Notes for Functional Analysis

Wang Zuoqin (typed by Xiyu Zhai)

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1 Lecture 09

1.1 Equicontinuity

First let's recall the conception of "equicontinuity" for family of functions that we learned in classical analysis: A family of continuous functions defined on a region Ω,

$$\Lambda = \{f_\alpha\} \subset C(\Omega),$$

is called an equicontinuous family if $\forall \epsilon > 0, \exists \delta > 0$ such that for any $x_1, x_2 \in \Omega$ with $|x_1 - x_2| < \delta$, and for any $f_\alpha \in \Lambda$, we have

$$|f_\alpha(x_1) - f_\alpha(x_2)| < \epsilon.$$

This conception of equicontinuity can be easily generalized to maps between topological vector spaces. For simplicity we only consider linear maps between topological vector space , in which case the continuity (and in fact the *uniform continuity*) at an arbitrary point is reduced to the continuity at 0.

Definition 1.1. Let X, Y be topological vector spaces, and Λ be a family of continuous linear operators. We say Λ is an *equicontinuous* family if for any neighborhood V of 0 in Y, there is a neighborhood U of 0 in X, such that

$$L(U) \subset V$$

for all $L \in \Lambda$.

Remark 1.2. Equivalently, this means if $x_1, x_2 \in X$ and $x_1 - x_2 \in U$, then

$$L(x_1) - L(x_2) \in V$$

for all $L \in \Lambda$.

Remark 1.3. If $\Lambda = \{L\}$ contains only one continuous linear operator, then it is always equicontinuous.

If Λ is an equicontinuous family, then each $L \in \Lambda$ is continuous and thus bounded. In fact, this boundedness is uniform:

Proposition 1.4. *Let X,Y be topological vector spaces, and Λ be an equicontinuous family of linear operators from X to Y. Then for each bounded set E in X, there exists a bounded set F in Y, such that $L(E) \subset F$, for all $L \in \Lambda$.*

Proof. Let $F = \bigcup_{L\in\Lambda} L(E)$. In what follows we will show that F is bounded in Y.

By equicontinuity of Λ, for any neighborhood V of 0 in Y, one can find a neighborhood U of 0 in X such that $L(U) \subset V$, for all $L \in \Lambda$. On the other hand, since E is bounded in X, so there exists $s > 0$, such that $\forall t > s$,

$$E \subset tU.$$

It follows that for any $L \in \Lambda$,

$$L(E) \subset L(tU) = tL(U) \subset tV,$$

and therefore $F = \bigcup_{L\in\Lambda} L(E) \subset tV$ for all $t > s$. In other words, F is bounded in Y. $\square$

1.2 The Banach-Steinhaus Theorem

In (linear) functional analysis there are three theorems, known as "THE BIG THREE", sitting at the foundation of the subject. They are all applications of the Baire's category theorem.

Recall: A set $A \subset X$ is

- of the *first category* if it is the union of countably many nowhere dense subsets.
- of the *second category* if it is not of the first category.

and the famous Baire's category theorem we proved in lecture 3 claims

$$\boxed{\text{Any complete metric space is of the second category.}}$$

Since the completeness property is so important, it is worth to give topological vector space with this property a special name.

Definition 1.5. A topological vector space X is called an *F-space* if the topology is induced by a complete translation-invariant metric.

Remark 1.6. In lecture 7 we actually proved "Fréchet space=locally convex F-space". So

$$\boxed{\text{Metrizable TVS} \supset \text{F-space} \supset \text{Fréchet} \supset \text{Banach} \supset \text{Hilbert.}}$$

Now we are ready to state the first of the BIG THREE:

Theorem 1.7. *(The Banach-Steinhaus theorem) Let X be an F-space, Y be any topological vector space and Λ be a family of continuous linear operators from X to Y. Suppose the family Λ is "pointwise bounded", i.e. for any $x \in X$, the set*

$$\Lambda_x = \{L(x) | L \in \Lambda\}$$

is bounded in Y. Then the family Λ is equicontinuous.

Remark 1.8. According to proposition 1.4, and equicontinuous family is uniformly bounded. So the Banach-Steinhaus theorem tells us any "pointwise bounded family" of continuous linear operators is in fact "uniformly bounded".

Remark 1.9. The converse of theorem 1.7 is a consequence of proposition 1.4, because the single point set $\{x\}$ is a bounded set in X. (If Λ is an equicontinuous family, then there exists a bounded set in Y that contains all $L(x)$'s, and thus contains the set Λ_x. So Λ_x is bounded.)

We need a small lemma before we prove the theorem. (Thank Qian Jian for pointing out a mistake in an earlier version.)

Lemma 1.10. *(1) For any set S in a topological vector space X and any open neighborhood U of 0 in X, one has $\overline{S} \subset S + U$.*

(2) For any neighborhood V of 0 in a topological vector space, there is a balanced neighborhood W of 0 such that $\overline{W} \subset V$.

Proof. (1) If $x \in \overline{S}$, then by definition of the closure in topological space, any neighborhood of x contains a point in S. In particular for the open neighborhood $x - U$ of x, one has

$$(x - U) \cap S \neq \emptyset,$$

which implies $x \in S + U$, and the fact follows.

(2) We only need to take a balanced neighborhood W of 0 such that $W + W \subset V$. By part (1), $\overline{W} \subset W + W$. So $\overline{W} \subset V$. □

Now we are ready to prove the main theorem.

Proof of the Banach-Steinhaus theorem. For any neighborhood V of 0 in Y, we want to find a neighborhood U of 0 such that $L(U) \subset V$ for all $L \in \Lambda$.

We first apply lemma 1.10 to choose a closed balanced neighborhood (the closure of a balanced neighborhood) W of 0 in Y such that

$$W + W \subset V.$$

By pointwise boundedness, for any $x \in X$, there exists $s(x) > 0$, s.t. for all $t > s(x)$,

$$\Lambda_x \subset tW.$$

Let $n(x) = [s(x)] + 1 \in \mathbb{N}$. Then for all $L \in \Lambda$,

$$L(x) \in n(x)W.$$

Now for each $n \in \mathbb{N}$, we define

$$X_n = \{x \in X \mid L(x) \in nW, \ \forall L \in \Lambda\}.$$

Then

$$X = \bigcup_{n=1}^{\infty} X_n.$$

Observe that each X_n is closed in X, since

$$X_n = \bigcap_{L \in \Lambda} L^{-1}(nW)$$

and each $L^{-1}(nW)$ is closed due to the continuity of L and the closedness of W.

Since X is an F-space, it is complete. By Baire's category theorem, it cannot be written as a countable union of nowhere dense subsets. So there exists n such that the interior of X_n is not empty. Let

$$U_0 = \frac{1}{n}\mathrm{Int}(X_n),$$

then U_0 is open in X, and for any $L \in \Lambda$,

$$L(U_0) = \frac{1}{n}L(\mathrm{Int}(X_n)) \subset \frac{1}{n}L(X_n) \subset W.$$

So if we let

$$U = U_0 + (-U_0),$$

then U is open in X and contains 0, and

$$L(U) \subset L(U_0) + L(-U_0) \subset W + (-W) = W + W \subset V$$

for all $L \in \Lambda$. This completes the proof. □

By using exactly the same argument, one can prove the following more general result, which is also known as the Banach-Steinhaus theorem:

Theorem 1.11. *(The Banach-Steinhaus theroem) Let X, Y be topological vector spaces, and Λ a family of continuous linear operators from X to Y. If the set*

$$B = \{x \in X : \Lambda_x \text{ is bounded in } Y\}$$

is of the second category in X, then Λ is an equicontinuous family and, as a consequence of proposition 1.4, $B = X$.

A very useful consequence of the Banach-Steinhaus theorem is

Corollary 1.12. *Let X be an F-space, Y be a topological vector space. Suppose $\{L_n\}$ is a sequence of continuous linear operators from X to Y, and suppose for each $x \in X$, the limit*

$$L(x) := \lim_{n\to\infty} L_n(x)$$

exists. Then L is a continuous linear operator from X to Y.

Proof. Since each L_n is linear, the limit L is linear. So it is enough to prove the continuity of L at 0. For any neighborhood V of 0 in Y, we take a smaller closed balanced neighborhood W of 0 in Y such that $W \subset V$. The existence of such W follows from lemma 1.10.

Since any convergent sequence in a topological vector space is bounded, the Banach-Steinhaus theorem implies that the family $\Lambda = \{L_n\}$ is equicontinuous. So for the neighborhood W of 0 in Y, we can find a neighborhood U of 0 in X such that $L_n(U) \subset W$ for all n. Since W is closed, by passing to the limit we get

$$L(U) \subset W \subset V.$$

So L is continuous. □

1.3 The Banach-Steinhaus theorem for normed topological vector spaces

In most applications the topological vector spaces have norm structures. We now introduce the version of the Banach-Steinhaus theorem in this special situation.

Let X, Y be normed vector spaces. Note that the norm structure gives us convenience in studying bounded sets: A subset $E \subset X$ is bounded iff there exists C such that

$$\|x\| < C, \ \forall x \in E.$$

As a consequence, we get

Proposition 1.13. *Let $L : X \to Y$ be a linear operator between normed vector spaces. Then the following are equivalent*

1. *L is bounded*

2. *for any* $r > 0$, *there exists* $s > 0$ *such that* $L(B_X(0,r)) \subset B_Y(0,s)$,

3. *there exists* $r > 0$ *such that* $L(B_X(0,1)) \subset B_Y(0,r)$.

Now we let

$$\mathfrak{L}(X,Y) = \{L : X \to Y \mid L \text{ is a continuous linear operator}\}.$$

Obviously $\mathfrak{L}(X,Y)$ is a vector space.

For any $L \in \mathfrak{L}(X,Y)$, we define

$$||L|| = \sup_{||x||_X=1} ||Lx||_Y = \sup_{x\neq 0} \frac{\|Lx\|_Y}{\|x\|_X}.$$

Proposition 1.14. $||\cdot||$ *is a norm on* $\mathfrak{L}(X,Y)$.

Proof. Quiz. □

With the help of the operator norm defined above, one can state the Banach-Steinhaus theorem as follows, which is usually referred to as the uniformly boundedness principle:

Theorem 1.15. (The Uniform Boundedness Principle) *Let* X *be a Banach space,* Y *be a normed vector space, and* Λ *be a family of continuous linear operators from* X *to* Y*. Suppose the family* Λ *is "pointwise bounded", i.e.,*

$$\forall x \in X,\ \sup_{L\in\Lambda}\{\|Lx\|\} < \infty.$$

then there exists a constant C *such that*

$$\|L\| < C,\ \forall L \in \Lambda.$$

Proof. By the Banach-Steinhaus theorem, Λ is an equicontinuous family. So for the bounded set

$$E = \{x \in X : \|x\|_X = 1\}$$

in X one can find $C > 0$ such that

$$L(E) \subset B_Y(0,C),\ \forall L \in \Lambda.$$

In other words, $\|L\| < C$, for all $L \in \Lambda$. □

Equivalently, one can state the uniform boundedness principle as

Theorem 1.16 (The Resonance Theorem)**.** *Let* X *be a Banach space,* Y *be a normed vector space, and* Λ *be a family of continuous linear operators from* X *to* Y*. If* $\sup_{L\in\Lambda}\|L\| = \infty$*, then* $\exists x \in X$ *such that*

$$\sup_{L\in\Lambda} ||L(x)|| = \infty.$$