Gamma_2 \subset V_2 \times V_3^- \) linear canonical relations. Let

\[
\pi : \Gamma_1 \to V_2, (v_1, v_2) \mapsto v_2
\]

and

\[
\rho : \Gamma_2 \to V_2, (v_2, v_3) \mapsto v_2
\]

be the canonical projections onto \( V_2 \). Let \( F \) be the fiber product of \( \pi \) and \( \rho \), namely

\[
F = \{(\gamma_1, \gamma_2) \in \Gamma_1 \times \Gamma_2 \mid \pi(\gamma_1) = \rho(\gamma_2)\}
= \{(v_1, v_2, v_3) \mid (v_1, v_2) \in \Gamma_1, (v_2, v_3) \in \Gamma_2\}.
\]

Let \( \tau : \Gamma_1 \times \Gamma_2 \to V_2 \) be the map

\[
\tau(\gamma_1, \gamma_2) = \pi(\gamma_1) - \rho(\gamma_2).
\]

Then we have a short exact sequence

\[
0 \to F \hookrightarrow \Gamma_1 \times \Gamma_2 \xrightarrow{\tau} \text{Im}(\tau) \to 0
\]

and thus a canonical isomorphism

\[
|F|^{\frac{1}{2}} \otimes |\text{Im}(\tau)|^{\frac{1}{2}} \simeq |\Gamma_1|^{\frac{1}{2}} \otimes |\Gamma_2|^{\frac{1}{2}}.
\]

If we let \( \alpha \) be the map

\[
\alpha : F \to V_1 \times V_3, (v_1, v_2, v_3) \mapsto (v_1, v_3),
\]
then by definition $\text{Im}(\alpha) = \Gamma_2 \circ \Gamma_1$ so we have a short exact sequence

$$0 \to \ker(\alpha) \hookrightarrow F \to \Gamma_2 \circ \Gamma_1$$

from which we get a canonical isomorphism

$$|F|^\frac{1}{2} \simeq |\Gamma_2 \circ \Gamma_1|^\frac{1}{2} \otimes |\ker(\alpha)|^\frac{1}{2}. \quad (16)$$

Obviously one can identify $\ker(\alpha)$ as a subspace of $V_2$.

**Exercise 1.** Check: $\ker(\alpha) = (\text{Im}(\tau))^\Omega_2$, where $\Omega_2$ is the symplectic form on $V_2$.

As an consequence, we get an identification

$$V_2/\ker(\alpha) = V_2/(\text{Im}(\tau))^\Omega_2 \simeq (\text{Im}(\tau))^*$$

and thus a canonical isomorphism

$$|V_2/\ker(\alpha)|^{-\frac{1}{2}} \simeq |\text{Im}(\tau)|^{\frac{1}{2}}. \quad (17)$$

On the other hand side, from the short exact sequence

$$0 \to \ker(\alpha) \hookrightarrow V_2 \to V_2/\ker(\alpha) \to 0$$

we get

$$|V_2|^\frac{1}{2} \simeq |\ker(\alpha)|^\frac{1}{2} \otimes |V_2/\ker(\alpha)|^\frac{1}{2}.$$ 

Using the symplectic form (and thus the Liouville volume form) on $V_2$ one can identify $|V_2|^\frac{1}{2} \simeq \mathbb{C}$. As an consequence, we get from (17) a canonical isomorphism

$$|\ker(\alpha)|^\frac{1}{2} \simeq |V_2/\ker(\alpha)|^{-\frac{1}{2}} \simeq |\text{Im}(\tau)|^\frac{1}{2}. \quad (18)$$

Combining (15), (16) and (18), we get

**Theorem 3.1.** There is a canonical isomorphism

$$|\Gamma_1|^\frac{1}{2} \otimes |\Gamma_2|^\frac{1}{2} \simeq |\ker(\alpha)| \otimes |\Gamma_2 \circ \Gamma_1|^\frac{1}{2}. \quad (19)$$

In particular we see that if $\tau$ is surjective, then by the exercise above, $\ker(\alpha) = 0$. The canonical isomorphism in the theorem becomes

$$|\Gamma_1|^\frac{1}{2} \otimes |\Gamma_2|^\frac{1}{2} \simeq |\Gamma_2 \circ \Gamma_1|^\frac{1}{2}.$$ 

In other words, given half densities $\sigma_i$ on $\Gamma_i$ one can canonically obtain a half density $\sigma_2 \circ \sigma_1$ on $\Gamma_2 \circ \Gamma_1$. 
Then enhanced symplectic “category”.

Now suppose $M_i, i = 1, 2, 3$ are symplectic manifolds, and $\Gamma_i \subset M_i \times M_{i+1}$ canonical relations. We will assume as before that $\Gamma_1$ and $\Gamma_2$ intersect cleanly, or more precisely, the fiber product

$$ F = \{(m_1, m_2, m_3) \mid (m_1, m_2) \in \Gamma_1, (m_2, m_3) \in \Gamma_2\} $$

is a manifold, and the tangent space of $F$ at $m = (m_1, m_2, m_3)$ is

$$ T_m F = \{(v_1, v_2, v_3) \mid v_i \in T_{m_i} M_i, (v_i, v_{i+1}) \in T_{(m_i, m_{i+1})} \Gamma_i\}. $$

As in the linear case we define the map

$$ \alpha : F \to M_1 \times M_3, \quad (m_1, m_2, m_3) \mapsto (m_1, m_3). $$

The clean intersection condition implies that $\alpha$ is a constant rank map and that $\Gamma_2 \circ \Gamma_1$ is an immersed canonical relation. We will further assume

- $\alpha$ is proper,
- the level sets of $\alpha$ are connected,

then $\Gamma_2 \circ \Gamma_1$ is an embedded Lagrangian submanifold of $M_1 \times M_3$, and

$$ \alpha : F \to \Gamma_2 \circ \Gamma_1 $$

is a fiber mapping with compact fibers.

For any point $m \in F$ we denote $q = \alpha(m) \in \Gamma_2 \circ \Gamma_1$. Then the fiber $F_q = \alpha^{-1}(q)$ is compact and $m \in F_q$. More over, by definition

$$ T_m F_q = \ker(d\alpha_m). $$

According to theorem 3.1, we get a canonical identification

$$ |T_m F_q| \otimes |T_q \Gamma_2 \circ \Gamma_1|^{\frac{1}{2}} \simeq |T_{(m_1, m_2)} \Gamma_1|^{\frac{1}{2}} \otimes |T_{(m_2, m_3)} \Gamma_2|^{\frac{1}{2}}. $$

Fix any non-zero half density $\sigma_q \in |T_q \Gamma_2 \circ \Gamma_1|^{\frac{1}{2}}$. Let $\sigma_1, \sigma_2$ be half densities on $\Gamma_1$ and $\Gamma_2$ respectively. By the identification above, there is a unique one density $\nu_m \in |T_m F_q|$ which depends on $m$ smoothly so that

$$ \nu_m \otimes \sigma_q = \sigma_1^{(m_1, m_2)} \otimes \sigma_2^{(m_2, m_3)}. $$

Since $F_q$ is compact, we can integrate $\nu$ over $F_q$.

**Definition 3.2.** We define the composition of $\sigma_1$ and $\sigma_2$ to be

$$ (\sigma_2 \circ \sigma_1)(q) = \left( \int_{F_q} \nu \right) \sigma_q. $$

Note: if we change $\sigma_q$ to $c \sigma_q$, then $\nu$ is changed to $\frac{1}{c} \nu$, so the right hand side of (21) is independent of the choice of $\sigma_q$ and thus gives a well-defined half-density on $\Gamma_2 \circ \Gamma_1$.

Now we can describe the enhanced symplectic “category” $\mathcal{E}S$:
Objects = symplectic manifolds
• \( \text{Mor}(M_1, M_2) = \text{pairs } (\Gamma, \sigma) \), where \( \Gamma \) is a canonical relation from \( M_1 \) to \( M_2 \), and \( \sigma \in C^\infty(|T\Gamma|^{1/2}) \) is a half-density on \( \Gamma \).

The composition of morphisms is given by
\[
(\Gamma_2, \sigma_2) \circ (\Gamma_1, \sigma_1) = (\Gamma_2 \circ \Gamma_1, \sigma_2 \circ \sigma_1).
\]

We will leave the associativity of the composition as an exercise.

Remark. If \( M_2 = M_3 \), \( \Gamma_2 = \Delta_{M_2} \) is the diagonal (which could be identified with \( M_2 \)), and \( \sigma_2 = \sigma_\Delta \) is the canonical half density that corresponds to the symplectic volume form on \( M_2 \), then for any \( (\Gamma_1, \sigma_1) \), one has
\[
(\Delta_{M_2}, \sigma_\Delta) \circ (\Gamma_1, \sigma_1) = (\Gamma_1, \sigma_1).
\]

Similarly one has
\[
(\Gamma_2, \sigma_2) \circ (\Delta_{M_1}, \sigma_\Delta) = (\Gamma_2, \sigma_2).
\]

4. Oscillatory half densities

Oscillatory half densities.

Now assume \( X \) is a smooth manifold, \( \Lambda \subset T^*X \) a Lagrangian submanifold. Let \( \varphi \in C^\infty(Z) \) be a (global!) generating function for \( \Lambda \) with respect to a fibration \( \pi: Z \to X \).

We will assume that \( \Lambda \) is exact in the sense that
\[
\iota^*_\Lambda \alpha_{T^*X} = d\varphi_\Lambda
\]
for some \( \varphi_\Lambda \in C^\infty(\Lambda) \), where \( \alpha_{T^*X} \) is the canonical 1 form on \( T^*X \).

In what follows we will fix a choice of such phase function \( \varphi_\Lambda \), and we will fix the constant in the generating function \( \varphi \) by requiring
\[
\iota^* \varphi = p^*_\varphi \varphi_\Lambda,
\]
where \( \iota: C_\varphi \hookrightarrow Z \) is the inclusion, and
\[
p_\varphi: C_\varphi \to \Lambda
\]
is the map (4).

Let \( d = \dim Z - \dim X \) be the fiber dimension. For any \( k \in \mathbb{Z} \), we define \( I^k_0(X, \Lambda) \), the space of compactly supported oscillatory half densities on \( X \) associated with \( \Lambda \), to be
\[
I^k_0(X, \Lambda) = \{ \mu = h^{k-d/2} \pi_*(a(z, h)e^{i\varphi_\Lambda(z)/h}) \mid a \in C^\infty_0(Z \times \mathbb{R}) \},
\]
where \( \tau \) is a nowhere vanishing half density on \( Z \). (Obviously the definition is independent of the choice of \( \tau \).) Similarly we define \( I^k(X, \Lambda) \), the space of oscillatory half densities on \( X \) associated with \( \Lambda \), to be the set consists of those half densities \( \mu \) so that \( \rho \mu \in I^k_0(X, \Lambda) \) for all \( \rho \in C^\infty_0(X) \).
Locally we may assume $Z = X \times S$, where $S$ is an open set in $\mathbb{R}^d$. We may choose our fiber half density to be the Euclidean one $ds^\frac{1}{2}$ and choose $\tau$ to be $\tau_0 \otimes ds^\frac{1}{2}$ with $\tau_0$ a nowhere vanishing half density on $X$. Then $\mu \in I^k_0(X, \Lambda)$ is of the form

$$\hbar^{k-\frac{d}{2}} \left( \int_S a(x, s, \hbar) e^{\frac{i}{\hbar} \varphi(x, s)} ds \right) \tau_0.$$ 

**Independence of generating function.**

We must show that the above definition is also independent of the choices of generating functions. Let $\pi : Z_i \to X$, $i = 1, 2$ be two fibrations, and $\varphi_i$ be a generating function of $\Lambda$ with respect to $\pi_i$.

It is enough to do this locally. Recall that the two generating functions $\varphi_1$ and $\varphi_2$ are related by

(a) Replace $\varphi$ by $\varphi + c$.

(b) For a diffeomorphism $g : Z \to \tilde{Z}$, replace $\pi$ by $g^* \pi$ and $\varphi$ by $g^* \varphi$.

(c) Replace $Z$ by $Z = Z \times \mathbb{R}^d$ and $\varphi$ by $\varphi(z) + \frac{1}{2} \langle Az, z \rangle$, where $A$ is a non-degenerate $d \times d$ matrix.

We have already get rid of type (a) by choosing only generating functions satisfying the normalization condition (24). If two densities are related by a type (b) change, then by a change of variable argument it is not hard to prove

$$(\pi_2)_*(ae^{\frac{i}{\hbar} \varphi_2} g_* \tau_1) = (\pi_1)_*(g^* ae^{\frac{i}{\hbar} \varphi_1} \tau_1)$$

so the spaces defined via $\varphi_1$ and via $\varphi_2$ are the same.

Now suppose $\varphi_1$ and $\varphi_2$ are related by a type (c) change. Without loss of generality, we may assume $Z_2 = Z_1 \times S$, where $S$ is an open subset of $\mathbb{R}^m$, and

$$\varphi_2(z, s) = \varphi_1(z) + \frac{1}{2} s^T As,$$

where $A$ is a symmetric non-degenerate $m \times m$ matrix. Let $d$ be the fiber dimension of $Z_1 \to X$, then the fiber dimension of $Z_2 \to X$ is $d + m$. Let $\tau_1$ be a nowhere vanishing half density on $Z_1$, then $\tau_1 \otimes ds^\frac{1}{2}$ is a nowhere vanishing half density on $Z_2$. Using the generating function $\varphi_2$ we get the expressions

$$\hbar^{k-d-m/2} (\pi_2)_* a_2(z, s, \hbar) e^{\frac{i}{\hbar} \varphi_2(z, s)} \tau_1 \otimes ds^\frac{1}{2}.$$

Let $\pi_{2,1} : Z_2 \to Z_1$ be the projection on to the first factor so that $(\pi_2)_* = (\pi_1)_* \circ (\pi_{2,1})_*$. Then by definition, $(\pi_{2,1})_*$ acts as

$$(\pi_{2,1})_*(a_2(z, s, \hbar) e^{\frac{i}{\hbar} \varphi_2(z, s)} \tau_1 \otimes ds^\frac{1}{2}) = \left( \int a_2(z, s, \hbar) e^{\frac{i}{\hbar} s^T As} ds \right) e^{\frac{i}{\hbar} \varphi_1 \tau_1}.$$ 

Now the conclusion follows from the lemma of stationary phase (c.f. line 2 on page 11 in lecture 5).

In conclusion, we proved
Theorem 4.1. The space $I^k_0(X, \Lambda)$ (and thus $I^k(X, \Lambda)$) is intrinsically defined (provided we fix a choice of $\varphi_\Lambda$ on $\Lambda$).

5. Semiclassical Fourier integral operators

Recall:
- Let $X$ be a smooth manifold. Let $\iota_\Lambda : \Lambda \hookrightarrow T^*X$ be a connected exact Lagrangian submanifold, i.e.
  \[ \iota_\Lambda^*\alpha_{T^*X} = d\varphi_\Lambda, \]
  where $\varphi_\Lambda \in C^\infty(\Lambda)$ is called a phase function of $\Lambda$ and will be fixed in what follows.
- Let $\pi : Z \to X$ be a fibration. Let $\varphi \in C^\infty(Z)$ be a generating function for $\Lambda$ with respect to $\pi$, i.e.
  \[ \Lambda = \Gamma_\pi \circ \Lambda_{\varphi}, \]
  where $\Lambda_{\varphi} = \{(z, \xi, x, \eta) \mid x = \pi(z), \xi = (d\pi_z)^*\eta\}$ is a canonical relation from $T^*Z$ to $T^*X$, and $\Lambda_{\varphi} = \{(z, d\varphi(z)) \mid z \in Z\}$ is a horizontal Lagrangian submanifold of $T^*Z$ generated by $\varphi$. And we will assume for simplicity that $\Gamma_\pi$ and $\Lambda_{\varphi}$ are transversally composable.
- If we set $C_{\varphi} = \{z \in Z \mid (d_{vert}\varphi)_z = 0\}$, then the transversality condition gives a diffeomorphism $p_{\varphi} : C_{\varphi} \to \Lambda, z \mapsto (\pi(z), \eta)$, where $\eta \in T^*_{\pi(z)}\Lambda$ is determined by the equation $d\varphi(z) = (d\pi_z)^*\eta$.
- In local coordinates, we may assume $Z = X \times S \subset \mathbb{R}^n \times \mathbb{R}^k$, then
  \[ C_{\varphi} = \{(x, s) \mid \frac{\partial \varphi}{\partial s_i}(x, s) = 0, 1 \leq i \leq k\}, \]
  and the map $p_{\varphi}$ is just the map
  \[ p_{\varphi}(x, s) = (x, \frac{\partial \varphi}{\partial x}(x, s)). \]
- Let $\iota : C_{\varphi} \hookrightarrow Z$ be the inclusion. We claim $d(\iota^*\varphi - p_{\varphi}^*\varphi_\Lambda) = 0$.

Proof. In fact, by definition of phase function $\varphi_\Lambda$,
\[ d(\iota^*\varphi - p_{\varphi}^*\varphi_\Lambda) = \iota^*d\varphi - (\iota_\Lambda \circ p_{\varphi})^*\alpha_{T^*X}. \]
In local coordinates
\[ \iota^*d\varphi = \iota^*(\sum \frac{\partial \varphi}{\partial x_i} dx_i + \frac{\partial \varphi}{\partial s_i} ds_i) = \sum \frac{\partial \varphi}{\partial x_i} dx_i. \]
On the other hand, since \( t_\Lambda \circ p_\varphi(x, s) = (x, \frac{\partial \varphi}{\partial x}) \),

\[(t_\Lambda \circ p_\varphi)^* \alpha_{T^*X} = (t_\Lambda \circ p_\varphi)^* \sum \xi_i dx_i = \sum \frac{\partial \varphi}{\partial x_i} dx_i. \]

\[\square\]

- As a consequence, the functions \( t^* \varphi \) and \( p_\varphi^* \varphi_\Lambda \) differ by a constant. In what follows we fix a phase function \( \varphi_\Lambda \) and normalize generating function \( \varphi \) by requiring

\[t^* \varphi = p_\varphi^* \varphi_\Lambda.\]

- The space of compactly supported oscillatory half densities on \( X \) associated with \( \Lambda \) is (strictly speaking one should write \( I_k^0(X, \Lambda, \varphi_\Lambda) \) instead)

\[I_k^0(X, \Lambda) = \{ \mu = h^{k-\frac{d}{2}} \pi_*(a(z, h)e^{i\varphi(z)\tau}) \mid a \in C_0^\infty(Z \times \mathbb{R}) \}, \]

where \( \tau \) is a nowhere vanishing half density on \( Z \), and \( d = \dim Z - \dim X \) is the dimension of the fibration. Locally we may fix a nowhere vanishing half density \( \tau_0 \) on \( X \) and write \( \mu \in I_k^0(X, \Lambda) \) as

\[h^{k-\frac{d}{2}} \left( \int_S a(x, s, h)e^{i\varphi(x, s)\tau_0} ds \right). \]

We have shown that this space is independent of the choices of generating functions.

- In case \( \Lambda \) does not admit a global generating function, one may find a locally finite open cover of \( \Lambda \) by open sets \( \Lambda_i \) such that each \( \Lambda_i \) is defined by a generating function \( \varphi_i \) relative to some fibrations \( \pi_i : Z_i \to U_i \), where \( U_i \) are open sets in \( X \). Then we define \( I_k^0(X, \Lambda) \) to be the space of those half densities that can be written as a finite sum \( \mu = \sum_{j=1}^N \mu_j \), where \( \mu_j \in I_k^0(U_j, \Lambda_j) \).

- The space \( I_k^k(X, \Lambda) \) consists of those half densities \( \mu \) such that \( \rho \mu \in I_k^0(X, \Lambda) \) for all \( \rho \in C_0^\infty(X) \) if \( \Lambda \) admits a global generating function.

- Take \( X = X_1 \times X_2 \), where \( X_1, X_2 \) are smooth manifolds. Suppose \( \Gamma \subset T^*X \times T^*X \) is an exact canonical relation. Then \( \Lambda = \sigma_2 \circ \Gamma \) is an exact Lagrangian submanifold of \( T^*X \). Playing the previous games, we get the space \( I_k^0(X, \Lambda) \) which consists of half densities of the form

\[\mu = h^{k-\frac{d}{2}} \left( \int_S a(x_1, x_2, s, h)e^{i\varphi(x_1, x_2, s)} ds \right) dx_1^\frac{d}{2} dx_2^\frac{d}{2}. \]

\[\square\] The definition.

Now suppose \( X_1, X_2 \) are manifolds. We will denote \( M_i = T^*X_i, i = 1, 2 \). Suppose \( \Gamma \subset M_1 \times M_2^{-} \) is an exact canonical relation. Then

\[\Lambda = \sigma_2 \circ \Gamma\]

is an exact Lagrangian submanifold of \( T^*X \), where \( X = X_1 \times X_2 \). Associated with \( \Lambda \) we have the space of compactly supported oscillatory half densities \( I_k^0(X, \Lambda) \). If we
fix a nowhere vanishing one density \( dx_1 \) on \( X_1 \) and a nowhere vanishing one density \( dx_2 \) on \( X_2 \), then a typical element in \( I^k_0(X, \Lambda) \) is of the form

\[
\mu = u(x_1, x_2, \hbar)dx_1^{\frac{1}{2}}dx_2^{\frac{1}{2}},
\]

where \( u \) is a compactly supported smooth function.

With some abuse of notion we let \( L^2(X_i) \) be the Hilbert space of \( L^2 \) half densities on \( X_i \). Then associated to each \( \mu \in I^k_0(X, \Lambda) \) we can define an integral operator

\[
F_{\mu}(f) = \left( \int f(x_1)u(x_1, x_2, \hbar)dx_1 \right)dx_2^{\frac{1}{2}}.
\]

Such operators are called compactly supported semi-classical Fourier integral operators. The space of these operators is denoted by \( F^m_0(\Gamma) \), where \( m = k + \frac{n_2}{2} \), where \( n_2 = \dim X_2 \).

**Remark.** We could loose the conditions on \( u \) by requiring only \( u(x_1, x_2, \hbar)dx_1^{\frac{1}{2}} \in L^2(X_1) \). In this case we drop the subscript 0.

**Example:** Semi-classical pseudo-differential operators.

Take \( X_1 = X_2 = \mathbb{R}^n \) and

\[
\Gamma = \Delta_M = \text{graph of the identity} = \{(x, s, x, s) \} \subset M \times M^-,
\]

where \( M = T^*\mathbb{R}^n \). Then

\[
\Lambda = \sigma_2 \circ \Gamma = \{(x, x, s, -s) \} \subset T^*(\mathbb{R}^n \times \mathbb{R}^n).
\]

On \( \Lambda \) one has \( \iota_\Lambda^* \alpha_{T^*X} = \sum s_i dx_i - s_i dx_i = 0 \), so one take phase function \( \varphi_\Lambda = 0 \).

To find a generating function, one use the fibration

\[
\pi : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n, \quad (x, y, s) \mapsto (x, y).
\]

Then by definition,

\[
\Gamma_\pi = \{(x, y, s, \eta_1, \eta_2, 0, x, y, \eta_1, \eta_2) \}.
\]

If we take \( \varphi \in C^\infty(\mathbb{R}^n) \) to be the function

\[
\varphi(x, y, s) = \sum (x_i - y_i)s_i,
\]

then

\[
\Lambda_\varphi = \{(x, y, s, -s, x - y) \}.
\]

and it is easy to see

\[
\Lambda = \Gamma_\pi \circ \Lambda_\varphi.
\]

Moreover, the set \( C_\varphi \) is defined by the equations \( \frac{\partial \varphi}{\partial s_i} = x_i - y_i = 0 \), i.e.

\[
C_\varphi = \{(x, x, s) \},
\]

and the map \( p_\varphi \) is given explicitly by

\[
p_\varphi : C_\varphi \to \Lambda, \quad (x, x, s) \mapsto (x, x, -s).
\]
So \( i^* \varphi = 0 = p^*_\varphi \varphi_\Lambda \). In other words, \( \varphi \) is the normalized generating function for \( \Lambda \).

What are the semi-classical Fourier integral operators associated to \( \Gamma \)? By definition, \( \mathcal{F}^m(\Gamma) \) consist of those operators that maps \( f(x)dx^\frac{1}{2} \) to
\[
\hbar^{m - \frac{n}{2} - \frac{n}{2}} \left( \int_{\mathbb{R}^n \times \mathbb{R}^n} e^{\frac{i}{\hbar} (x-y)^s a(x, y, s, \hbar)} f(x)dx d\xi \right) dy^\frac{1}{2}.
\]
They are the semi-classical pseudo-differential operators (of semi-classical order \( m \)) we learned earlier in this course! (We used special symbols of the form \( a(x, s, \hbar) \) or \( a(y, s, \hbar) \) or \( a(x + y, s, \hbar) \), but we could use general symbols \( a(x, y, s, \hbar) \). One can show that any general symbol corresponds to a unique left/right/Weyl symbol. c.f A. Martinez, page 37.)

In general, if \( X_1 = X_2 = X \) is a smooth manifold, then the construction above gives us semi-classical pseudo-differential operators on \( X \). In other words, \( \Psi^m(X) = \mathcal{F}(\Delta_M) \).

**Example; The semi-classical Fourier transform.**

Let \( X_1 = X_2 = \mathbb{R}^n \). Let \( \Gamma \) be the graph of the symplectomorphism
\[
J : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n, \quad (x, y) \mapsto (-y, x),
\]
i.e.
\[
\Gamma = \{(x, y, -y, x)\} \subset T^* \mathbb{R}^n \times T^* \mathbb{R}^n.
\]
Then
\[
\Lambda = \{(x, y, -y, -x)\} \subset T^*(\mathbb{R}^n \times \mathbb{R}^n).
\]
\( \Lambda \) is exact since \( i^* \alpha_{T^*X} = -\sum y_idx_i - \sum x_idy_i = -d(x \cdot y) \). We just choose the phase function \( \varphi_\Lambda = -x \cdot y \).

We don’t need a fibration to find a generating function, since \( \Lambda \) is already a horizontal Lagrangian, with (normalized!!) generating function
\[
\varphi(x, y) = -x \cdot y.
\]
(So in this example \( C_\varphi = X = Z \). What is the map \( p_\varphi \)?)

Let \( \mu = e^{-\frac{i}{\hbar} x \cdot y} dx^\frac{1}{2} dy^\frac{1}{2} \in \mathcal{I}^0(X, \Lambda) \). What is the corresponding semi-classical Fourier integral operator? By definition \( \mathcal{F}_\mu \) maps any \( f(x)dx^\frac{1}{2} \) (with \( f \in C_0^\infty(\mathbb{R}^n) \) for simplicity) to
\[
\left( \int_{\mathbb{R}^n} f(x) e^{-\frac{i}{\hbar} x \cdot y} dx \right) dy^\frac{1}{2},
\]
which is the semi-classical Fourier transform \( \mathcal{F}_\hbar \)!

What about the inverse (semi-classical) Fourier transform? Well, repeating the previous process one can see that \( \mathcal{F}_\hbar^{-1} \) is a semi-classical Fourier integral operator associated to the graph of the symplectomorphism (which is the inverse of the previous one)
\[
J^{-1} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n, \quad (x, y) \mapsto (y, -x).
\]
The composition of phase functions.

Now let $X_1, X_2$ and $X_3$ be smooth manifolds, $M_i = T^*X_i$. Let $\Gamma_i \in M_i \times M_{i+1}$ be exact Lagrangian submanifolds, with phase function $\varphi_{\Lambda_i}$. Suppose $\Gamma_1$ and $\Gamma_2$ are cleanly composable. Recall that this implies that the map

$$\alpha : F = \{(m_1, m_2, m_3) \mid (m_i, m_{i+1}) \in \Gamma_i\} \rightarrow M_1 \times M_3$$

is a constant rank map which maps onto $\Gamma_2 \circ \Gamma_1$. As before we will assume that $\alpha$ is proper and has connected level sets, so that $\Gamma_2 \circ \Gamma_1$ is an embedded Lagrangian submanifold.

**Theorem 5.1.** $\Gamma_2 \circ \Gamma_1$ is an exact Lagrangian submanifold of $M_1 \times M_3$.

**Proof.** Let $\rho_i : F \rightarrow \Gamma_i$ and $\pi_i : F \rightarrow M_i$ be the obvious projection maps. Then

$$\rho_i^*(\alpha_{T^*X_1 \times T^*X_2}\mid_{\Gamma_i}) = \pi_1^*\alpha_{T^*X_1} - \pi_2^*\alpha_{T^*X_2}.$$  

Similar expressions holds for $\rho_2^*(\alpha_{T^*X_2 \times T^*X_3}\mid_{\Gamma_2})$ and $\alpha^*(\alpha_{T^*X_1 \times T^*X_3}\mid_{\Gamma_2 \circ \Gamma_1})$, which implies

$$\rho_1^*(\alpha_{T^*X_1 \times T^*X_2}\mid_{\Gamma_1}) + \rho_2^*(\alpha_{T^*X_2 \times T^*X_3}\mid_{\Gamma_2}) = \alpha^*(\alpha_{T^*X_1 \times T^*X_3}\mid_{\Gamma_2 \circ \Gamma_1}).$$

On the other hand, by definition

$$\alpha_{T^*X_1 \times T^*X_{i+1}}\mid_{\Gamma_i} = d\varphi_{\Gamma_i}$$

for $i = 1, 2$. So if we let

$$\varphi = \rho_1^*\varphi_{\Gamma_1} + \rho_2^*\varphi_{\Gamma_2} \in C^\infty(F),$$

then

$$d\varphi = \rho_1^*d\varphi_{\Gamma_1} + \rho_2^*d\varphi_{\Gamma_2} = \alpha^*(\alpha_{T^*X_1 \times T^*X_3}\mid_{\Gamma_2 \circ \Gamma_1}).$$

For any $p \in \Gamma_2 \circ \Gamma_1$, let $F_p = \alpha^{-1}(p)$ be the connected compact fiber over $p$ and let $\iota_p : F_p \hookrightarrow F$ be the inclusion. Then $\alpha \circ \iota_p : F_p \rightarrow \Gamma_2 \circ \Gamma_1$ is the constant map. So

$$(\alpha \circ \iota_p)^*(\alpha_{T^*X_1 \times T^*X_3}\mid_{\Gamma_2 \circ \Gamma_1}) = 0.$$  

It follows

$$d\iota_p^*\varphi = \iota_p^*d\varphi = 0.$$  

Since $F_p$ is connected, $\iota_p^*\varphi$ is constant on $F_p$. In other words, $\varphi$ is constant on each fiber $F_b$. So one can find a function $\varphi_{\Gamma_2 \circ \Gamma_1} \in C^\infty(\Gamma_2 \circ \Gamma_1)$ so that

$$\varphi = \alpha^*\varphi_{\Gamma_2 \circ \Gamma_1}.$$  

Thus

$$\alpha^*d\varphi_{\Gamma_2 \circ \Gamma_1} = d\varphi = \alpha^*(\alpha_{T^*X_1 \times T^*X_3}\mid_{\Gamma_2 \circ \Gamma_1}).$$

Since $\alpha : F \rightarrow \Gamma_2 \circ \Gamma_1$ is a fibration, $\alpha^*$ is injective. It follows

$$d\varphi_{\Gamma_2 \circ \Gamma_1} = \alpha_{T^*X_1 \times T^*X_3}\mid_{\Gamma_2 \circ \Gamma_1},$$

i.e. $\varphi$ is a phase function for $\Gamma_2 \circ \Gamma_1$. \qed
Remark. In the proof we have seen that the three phase functions are related by
\[ d\alpha^* \varphi_{\Gamma_2 \circ \Gamma_1} = d\rho_1^* \varphi_{\Gamma_1} + d\rho_2^* \varphi_{\Gamma_2}. \]

In the future we will fix the constant in the phase function of \( \Gamma_2 \circ \Gamma_1 \) by requiring
\[ (27) \quad \alpha^* \varphi_{\Gamma_2 \circ \Gamma_1} = \rho_1^* \varphi_{\Gamma_1} + \rho_2^* \varphi_{\Gamma_2}. \]

\section*{The composition of semi-classical FIOs.}

Now suppose two canonical relations \( \Gamma_i \in M_i \times M_{i+1} \) are transversally composable. Then as we mentioned earlier, if \( \pi_i : Z_i \to X_i \times X_{i+1} \) are fibrations and \( \varphi_i \in C^\infty(Z_i) \) are generating functions for \( \Gamma_i \) with respect to \( \pi_i \), then if we set
\[ Z = (\pi_1 \times \pi_2)^{-1}(X_1 \times \Delta_{Z_2} \times Z_3) \subset Z_1 \times Z_2 \]
and let \( \pi : Z \to X_1 \times X_3 \) be the obvious fibration map, then
\[ \varphi(z_1, z_2) = \varphi_1(z_1) + \varphi_2(z_2) \]
is a generating function for \( \Gamma_2 \circ \Gamma_1 \) with respect to \( \pi \). More over, if we normalize \( \varphi_1 \) and \( \varphi_2 \) by fixing phase functions \( \varphi_{\Gamma_1} \) and \( \varphi_{\Gamma_2} \), then \( \varphi \) is normalized with respect to \( \varphi_{\Gamma_2 \circ \Gamma_1} \) described above.

\textbf{Theorem 5.2.} If \( F_i \in \mathcal{F}_0^{m_i}(\Gamma_i) \) for \( i = 1, 2 \), then
\[ (28) \quad F_2 \circ F_1 \in \mathcal{F}_0^{m_1+m_2}(\Gamma_2 \circ \Gamma_1). \]

\textbf{Proof.} By a partition of unity argument one can assume that we have fibrations
\[ \pi_1 : X_1 \times X_2 \times S_1 \to X_1 \times X_2 \]
and
\[ \pi_2 : X_2 \times X_3 \times S_2 \to X_2 \times X_3, \]
where \( S_1 \subset \mathbb{R}^{d_1} \) and \( S_2 \subset \mathbb{R}^{d_2} \) are open sets. Let \( \varphi_1 \) and \( \varphi_2 \) be generating functions of \( \Gamma_1 \) and \( \Gamma_2 \) with respect to these fibrations. Then by definition, \( F_2 \circ F_1 \) maps \( f dx_1^\frac{d_1}{2} \) to
\[ \hbar^{k_1 - \frac{d_1}{2} + k_2 - \frac{d_2}{2}} \left( \int_{X_2} \int_{S_2} \int_{X_1} \int_{S_1} e^{\frac{i}{\hbar} (\varphi_1(x_1, x_2, s_1) + \varphi_2(x_2, x_3, s_2))} a_1(x_1, x_2, s_1, \hbar) \right. \]
\[ \times a_2(x_2, x_3, s_2, \hbar) f(x_1) ds_1 dx_1 ds_2 dx_2 \left. \right) dx_3^\frac{d_3}{2}. \]
Since \( \varphi_1(x_1, x_2, s_1) + \varphi_2(x_2, x_3, s_2) \) is a normalized generating function of \( \Gamma_2 \circ \Gamma_1 \) with respect to the fibration
\[ X_1 \times X_3 \times (X_2 \times S_1 \times S_2) \to X_1 \times X_2 \]
with fiber dimension \( d_1 + d_2 + n_2 \), the conclusion follows. \( \square \)

As a consequence, we see
- If \( P_i \in \Psi^{m_i}(X) \), then \( P_1 P_2 \in \Psi^{m_1+m_2}(X) \).
6. The symbols

The symbol of an oscillatory half density.

Let $\mu = u \nu \in I^k(X, \Lambda)$, where $\nu$ is a nowhere vanishing half density on $X$. The locally

$$u(x, h) = h^{k - \frac{d}{2}} \int a(x, s, h) e^{i \frac{\varphi(x, s)}{h}} ds,$$

where $\varphi = \varphi(x, s)$ is a generating function with respect to a fibration $X \times S \to X$, and $a \in C_0^\infty(X \times S \times h)$. Recall that the critical set $C_{\varphi}$ is the subset of $X \times S$ defined by the equations

$$\frac{\partial \varphi}{\partial s_i} = 0, \quad i = 1, 2, \ldots, d.$$

Proposition 6.1. If $a(x, s, 0) = 0$ on $C_{\varphi}$, then $\mu \in I_{0}^{k+1}(X, \Lambda)$.

Proof. If $a(x, s, 0) = 0$ on $C_{\varphi}$, then we can write

$$a(x, s, h) = \sum a_j(x, s, h) \frac{\partial \varphi}{\partial s_j} + a_0(x, s, h) h$$

for smooth functions $a_0, \ldots, a_d$. So we can rewrite the integral expression of $u(x, h)$ as $u_0 + u_1 + \cdots + u_d$, where

$$u_0(x, h) = h^{k+1-\frac{d}{2}} \int a_0(x, s, h) e^{i \varphi(x, s)} ds,$$

and

$$u_j = h^{k - \frac{d}{2}} \int a_j(x, s, h) \frac{\partial \varphi}{\partial s_j} e^{i \varphi} ds$$

$$= -i h^{k+1-\frac{d}{2}} \int a_j(x, s, h) \frac{\partial}{\partial s_j} e^{i \varphi} ds$$

$$= i h^{k+1-\frac{d}{2}} \int \left( \frac{\partial}{\partial s_j} a_j(x, s, h) \right) e^{i \varphi} ds.$$

 Remark. Apply the proof inductively, one can see that if for $i = 0, 1, \cdots, 2l - 1$, the functions $\frac{\partial}{\partial s_i} a_j(x, s, 0)$ vanishes to order $2l - i$ on $C_{\varphi}$, then $\mu \in I_0^{k+2l+1}(X, \Lambda)$. In particular, if $a$ vanishes to infinite order on $C_{\varphi}$, then $\mu \in I^\infty(X, \Lambda) = \cap_k I^k(X, \Lambda)$.

Now suppose we have all the data as before. Recall that $p_\varphi$ is a diffeomorphism from $C_{\varphi}$ onto $\Lambda$. For any $\mu \in I^k(X, \Lambda)$ we tentatively define its “symbol” to be the function $\sigma_\varphi(\mu) \in C^\infty(\Lambda)$ such that

$$\sigma_\varphi(\mu)(x, \xi) = a(x, s, 0), \quad \text{if } (x, s) \in C_{\varphi} \text{ and } p_\varphi(x, s) = (x, \xi).$$

Of course the function $\sigma_\varphi(\mu)$ depends on the choice of fibration, the choice of generating function, and also on the choice of the non-vanishing half density on $X$. 


Proposition 6.2. For any $p \in \Lambda$ the assertion $\sigma_{\varphi}(\mu)(p) = 0$ is intrinsic, i.e. independent of all the choices above.

Proof. Exercise. \hfill \Box

It follows that

$$I_p^k(X, \Lambda) = \{ \mu \in I^k(X, \Lambda) \mid \sigma_{\varphi}(\mu)(p) = 0 \}$$

is intrinsically defined. We thus get a line bundle $\mathbb{L} \to \Lambda$, whose fiber over $p$ is

$$\mathbb{L}_p = I^k(X, \Lambda)/I^k_p(X, \Lambda).$$

Note that the fiber $\mathbb{L}_p$ is independent of $k$, since multiplication by $h^{l-k}$ is an isomorphism from $I^k(X, \Lambda)$ to $I^l(X, \Lambda)$ which maps $I^k_p(X, \Lambda)$ to $I^l_p(X, \Lambda)$. Moreover, a choice of data above gives a trivialization of $\mathbb{L} \to \Lambda$. So it is a smooth line bundle.

Now we can define the intrinsic symbol of an oscillatory half density:

Definition 6.3. The intrinsic symbol of $\mu \in I^k(X, \Lambda)$ is a section $\sigma(\mu)$ of the line bundle $\mathbb{L}$ given by

$$\sigma(\mu)(p) = [\mu_p] \in \mathbb{L}_p.$$  

The following result is now obvious:

Proposition 6.4. If $\mu \in I^k(X, \Lambda)$ and $\sigma(\mu) = 0$, then $\mu \in I^{k+1}(X, \Lambda)$.

The symbol of a semi-classical FIO.

For a semi-classical FIO $F = F_{\mu} \in \mathcal{F}_0^{\alpha}(\Gamma)$. We can define a line bundle $\mathbb{L}_{\Gamma} \to \Gamma$ to be the pull back of the line bundle $\mathbb{L}_{\Lambda} \to \Lambda$ via the map $id \times \sigma_2$. We then define the symbol of $F$ to be the section of this new line bundle

$$\sigma(F) = (id \times \sigma_2)^* \sigma(\mu).$$

Composition law for symbols of a semi-classical FIO.

Now let $\Gamma_1 \in T^*X_1 \times T^*X_2^-$ and $\Gamma_2 \in T^*X_2 \times T^*X_3^-$ be exact canonical relations that are transversally composable and let $\Gamma = \Gamma_2 \circ \Gamma_1$. Then the map

$$\alpha : F \to \Gamma, \quad (m_1, m_2, m_3) \mapsto (m_1, m_3)$$

is a diffeomorphism. Let

$$j : F \to \Gamma_1 \times \Gamma_2, (m_1, m_2, m_3) \mapsto ((m_1, m_2), (m_2, m_3))$$

be the canonical embedding.

Recall that the function

$$\varphi(z_1, z_2) = \varphi_1(z_1) + \varphi_2(z_2)$$

is a generating function of $\Gamma$ with respect to the fibration $\pi : Z \to X_1 \times X_3$, where $\varphi_i$ is a generating function of $\Gamma_i$ with respect to a fibration $\pi_i : Z_i \to X_i \times X_{i+1}$ and
Note that in particular this implies as maps from $F$ we can rewrite the formula as

$$C$$

Similar formula holds for $Z$'s. By (34), we get an embedding

$$\iota : C_{\varphi} \rightarrow C_{\varphi_1} \times C_{\varphi_2}, \quad (x_1, x_3, s_1, s_2, x_2) \mapsto ((x_1, x_2, s_1), (x_2, x_3, s_2)).$$

Last time we have shown that $F_2 \circ F_1$ is a semi-classical Fourier integral operator associated to the oscillatory half density $\mu$ of the form

$$h^{k_1-\frac{d_1}{2}+k_2-\frac{d_2}{2}} \left( \int_{X_2} \int_{S_2} \int_{S_1} e^{\frac{i}{h}(\varphi_1(x_1, x_2, s_1)+\varphi_2(x_2, x_3, s_2))} a_1(x_1, x_2, s_1, h) \right. \left. \times a_2(x_2, x_3, s_2, h) ds_1 ds_2 dx_2 \right) dx_1^\frac{1}{2} dx_3^\frac{1}{2}.$$

In particular, the amplitude is

$$a(x_1, x_3, s_1, s_2, x_2, h) = a_1(x_1, x_2, s_1, h) a_2(x_2, x_3, s_2, h).$$

So

$$a|_{C_{\varphi}} = \iota^*(a_1|_{C_{\varphi_1}} a_2|_{C_{\varphi_2}}).$$

By definition

$$\sigma_{\varphi}(F) = (p_{\varphi}^{-1})^* a|_{C_{\varphi}, h=0}.$$ Similar formulae hold for $\sigma_{\varphi_1}(F_1)$ and $\sigma_{\varphi_2}(F_2)$. Note that as maps from $C_{\varphi}$ to $\Gamma_1 \times \Gamma_2$,

$$j \circ \alpha^{-1} \circ p_{\varphi} = (p_{\varphi_1} \times p_{\varphi_2}) \circ \iota,$$

since both maps a point $(x_1, x_3, s_1, s_2, x_2)$ to the point $((x_1, x_2, \frac{\partial \varphi_1}{\partial x_1}, \frac{\partial \varphi_1}{\partial x_2}), (x_2, x_3, \frac{\partial \varphi_2}{\partial x_2}, \frac{\partial \varphi_2}{\partial x_3}))$. We can rewrite the formula as

$$\iota \circ p_{\varphi}^{-1} \circ \alpha = (p_{\varphi_1} \times p_{\varphi_2})^{-1} \circ j$$

as maps from $F$ to $C_{\varphi_1} \times C_{\varphi_2}$. Setting $h = 0$ in (35) we get

$$\alpha^* \sigma_{\varphi}(F_2 \circ F_1) = j^*(\sigma_{\varphi_1}(F_1)\sigma_{\varphi_2}(F_2)).$$

Note that in particular this implies

$$\sigma_{\varphi}(F_2 \circ F_1)(m_1, m_3) = 0 \Leftrightarrow \text{either } \sigma_{\varphi_1}(F_1)(m_1, m_2) = 0 \text{ or } \sigma_{\varphi_2}(F_2)(m_2, m_3) = 0.$$
Passing to the intrinsic symbol, we get

**Theorem 6.5.** If \( F_i \in \mathcal{F}^{m_i}(\Gamma_i) \), then

\[
\alpha^*(\sigma(F_2 \circ F_1)) = j^*(\sigma(F_1)\sigma(F_2)).
\]

\(\square\)

**Vanishing symbols.**

Last time we showed that if \( \mu \in I^k(X, \Lambda) \) and \( \sigma(\mu) = 0 \), then \( \mu \in I^{k+1}(X, \Lambda) \). Now we can prove the inverse:

**Theorem 6.6.** Let \( \mu \) be an element of \( I^{k+1}(X, \Lambda) \). Then as an element of \( I^k(X, \Lambda) \), \( \sigma(\mu) = 0 \).

**Proof.** Let’s first prove the theorem in the case \( \Lambda = \Lambda_\varphi \) is horizontal, so that there is no fibration needed in defining the generating function. As a consequence, the critical set \( C_\varphi \) is \( X \) itself, and the diffeomorphism \( p_\varphi : X \to \Lambda_\varphi \) is just the map \( x \mapsto (x, d\varphi_x) \). Any \( \mu \in I^{k+1}(X, \Lambda) \) is of the form

\[
\hbar^{k+1} a(x, \hbar) e^{\frac{i}{\hbar} \varphi} dx_1^\frac{1}{2}.
\]

So as an element of \( I^k(X, \Lambda) \),

\[
\mu = \hbar^k a(x, \hbar) e^{\frac{i}{\hbar} \varphi} dx_1^\frac{1}{2}.
\]

Thus by definition,

\[
\sigma_\varphi(\mu) = (p_\varphi^{-1})^*(\hbar a(x, \hbar))|_{\hbar=0} = 0.
\]

For the general case, we need

**Exercise 2.** Given any Lagrangian submanifold \( \Lambda \subset T^*X \) and any \( p \in \Lambda \), one can find a horizontal Lagrangian submanifold \( \Lambda_\varphi \) such that \( \Lambda_\varphi \cap \Lambda = \{p\} \) and such that the intersection is transverse.

Now let \( \mu_1 \in I^0(X, \Lambda_\varphi) \) be \( \mu_1 = a_1(x, \hbar) e^{\frac{i}{\hbar} \varphi} dx_1^\frac{1}{2} \) such that \( a_1(x, 0) \neq 0 \). In other words, \( \sigma(\mu_1) \) is nowhere vanishing. We can think of \( \Lambda \) as a morphism from \( pt \) to \( T^*X \), and \( \Lambda_\varphi^T \) as a morphism from \( T^*X \) to \( pt \). This is of course a transverse composition, so for \( \mu \in I^{k+1}(X, \Lambda) \), we have

\[
F_{\mu_1} \circ F_{\mu} = F_\nu
\]

for some \( \nu \in I^{k+1+\frac{n}{2}}(pt) \). It follows

\[
\nu = \hbar^{k+\frac{n}{2}+1} c(\hbar)
\]

and

\[
\sigma(\nu) = \sigma(\mu_1)(p)\sigma(\mu)(p) = c(0).
\]

So if we think \( \mu \) as an element of \( I^k(X, \Lambda) \),

\[
\nu = \hbar^{k+\frac{n}{2}} h c(\hbar),
\]

and

\[
\sigma(\nu) = (hc(\hbar))|_{\hbar=0} = 0.
\]
Since $\sigma(\mu_1)(p) \neq 0$, we conclude $\sigma(\mu)(p) = 0$. Since we can do this for every $p$, we must have $\sigma(\mu) = 0$. □

\[ \Delta_M \circ \Delta_M = \Delta_M. \]

It follows that if $P_i \in \Psi^{m_i}(X)$, then $P_2 \circ P_1 \in \Psi(m_1 + m_2)(X)$.

We can identify $M$ with $\Delta M$ and also with the set $F = \{(m,m,m)\}$ under the obvious maps. Then the maps $\alpha : F \to \Delta M \circ \Delta M = \Delta M$ and $pr_i : F \to \Delta M$ are all identities. So according to the composition law, we have a canonical isomorphism

\[ \mathbb{L}_M = \mathbb{L}_M \otimes \mathbb{L}_M, \]

where $\mathbb{L}_M$ is the pull back of $\mathbb{L}_{\Delta M}$ to $M$ under the diagonal embedding. In particular, $\mathbb{L}_M$ is a trivial line bundle, and thus the sections of $\mathbb{L}_M$ can be identified with $C^\infty(M)$. Under this identification, the symbol map is a map

\[ \sigma_m : \Psi^m(X) \to C^\infty(M) \]

and the kernel of this symbol map is $\Psi^{m+1}(X)$, and the composition law becomes

\[ \sigma_{m_1+m_2}(P_2 \circ P_1) = \sigma_{m_2}(P_2)\sigma_{m_1}(P_1). \]

As a consequence, we see

\[ \sigma_{m_1+m_2}(P_2 \circ P_1) = \sigma_{m_1+m_2}(P_1 \circ P_2). \]

In other words, the symbol of the commutator vanishes:

\[ \sigma_{m_1+m_2}(P_1 \circ P_2 - P_2 \circ P_1) = 0. \]

It follows

\[ P_1 \circ P_2 - P_2 \circ P_1 \in \Psi^{m_1+m_2+1}(X). \]

It is easy to check that as an element of $\Psi^{m_1+m_2+1}(X)$, the symbol $\sigma_{m_1+m_2+1}$ depends only $\sigma_{m_1}(P_1)$ and $\sigma_{m_2}(P_2)$, which is basically the Poison bracket on $C^\infty(M)$:

\[ \sigma_{m_1+m_2+1}(P_1 \circ P_2 - P_2 \circ P_1) = -i\{\sigma_{m_1}(P_1), \sigma_{m_2}(P_2)\}. \]
\[ I(X, \Lambda) \text{ as a module over } \Psi(X). \]

More generally, since
\[ \Delta_M \circ \Lambda = \Lambda, \]
we have the composition
\[ P \in \Psi^{k_1}(X), \mu \in I^{k_2}(X, \Lambda) \implies P\mu \in I^{k_1+k_2}(X, \Lambda), \]
where we think of \( \mu \) as a semi-classical Fourier integral operator from \( pt \) to \( X \). In other words, one can regard \( I(X, \Lambda) \) as a module over \( \Psi(X) \).

In this example the set \( F \) contains points of the form \((pt, p, p)\) for \( p \in M \). We can identify \( F \to \Lambda \), \((pt, p, p) \mapsto p\).

Under this identification, the map \( \alpha : F \to \Delta_M \circ \Lambda = \Lambda \) is the identity, the map \( pr_1 : F \to \Lambda \) is the identity, and the map
\[ pr_2 : F \to \Delta_M, (pt, p, p) \mapsto (p, p), \]
becomes the inclusion map \( \iota : \Lambda \to M \) where we identified \( \Delta_M \) with \( M \), and the map \( j : F \to \Lambda \times \Delta_M \) becomes the map \( id \times \iota \). The symbol composition law becomes
\[ \sigma(P\mu) = \iota^*(\sigma(P))\sigma(\mu). \]

\[ \text{¶ The Egorov's theorem.} \]

Now let \( \gamma : T^*X_1 \to T^*X_2 \) be a symplectomorphism, and set
\[ \Gamma_1 = \text{graph}(\gamma), \quad \Gamma_2 = \text{graph}(\gamma^{-1}). \]
Suppose \( F_1 \) is a semi-classical Fourier integral operator associated to \( \Gamma_1 \) and \( F_2 = F_1^{-1} \).

**Theorem 6.7.** For any \( A \in \Psi^k(X_2) \), we have
\[ F_2 \circ A \circ F_1 \in \Psi^k(X_1) \]
and
\[ \sigma(F_2 \circ A \circ F_1) = \gamma^*(\sigma(A)). \]

**Proof.** The first assertion follows from the fact
\[ \Gamma_2 \circ \Delta_{T^*X_2} \circ \Gamma_1 = \Delta_{T^*X_1}. \]
For the second one, we have
\[ \sigma(F_2AF_1)(x, \xi) = \sigma(F_2)(y, \eta, x, \xi)\sigma(A)(y, \eta)\sigma(F_1)(x, \xi, y, \eta) \]
for \((x, \xi, y, \eta) \in \Gamma_1\). Since \( \sigma(A) \) is just a scalar, we can pull the middle term out of the product. On the other hand, since \( F_2 \circ F_1 = 1 \), we have
\[ \sigma(F_2)(y, \eta, x, \xi)\sigma(F_1)(x, \xi, y, \eta) = 1. \]
It follows
\[ \sigma(F_2AF_1)(x, \xi) = \sigma(A)(y, \eta), \]
where \((y, \eta) = \gamma(x, \xi)\) since \((x, \xi, y, \eta) \in \Gamma_1\).

\[
\boxed{7. \text{ The trace of compactly supported FIOs}}
\]

\[\hfill \]The composition of Lagrangian submanifolds.

Let \(X\) be an \(n\) dimensional manifold and \((\Gamma_i, \psi_i), i = 1, 2\) be exact Lagrangian submanifold of \(T^*X\). Then \((\Lambda_1, \psi_1)\) can be think of as an exact canonical relation from \(pt\) to \(T^*X\), and \((\Lambda_2, -\psi_2)\) can be think of as an exact canonical relation from \(T^*X\) to \(pt\). The set \(F\) of these canonical relations contains points of the form \(\{(pt, m, pt) | (pt, m) \in \Lambda_1, (m, pt) \in \Lambda_2\}\), or in other words, \(F = \Lambda_1 \cap \Lambda_2\). We assume that \(\Lambda_1\) and \(\Lambda_2\) intersects cleanly in \(T^*X\), and let \(W_1, \cdots, W_N\) be the connected components of this intersection. They are isotropic submanifolds of \(T^*X\). Since \(\Lambda_2 \circ \Lambda_1\) is the trivial canonical relation from \(pt\) to \(pt\), the composed phase function \(\psi\),

\[
\alpha^* \psi = pr_1^* \psi_1 - pr_2^* \psi_2,
\]

when restricted to each \(W_r\), is a constant. We denote this by \(\psi^r\).

Now let \(S_i, i = 1, 2\) be \(d_i\) dimensional manifolds and \(\varphi_i \in C^\infty(X \times S_i)\) a generating function for \(\Lambda_i\) with respect to the fibration \(X \times S_i \to X\). Then

\[
\varphi(x, s_1, s_2) = \varphi(x, s_1) - \varphi(x, s_2)
\]

a generating function for \(pt\) with respect to the fibration \(X \times S_1 \times S_2 \to pt\). The generating function and the phase function is related by

\[
p_{\varphi}^* \alpha^* \psi = \varphi.
\]

where \(p_{\varphi} : C_{\varphi} \to F\).

Fact: \(\varphi\) is a Bott-Morse function.

Some missing pieces:

- A smooth function \(\varphi\) on \(X\) is called a Bott-Morse function if its critical set \(C_{\varphi} = \{x \mid d\varphi_x = 0\}\) is a smooth submanifold of \(X\), and for every \(p \in C_{\varphi}\), the Hessian \(d^2 \varphi_p\) is non-degenerate on the normal space \(N_pC_{\varphi} = T_pX/T_pC_{\varphi}\).
- For transversal generating function, increasing fiber variables gives Morse functions, while for clean generating function, increasing fiber variables gives Bott-Morse function.
- The lemma of stationary phase for Morse functions can be generalized to a lemma of stationary phase for Bott-Morse functions: Let \(W_1, \cdots, W_N\) be the connected components of \(C_{\varphi}\). Note that the value of \(\varphi\) is a constant on \(W_r\), which we will denote by \(\gamma_r\). Similarly the signature of \(d^2 \varphi\) is a constant on \(W_r\). More over, a density \(\mu\) on \(X\) determines a density \(\mu_r\) on each \(W_r\). The
lemma of stationary phase for Bott-Morse functions says that for a density $\mu$ of compact support on $X$,

$$\int e^{i\frac{\omega}{\hbar}} \mu = \sum_r (2\pi\hbar)^{d_r/2} e^{i\frac{\omega}{\hbar}} \int_{W_r} \nu + O(\hbar),$$

where $d_r$ is the codimension of $W_r$.

Return to the composition. Let $C_{\varphi}$ be the critical set of $\varphi$ above and let $C_{r}, r = 1, \cdots , N$ be its connected components. Then the map $p_{\varphi} : C_{\varphi} \to F$ maps each $C_{r}$ onto a component $W_{r}$ of $F = \Lambda_1 \cap \Lambda_2$, and $\varphi|_{C_{r}} = \psi_{r}^{\#}$.

The $L^2$ Pair of two Lagrangian distributions.

Now let $f_{i}, i = 1, 2$ be elements in $I_{0}^{m_{i}}(X, \Lambda_{i})$. Then $f_{1}$ can be think of as the Schwartz kernel of the Fourier integral operator

$$F_{1} : C^{\infty}(pt) \to C^{\infty}(X), \quad c \mapsto cf_{1}$$

and $\tilde{f}_{2}$ can be think of as the kernel of $F_{2}^{*} : C_{0}^{\infty}(X) \to C^{\infty}(pt)$, where $F_{2}$ is defined as $F_{1}$. It follows that the Schwartz kernel of the operator $F_{2}^{*}F_{1}$, as an operator from $C^{\infty}(pt)$ to $C^{\infty}(pt)$, is just the $L_{2}$ norm $\langle f_{1}, f_{2} \rangle$.

Using the local data above, we write

$$f_{i}(x, \hbar) = \hbar^{m_{i} - \frac{d}{2}} \int a_{i}(x, s_{i}, \hbar) e^{\frac{i}{\hbar}\varphi_{i}} ds_{i}.$$  

So with $m = m_{1} + m_{2}$ and $d = d_{1} + d_{2}$, we have

$$\langle f_{1}, f_{2} \rangle = \hbar^{m - \frac{d}{2}} \int a(x, s_{1}, s_{2}, \hbar) e^{\frac{i}{\hbar}\varphi(x, s_{1}, s_{2})} dx ds_{1} ds_{2},$$

where $a(x, s_{1}, s_{2}, \hbar) = a_{1}(x, s_{1}, \hbar) \tilde{a}_{2}(x, s_{2}, \hbar)$. Since $\varphi$ is Bott-Morse, the right hand side integral can be evaluated via the lemma of stationary phase. Since the value of $\varphi$ on the $r$th connected component of the critical set is $\psi_{r}^{\#}$, we end with

**Theorem 7.1.** We have

$$\langle f_{1}, f_{2} \rangle \sim \hbar^{m - \frac{d}{2}} \sum_{r=1}^{N} \hbar^{\frac{d_{r}}{2}} e^{\frac{i}{\hbar}\psi_{r}^{\#}} \sum a_{r,j} \hbar^{j}.$$  

As a special case, we let $Y$ be a codimension $k$ submanifold of $X$. Let $\Lambda = \Lambda_{1}$ and let $N^{*}Y = \Lambda_{2}$. Let $\mu$ be a smooth density on $Y$. Then for each $f \in I^{m}(X, \Lambda)$,

$$\int_{Y} f d\mu = \langle f, \delta_{\mu} \rangle.$$  

Moreover, one can show that modulo $O(\hbar^{\infty})$, $\delta_{\mu}$ is an element in $I^{-\frac{k}{2}}(X, N^{*}Y)$. So if $\Lambda$ and $N^{*}Y$ intersects cleanly, we get an asymptotic expansion

$$\int_{Y} f d\mu \sim \hbar^{m - \frac{d}{2}} \sum_{r=1}^{N} \hbar^{\frac{d_{r}}{2}} e^{\frac{i}{\hbar}\psi_{r}^{\#}} \sum a_{r,j} \hbar^{j}.$$
The trace of compactly supported FIOs.

Now let $X$ be an $n$ dimensional manifold, $M = T^*X$ and $\Gamma \subset M \times M$ a canonical relation. Let $\Delta_M$ be the diagonal. We will assume that $\Gamma$ and $\Delta_M$ intersect transversally. Note that this implies in particular that $\Gamma \cap \Delta_M$ is a discrete set. Let $F \in \mathcal{F}_0^k(\Gamma)$ be a semi-classical Fourier integral operator.

Let’s fix a non-vanishing density $dx$ on $X$. Then the Schwartz kernel of $F$ is of the form
\[ \mu = u(x, y, \hbar)dx^{\frac{1}{2}}dy^{\frac{1}{2}} \in I_{0}^{k-\frac{n}{2}}(X \times X, \Lambda). \]
Since $u$ is compactly supported, $F$ is a trace class operator on $L^2(X)$, and as we have mentioned in lecture 13, the trace of $F$ is given by the integral
\[ \text{tr}(F) = \int_X u(x, x, \hbar)dx. \]
One can think of this as the integral of $u$ over the submanifold $Y = \Delta_X$ of $X \times X$.

Applying the formula above, we get
\[ \text{tr}(F) = \hbar^k \sum_{p \in \Gamma \cap \Delta_M} e^{\frac{i}{\hbar} \tau_{p}^\#} \sum a_{r,j} \hbar^j. \]

8. The Gutzwiller trace formula

Flow out by Hamiltonian flow.

Let $p(x, \xi)$ be a real-valued smooth function on $T^*X$. Let $\Xi_p$ be the Hamiltonian flow of $p$. Define
\[ \Lambda = \{(x, \xi, y, \eta, t, \tau) \mid (y, \eta) = \exp(t\Xi_p), \tau = p(x, \xi) \} \subset T^*(X \times X \times \mathbb{R}). \]

Lemma 8.1. $\Lambda$ is a Lagrangian submanifold in $T^*X \times T^*X^- \times T^*\mathbb{R}$.

Proof. We have seen that for each $t$,
\[ \exp(t\Xi_p) : T^*X \to T^*X \]
is a symplectomorphism. So its graph $\Lambda_t$ is a Lagrangian submanifold in $T^*X \times T^*X^-$, and thus an isotropic subspace of $T^*X \times T^*X^- \times T^*\mathbb{R}$. At any point $(x, \xi, y, \eta, t, \tau) \in \Lambda$, the tangent space is spanned by the tangent vectors $\Lambda_t$ at $(x, \xi, y, \eta)$ together with the vector
\[ v(t) = \Xi_p(y, \eta) + \frac{\partial}{\partial t}. \]
Since
\[ \iota_{v(t)}(-\omega + dt \wedge d\tau) = -dp(y, \eta) + d\tau = 0, \]
we conclude that $\Lambda$ is an isotropic, and thus by dimension counting a Lagrangian submanifold of $T^*X \times T^*X^- \times T^*\mathbb{R}$. \qed
Let $\alpha$ be the canonical one form on $T^*X$. Then

$$\tilde{\alpha} = pr_1^*\alpha - pr_2^*\alpha + \tau dt$$

is the canonical one form on $T^*X \times T^*X^- \times T^*\mathbb{R}$.

Let $\iota_A : T^*X \times \mathbb{R} \to T^*X \times T^*X^- \times T^*\mathbb{R}$ be the map

$$(x, \xi, t) \mapsto (x, \xi, \exp(t\Xi_p)(x, \xi), t, H(x, \xi)).$$

Then $\iota_A$ is injective with image $\Lambda$. In other words, one could identify $\Lambda$ with $T^*X \times \mathbb{R}$ via the map $\iota_A$.

**Lemma 8.2.** $\iota_A^* \tilde{\alpha} = \alpha - \exp(t\Xi_p)^*\alpha - (\exp t\Xi_p)^*\iota(\Xi_p)\alpha dt + p dt$.

**Proof.** Fixing $t = t_0$, the restriction of $\iota_A^* \tilde{\alpha}$ to $M \times \{t_0\}$ is

$$\iota_A^* pr_1^*\alpha - \iota_A^* pr_2^*\alpha = \alpha - (\exp t\Xi_p)^*\alpha,$$

which gives the first two terms. It remains to check the formula on the flow-out vector $v(t)$ above,

$$\iota(v(t))\iota_A \tilde{\alpha} = -\iota_A^* pr_2^* \iota(\Xi_p)\alpha + p = -(\exp t\Xi_p)^* \iota(\Xi_p)\alpha + p,$$

which gives the last two terms. \(\square\)

Define $\phi \in C^\infty(T^*X \times \mathbb{R})$ be the function

$$\phi(x, \xi, t) = -\int_0^t (\exp s\Xi_p)^* \iota(\Xi_p)\alpha ds + tp(x, \xi).$$

We will prove that $\Lambda$ is an exact Lagrangian submanifold of $T^*X \times T^*X^- \times T^*\mathbb{R}$ with phase function $\phi$.

**Theorem 8.3.** $\iota_A^* \tilde{\alpha} = d\phi$.

**Proof.** We have

$$\iota_A^* \tilde{\alpha} = \alpha - \exp(t\Xi_p)^*\alpha - (\exp t\Xi_p)^*\iota(\Xi_p)\alpha dt + p dt$$

$$= -\int_0^t \frac{d}{ds} (\exp s\Xi_p)^* \alpha ds - (\exp t\Xi_p)^* \iota(\Xi_p)\alpha dt + p dt.$$
But
\[ \int_0^t \frac{d}{ds} (\exp s \Xi)^* \alpha = \int_0^t (\exp s \Xi_p)^* L_{\Xi_p} \alpha ds \]
\[ = \int_0^t (\exp s \Xi_p)^* d_M t(\Xi_p) \alpha ds + \int_0^t (\exp s \Xi_p)^* t(\Xi_p) d_M \alpha ds \]
\[ = \int_0^t (\exp s \Xi_p)^* d_M t(\Xi_p) \alpha ds + \int_0^t (\exp s \Xi_p)^* (-dp) ds \]
\[ = d_{M \times \mathbb{R}} \int_0^t (\exp s \Xi_p)^* dt(\Xi_p) \alpha ds - \frac{d}{dt} \int_0^t (\exp s \Xi_p)^* t(\Xi_p) \alpha ds dt - tdp \]
\[ = d_{M \times \mathbb{R}} \int_0^t (\exp s \Xi_p)^* dt(\Xi_p) \alpha ds - (\exp t \Xi_p)^* t(\Xi_p) \alpha dt - tdp, \]
so \( t^* \tilde{\alpha} = d\phi. \]

\[ \triangleq \text{The Gutzwiller trace formula.} \]

Let \( X \) be a smooth manifold and \( P \) a self-adjoint semi-classical pseudo-differential operator with leading symbol \( p(x, \xi) \). We will assume that the function \( p \) is proper. Let \( f \in C^\infty_0(\mathbb{R}) \) be a smooth function with compact support. Let
\[ U(t) = e^{\frac{itP}{\hbar}} f(P). \]
We would like to find an asymptotic expansion of the trace of \( U(t) \) in the \( \hbar \to 0 \) limit.

Let \( u(x, y, t, \hbar) \) be the Schwartz kernel of \( U(t) \).

**Theorem 8.4.** \( u \in I^{-n}(X \times X \times \mathbb{R}, \Lambda) \).


Let \( \gamma \) be a periodic trajectory of \( \Xi_p \) with least period \( T \), and let \( q = \gamma(0) = \gamma(T) \). Then \( q \) is a fixed point of the map \( \exp T\Xi_p : M \to M \). The differential of this map at \( q \),
\[ d \exp T\Xi_p : T_q M \to T_q M, \]
maps the subspace
\[ W_q = \{ \eta \mid dp_q(\eta) = 0 \} \subset T_q M \]
into itself and maps \( \Xi_p(q) \in W_q \) into itself. So we get a map, called the reduced Poincare map,
\[ P_\gamma : W_q / \{ \Xi_p \} \to W_q / \{ \Xi_p \}. \]

**Definition 8.5.** We say that the trajectory \( \gamma \) is non-degenerate if \( \det(I - P_\gamma) \neq 0 \).
We will denote
\begin{equation}
S_\gamma = \int_0^T \gamma^* \alpha.
\end{equation}

Note: on \( p(x, \xi) = 0 \), this is exactly the phase function \( \phi \) (modulo a negative sign) above evaluated at \( (q, T) \).

Now suppose there are only finitely many periodic trajectories, \( \gamma_1, \ldots, \gamma_N \) of \( \Xi_p \) lying on the energy surface \( p(x, \xi) = 0 \), and suppose each trajectory \( \gamma_j \) is non-degenerate, with period \( T_j \), and suppose \( T_j \in (a, b) \).

**Theorem 8.6** (Gutzwiller trace formula). For each \( \psi \in C^\infty_0((a, b)) \), the trace of the operator
\begin{equation}
\text{Tr} \left( \int_{\mathbb{R}} \hat{\psi}(t) e^{\frac{itP}{\hbar}} f(P) dt \right) \sim \hbar^\frac{n}{2} \sum_{j=1}^{N} e^{\frac{iS_\gamma_j}{\hbar}} \sum_{i=0}^{\infty} a_{j,i} \hbar^i.
\end{equation}

**Proof.** We can write the trace as
\[
\int \hat{\psi}(t) u(x, x, t, \hbar) dt,
\]
where \( \hat{\psi}(t) u(x, y, t, \hbar) \in I^{-\frac{n}{2}} (X \times X \times \mathbb{R}, \Lambda) \). In other words, the trace is an integral of \( \hat{\psi}(t) u(x, y, t, \hbar) \) over the submanifold \( Y = \Delta_X \times \mathbb{R} \) of \( X \times X \times \mathbb{R} \). The conormal bundle of \( \Gamma \) of \( Y \) in \( M \times M^- \times T^* \mathbb{R} \) is the subset of points
\[
\{(x, \xi, y, \eta, t, \tau) \mid x = y, \xi = \eta, \tau = 0 \}.
\]
This intersects \( \Lambda \) in the set of points
\[
\{(x, \xi, y, \eta, t, \tau) \mid (\exp t \Xi_p)(x, \xi) = \xi, p = \tau = 0 \}.
\]
For \( a < t < b \) this is exactly the union of points on the periodic orbits in this interval. The non-degeneracy condition implies that \( \Gamma \) intersects \( \Lambda \) cleanly in the union there \( \gamma \)'s. As we have seen, for \( p(x, \xi) = 0 \), \( S_\gamma \) is the restriction of the phase function \( \psi \) to \( \gamma \). Thus the Gutzwiller formula is a special case of the general trace formula above. \( \square \)