# Notes for Functional Analysis

Wang Zuoqin (typed by Xiyu Zhai)

October 16, 2015

# 1 Lecture 11

## 1.1 The closed graph theorem

**Definition 1.1.** Let $f : X \to Y$ be any map between topological spaces. We define its *graph* to be the set

$$\Gamma_f = \{(x, f(x)) : x \in X\} \subset X \times Y.$$

We say $f$ *has closed graph* if $\Gamma_f$ is closed in $X \times Y$ (equipped with the product topology).

The closedness of the graph $\Gamma_f$ is closely related to the continuity of $f$:

**Proposition 1.2.** *Let $X$ be a topological space, $Y$ be a Hausdorff space. Suppose $f : X \to Y$ is continuous. Then $\Gamma_f$ is closed.*

*Proof.* Let $\Omega = X \times Y \setminus \Gamma_f$ be the complement of the set $\Gamma_f$ in $X \times Y$. We want to prove that $\Omega$ is open.

Fix any $(x_0, y_0) \in \Omega$, then by definition, $(x_0, y_0) \notin \Gamma_f$, i.e.

$$y_0 \neq f(x_0).$$

Since $Y$ is Hausdorff, one can find a neighborhood $V$ of $y_0$ and a neighborhood $W$ of $f(x_0)$ such that

$$V \cap W = \emptyset.$$

By continuity of $f$, one can find a neighborhood $U$ of $X$ such that $U \subset f^{-1}(W)$, i.e.,

$$f(U) \subset W.$$

So $f(U) \cap V = \emptyset$, i.e.

$$(U \times V) \cap \Gamma_f = \emptyset.$$

It follows that $U \times V \subset \Omega$ is a neighborhood of $(x_0, y_0)$ in $X \times Y$. So $\Omega$ is open. □

*Remark* 1.3. The Hausdorff property of the target space $Y$ is necessary. In fact, if we consider

$$f = \mathrm{Id} : X \to X,$$

where $X$ is any topological space, then $f$ is obviously continuous. The graph $\Gamma_f$ of $f$ is the diagonal

$$\triangle = \{(x, x)\} \subset X \times X.$$

It is easy to see that $\Gamma_f$ is closed iff $X$ is Hausdorff.

*Remark* 1.4. One can prove: If $K$ is a compact set in a metric space, and $f : K \to K$ has closed graph, then $f$ is continuous. (A widely used compact-Hausdorff argument)

As an application of the open mapping theorem, we can prove

**Theorem 1.5** (The closed graph theorem)**.** *Let $X, Y$ be F-spaces, and $L : X \to Y$ be a linear operator. Suppose $\Gamma_L$ is closed in $X \times Y$, then $L$ is continuous.*

*Proof.* Let $d_X, d_Y$ be compatible complete translation-invariant metrics on $X$ and $Y$ respectively. Then (exercise)

$$d((x_1, y_1), (x_2, y_2)) := d_X(x_1, x_2) + d_Y(y_1, y_2)$$

is a compatible complete translation-invariant metric on $X \times Y$, making $X \times Y$ into an F-space. (Note: need to check that the metric topology induced by $d$ is the same thing as the product topology.)

Now consider the graph

$$\Gamma_L = \{(x, Lx) | x \in X\}.$$

Since $L$ is linear, it is a vector subspace of $X \times Y$. As a closed subset of the complete metric space $X \times Y$ [Recall PSet 2-1 1(1)], the induced metric on $\Gamma_f$ is complete, making $\Gamma_f$ an F-space. (Note again that the induced topology of $\Gamma_f$ as a subset of $X \times Y$ is compatible with the induced metric.)

Now consider the projection maps

$$\pi_1 : \Gamma_L \to X, (x, Lx) \mapsto x$$

and

$$\pi_Y : X \times Y \to Y, (x, y) \mapsto y.$$

Obviously $\pi_1$ is linear and invertible. It is also continuous since it can be written as the composition of two continuous maps,

$$\pi_1 = \pi_X \circ \iota,$$

where $\iota : \Gamma_L \to X \times Y$ is the inclusion map, which is continuous since the topology on $\Gamma_L$ is the subspace topology inherited from $X \times Y$, and $\pi_X : X \times Y \to X$ is the projection map, which is continuous since we are using product topology on $X \times Y$. By the Banach's inverse mapping theorem, $\pi^{-1}$ is continuous.

Finally, by definition, we have

$$L = \pi_Y \circ \pi_1^{-1}.$$

Since both $\pi_Y$ and $\pi_1^{-1}$ are continuous, we conclude that $L$ is continuous. □

*Remark* 1.6. By definition,

$$\Gamma_L \text{ is closed} \iff \text{if } x_n \in X, x_n \to x, Lx_n \to y, \text{ then } Lx = y.$$

while

$$L \text{ is continuous} \iff \text{if } x_n \in X, x_n \to x, \text{ then } Lx_n \to y \text{ and } Lx = y.$$

So the closed graph theorem is a powerful tool in proving continuity.

*Remark* 1.7. One can use the closed graph theorem to prove the Banach inverse mapping theorem. The argument is as follows:

$$\begin{aligned} & X, Y \text{ are F-spaces, and } L : X \to Y \text{ is bijective,linear, continuous} \\ \Longrightarrow & \Gamma_L \text{ is closed in } X \times Y \\ \Longrightarrow & \Gamma_{L^{-1}} \text{ is closed in } Y \times X \\ \Longrightarrow & L^{-1} \text{ is continuous.} \end{aligned}$$

One can also use the closed graph theorem to prove the following weaker version of the open mapping theorem:

Let $X, Y$ be F-spaces, and $L : X \to Y$ be surjective, linear, continuous. Then $L$ is open.

So there three statements are equivalent.

## 2 Closed graph operator

For simplicity, in what follows we assume that $X, Y$ are Banach spaces. Let $L$ be a linear map "from" $X$ to $Y$. In many applications, the domain of $L$ is NOT the whole space $X$. We will denote the domain of $L$ by $Dom(L)$, which is always assumed to be a vector subspace of $X$. So like before we have a linear operator

$$L : Dom(L) \subset X \to Y.$$

We can define the graph of $L$ to be

$$\Gamma_L = \{(x, Lx) | x \in Dom(L)\} \subset X \times Y.$$

**Definition 2.1.** We say $L$ is a *closed graph operator* if $\Gamma_L$ is closed in $X \times Y$.

Note that by definition,

$$L : Dom(L) \subset X \to Y \text{ is a closed graph operator}$$
$$\Longleftrightarrow \text{if } x_n \to x, Lx_n \to Y, \text{ then } x \in Dom(L), \text{ and } y = Lx.$$

This is slightly different from the closedness of the graph of $L$ whose domain is $X$.

*Example* 2.1. If $L : X \to Y$ is injective, continuous and linear (but not necessary bijective), then one can define an inverse $L^{-1}$ with domain $Dom(L^{-1}) = Im(L)$, which is again a linear map. Moreover, in this case

$$L^{-1} : Im(L) \subset Y \to X$$

is also a closed graph operator.

*Example* 2.2. Let $X = C([0,1])$ equipped with the norm

$$\|f\| = \max_{x \in [0,1]} |f|.$$

Let

$$L = \frac{d}{dx}$$

be the differential operator. Then the domain of $L$ is

$$Dom(L) = \left\{ f \in C([0,1]) \,\middle|\, \frac{df}{dx} \in C([0,1]) \right\} = C^1([0,1]) \subset C([0,1]).$$

Then $\frac{d}{dx}$ is a closed graph operator. In fact, if $f_n \to f$ and $f_n' \to g$ under the norm given above, then the convergences are uniform. So from what we learned in mathematical analysis we conclude that $f \in C^1([0,1])$, and $\frac{df}{dx} = g$.

It is very important to notice that $L$ is NOT a continuous operator, since it is not bounded. In fact, if we take $f(x) = x^n$, then

$$\|(x^n)'\| = n\|x^{n-1}\| = n = n\|x^n\| \Longrightarrow \|L\| \geq n.$$

So the operator norm of $L$ is infinity.

In general, the domain $Dom(L)$ of an operator might not be a closed subset in $X$. This can be easily seen from the the example above, since $C^1([0,1])$ is not closed in $C([0,1])$. However, under suitable assumptions one can extend $L$ to the closure of $Dom(L)$ without changing the operator norm, and thus in those cases we can assume that the domain is closed.

**Proposition 2.2.** *Let $X$ be a normed vector space, $Y$ be a Banach space, and*

$$L : Dom(L) \subset X \to Y$$

*be a continuous linear map. Then $L$ can be extended to a continuous linear operator*

$$\overline{L} : \overline{Dom(L)} \subset X \to Y$$

*such that*

*(1)* $\overline{L}|_{Dom(L)} = L$;

*(2)* $\|\overline{L}\| = \|L\|$.

*Proof.* For any $x \in \overline{Dom(L)}$, one can find a sequence $x_n \to x$ with $x_n \in Dom(L)$. We want to define the value of $\overline{L}(x)$.

Since $L : Dom(L) \to Y$ is continuous, it is bounded. So

$$\|Lx\| \leq \|L\| \cdot \|x\|.$$

It follows

$$\|Lx_n - Lx_m\| \leq \|L\| \cdot \|x_n - x_m\|.$$

Since $\{x_n\}$ is a convergent sequence and thus a Cauchy sequence, the sequence $\{Lx_n\}$ is also a Cauchy sequence. By the completeness of $Y$, there exists $y \in Y$ such that $Lx_n \to y$. We define

$$\overline{L}x = y.$$

We need to check that $\overline{L}$ is well-defined. In fact, if there is another sequence $\tilde{x}_n \to x$ with $\tilde{x}_n \in Dom(L)$, then by the above argument, there is an $\tilde{y} \in Y$ sucht that $L\tilde{x}_n \to \tilde{y}$. However, if we consider the mixed sequence

$$x_1, \tilde{x}_1, x_2, \tilde{x}_2, x_3, \tilde{x}_3, \cdots,$$

then it is also a sequence in $Dom(L)$ that converge to $x$. So the same arguments implies that the sequence

$$Lx_1, L\tilde{x}_1, Lx_2, L\tilde{x}_2, Lx_3, L\tilde{x}_3, \cdots$$

converges to some $\bar{y} \in Y$. Obviously one must have $y = \bar{y} = \tilde{y}$.

The linearity of $L$ can be proved in the same way: if $x_n^{(1)} \to x^{(1)}$ and $x_n^{(2)} \to x^{(2)}$, then $\alpha x_n^{(1)} + \beta x_n^{(2)}$ is a sequence in $Dom(L)$ that converges to $\alpha x^{(1)} + \beta x^{(2)}$. Then the continuity of vector addition implies that $L(\alpha x_n^{(1)} + \beta x_n^{(2)})$ converges to $\alpha \overline{L}(x^{(1)}) + \beta \overline{L}(x^{(2)})$.

Using the fact that if $x \in Dom(L)$, then the constant sequence $x, x, x, \cdots$ converges to $x$, we see $\overline{L}(x) = L(x)$. This proves (1).

From (1) we get $\|\overline{L}\| \geq \|L\|$. The other half of (2) follows from the continuity of norm:

$$\|\bar{L}x\| = \lim_{n\to\infty} \|Lx_n\| \leq \lim_{n\to\infty} \|L\| \cdot \|x_n\| = \|L\| \cdot \|x\|.$$

□

**Definition 2.3.** We say an operator $L : Dom(L) \subset X \to Y$ *admits a closure* if $\overline{\Gamma}_L \subset X \times Y$ is the graph of an operator $\bar{L} : \overline{Dom(L)} \subset X \to Y$.

For example, the above theorem implies that if $L : Dom(L) \subset X \to Y$ is continuous linear, then $L$ admits a closure.

Finally as an application of the closed graph theorem, we prove

**Theorem 2.4.** *(Hörmander) Let $X, Y, Z$ be Banach spaces, $L_1 : X \to Y$ and $L_2 : X \to Z$ be linear operators with $Dom(L_1) \subset Dom(L_2)$. Assume that $L_1$ is a closed graph operator and $L_2$ admits a closure, then there exists $C > 0$ such that for all $x \in Dom(L_1)$,*

$$\|L_2 x\|_Z \leq C(\|x\|_X + \|L_1 x\|_Y).$$

*Proof.* By the assumption, $\Gamma_{L_1}$ is a closed vector subspace of $X \times Y$, and thus a Banach space. Now we define a linear map $L_3 : \Gamma_{L_1} \to Z$ by

$$L_3((x, L_1 x)) := L_2(x).$$

We claim that the graph of $L_3$ is closd. In fact, if $\{x_n\} \in Dom(L_1)$ is a sequence such that

$$(x_n, L_1(x_n)) \to (x, y) \quad \text{and} \quad L_2(x_n) \to z,$$

then the fact that $L_1$ is a closed graph operator implies $y = L_1 x$. Moreover, since $L_2$ admits a closure and $Dom(L_1) \subset Dom(L_2)$, we see

$$z = \lim_{n\to\infty} L_2(x_n) = \overline{L}_2(x) = L_2(x).$$

Thus

$$L_3(x, y) = L_3(x, L_1(x)) = L_2(x) = z.$$

So we can apply the closed graph theorem to conclude that $L_3$ is continuous. In other words, there is a constant $C$ so that for any $x \in Dom(L_1)$,

$$\|L_2 x\|_Z = \|L_3((x, L_1 x))\|_Z \leq C\|(x, L_1 x)\|_{\Gamma_{L_1}} = C(\|x\|_X + \|L_1 x\|_Y).$$

□

*Example* 2.3. Let $X = Y = Z = C([0,1])$ and let $L_1 = \frac{d^2}{dx^2}$, $L_2 = \frac{d}{dx}$. then the theorem of Hormander implies that there exists $C > 0$ such that for any $f \in C^2([0,1])$,

$$\|f'\| \leq C(\|f\| + \|f''\|).$$