Notes for Functional Analysis

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1 Lecture 15

1.1 The weak $*$-topology on X^*

Let X be a topological vector space. We want to topologize the dual space X^* (in a natural way) to make it a (nice) topological vector space.

We have seen that if X is a normed vector space, then there is a natural way to assign a topology on X^* to make it not just a topological vector space, but in fact a Banach space. The norm we assigned to X^* is the operator norm

$$\|L\|_{X^*} = \sup_{\|x\|=1} |Lx|.$$

But this method does not work for more general topological vector spaces.

We will study the norm topology on X^* in more detail later. Today we will use the method of weak topology to construct a topology on X^*. Let X be any topological vector space (where we don't require X^* to separate points). Then each $x \in X$ induces a linear functional ev_x on X^*,

$$ev_x : X^* \to \mathbb{F}, \quad L \mapsto Lx \tag{1}$$

The linearity of ev_x is obvious:

$$ev_x(\alpha L_1 + \beta L_2) = (\alpha L_1 + \beta L_2)(x) = \alpha L_1(x) + \beta L_2(x) = \alpha ev_x(L_1) + \beta ev_x(L_2).$$

Moreover,

Lemma 1.1. *The family $\mathcal{X} = \{ev_x : x \in X\}$ separates "points" in X^*.*

Proof. For any $L_1 \neq L_2 \in X^*$, one can find $x_0 \in X$ such that

$$L_1 x_0 \neq L_2 x_0.$$

It follows that $ev_{x_0}(L_1) \neq ev_{x_0}(L_2)$. □

Definition 1.2. The $\mathcal{X}$-topology of X^* is called the *weak-$*$ topology.*

Remarks 1.3.

- For any topological vector space X, the weak-$*$ topology on X^* makes X^* a locally convex topological vector space. This is a consequence of the theorem we proved in lecture 14.
- A local base for the weak-$*$-topology consists of sets which are finitely intersections of sets of the form
$$V_{x,\epsilon} = \{L \in X^* \mid |Lx| < \epsilon\}.$$
- For a normal vector space X, we now have three topologies on X^*:
$$\text{the weak-}*\text{ topology} \subset \text{the weak topology} \subset \text{the norm topology} \tag{2}$$
(where the weak topology on X^* is the $(X^{*,norm})^*$-topology)
- For infinite dimensional normed vector spaces, the norm topology on X^* (which makes X^* a Banach space) and the weak-$*$ topology on X^* (which makes X^* a locally convex topological vector space which is not locally bounded) are always different.
- Moreover, if X is an infinitely dimensional Banach space, then with respect to the weak-$*$ topology, X^* is NOT even metrizable. See PSet 9-1 problem 4.

We have seen that the unit ball in an infinite dimensional Banach space need not be compact (for example, the space l^2). It turns out that the weak $*$-topology has the following compactness property. (This is one of the main reason that the weak-$*$ topology is very useful in practice.)

Theorem 1.4 (The Banach-Alaoglu Theorem)**.** *Let X be a topological vector space, $V \subset X$ a neighborhood of 0. Then the set (called the* polar *of V)*
$$K = \{L \in X^* \mid |Lx| \le 1, \forall x \in V\} \tag{3}$$
is compact in the weak-$*$-topology.

Proof. Since V is a neighborhood of 0, it is absorbing. So for any $x \in X$, one can find $c = c(x) > 0$ such that
$$x \in c(x)V.$$
By linearity, we see if $L \in K$, then
$$|Lx| \le c(x), \quad \forall x \in X. \tag{4}$$

We will take $c(x) = 1$ for $x \in V$. Then by definition,

$$L \in K \Longleftrightarrow L \in X^* \text{ and satisfies (4).}$$

For each $x \in X$, let

$$D_x = \{\alpha \in \mathbb{F} : |\alpha| \le c(x)\}.$$

Then D_x is compact. Define

$$\mathcal{P} = \prod_{x \in X} D_x,$$

equipped with the product topology. According to the Tychonoff's theorem, $\mathcal{P}$ is compact. Note each element $\{\alpha_x\}$ in $\mathcal{P}$ can be identified with a function f on X by

$$\{\alpha_x \in D_x | x \in X\} \quad \longleftrightarrow \quad f : X \to \mathbb{F}, \quad f(x) = \alpha_x.$$

By this way, we can identify $\mathcal{P}$ with the set of functions

$$\mathcal{P} = \{f : X \to \mathbb{F} \mid |f(x)| \le c(x), \forall x \in X\}.$$

It follows

$$K = X^* \cap \mathcal{P}.$$

So K admits two topologies, one inherited from the product topology on $\mathcal{P}$ and one from the weak-$*$ topology on X^*.

Claim 1 The product topology on $\mathcal{P}$ and the weak $*$-topology on X^* coincide on K.

As we know, a local base for the weak-$*$ topology at $L_0 \in K$ consists of sets of the form

$$W_{x_1,\cdots,x_k;\epsilon} = \{L \in X^* \mid |Lx_i - L_0x_i| < \epsilon, 1 \le i \le n\},$$

while a local base for the product topology at L_0 consists of sets of the form

$$V_{x_1,\cdots,x_k;\epsilon} = \{f \in \mathcal{P} \mid |f(x_i) - L_0(x_i)| < \epsilon, 1 \le i \le n\}.$$

Since $W_{x_1,\cdots,x_k} \cap K = V_{x_1,\cdots,x_k} \cap K$, the two topologies are the same when restricted to K.

Claim 2 K is closed (and thus compact) in $\mathcal{P}$.

Let f_0 be any element in the $\mathcal{P}$-topology closure of K. Since $\mathcal{P}$ is compact, $f_0 \in \mathcal{P}$. We shall prove $f_0 \in X^*$, thus $f_0 \in K$.

- Check f_0 is linear.

  Fix any $x, y \in X$, $\alpha, \beta \in \mathbb{F}$. For any $\epsilon > 0$, consider the $\mathcal{P}$-neighborhood

  $$U = \{f \in \mathcal{P} \mid |f(x) - f_0(x)| < \epsilon, |f(y) - f_0(y)| < \epsilon, |f(\alpha x + \beta y) - f_0(\alpha x + \beta y)| < \epsilon\}$$

of f_0. Since f_0 lies in the closure of K, one can find some $f \in K \cap U$. Then

$$\begin{aligned}|f_0(\alpha x + \beta y) - \alpha f_0(x) - \beta f_0(y)| &= |(f_0 - f)(\alpha x + \beta y) + \alpha(f - f_0)(x) + \beta(f - f_0)(y)| \\ &< (1 + |\alpha| + |\beta|)\epsilon\end{aligned}$$

Since ϵ is arbitrary, we get

$$f_0(\alpha x + \beta y) - \alpha f_0(x) - \beta f_0(y) = 0,$$

so f_0 is linear.

- Check f_0 is continuous.

  Fix any $x \in V$. Use the same $f \in K \cap U$ as above. Then $|f(x) - f_0(x)| < \epsilon$ implies

$$|f_0(x)| \leq 1 + \epsilon.$$

  Since ϵ is arbitrary chosen, we must have

$$|f_0(x)| \leq 1.$$

  Since f_0 is linear,

$$f_0^{-1}((-\epsilon, \epsilon)) \supset \frac{\epsilon}{2} V,$$

  which implies the continuity of f_0.

Finally combine these two claims, we see K is compact with respect to the $\mathcal{P}$-topology, and thus compact with respect to the weak $*$-topology. □

1.2 Some applications

We will introduce a number of applications of the Banach-Alaoglu theorem, not only to the weak-$*$ topology on X^*, but also to the original and weak topology on X.

Corollary 1.5. *Let X be a normed vector space, then the closed unit ball in X^**

$$B = \{L \in X^* : \|L\|_{X^*} \leq 1\} \tag{5}$$

is compact in the weak $*$-topology.

Proof. Take $V = \{x \in X : \|x\|_X < 1\}$, then we can check its polar set $K = B$. In fact, if $L \in B$, i.e. $\|L\|_{X^*} \leq 1$, then

$$|Lx| \leq \|L\|_{X^*} \|x\| \leq 1, \quad \forall x \in V,$$

i.e. $L \in K$. On the other hand, if $L \notin B$, then $\exists \varepsilon > 0$ such that $\|L\|_{X^*} \geq 1 + \varepsilon$. So $\exists x \in X$ with $\|x\| = 1$ such that

$$|Lx| \geq \|L\|_{X^*} - \frac{\varepsilon}{2} > 1,$$

so $L \notin K$. □

The next application is a structural theorem for Banach spaces:

Theorem 1.6. *Let X be any Banach space. Then there exists a compact Hausdorff topological space Ω and a norm-preserving linear isomorphism*

$$\phi : X \to \phi(X) \subset (C(\Omega), \|\cdot\|_0). \tag{6}$$

In other words, any Banach space can be viewed as a closed vector subspace of some $(C(\Omega), ||\cdot||_0)$ for some compact Hausdorff space Ω.

Proof. Let

$$\Omega = \{L \in X^* \mid \|L\|_{X^*} \le 1\}$$

with the weak $*$-topology, then Ω is Hausdorff and compact. Define the map

$$\phi : X \to C(\Omega), \quad x \mapsto \phi(x),$$

where $\phi(x) : \Omega \to \mathbb{F}$ is given by

$$\phi(x)(L) = Lx.$$

Then

- $\phi(x)$ is continuous as a function on Ω: For any $a \in \mathbb{F}$, if $\exists \varepsilon > 0$ such that

  $$B(a, \varepsilon) \cap \mathrm{Im}(\phi(x)) = \emptyset,$$

  then $\phi(x)^{-1}(B(a, \varepsilon)) = \emptyset$ is open in Ω. Otherwise, for any $\varepsilon > 0$, choose $L_0 \in \Omega$ such that $\phi(x)(L_0) \in B(a, \varepsilon/2)$, then

  $$\phi(x)^{-1}(B(a, \varepsilon)) \supset \phi(x)^{-1}(B(\phi(x)(L_0), \varepsilon/2)) = \{L \in \Omega \mid |Lx - L_0x| < \varepsilon/2\}$$

  is an open set in Ω.

- ϕ is linear as a map from X to $C(\Omega)$: For any $x_1, x_2 \in X$ and $\alpha, \beta \in \mathbb{F}$,

  $$\phi(\alpha x_1 + \beta x_2)(L) = L(\alpha x_1 + \beta x_2) = \alpha L(x_1) + \beta L(x_2) = \alpha \phi(x_1)(L) + \beta \phi(x_2)(L).$$

- ϕ preserves the norm: For any $x \in X$,

  $$\|\phi(x)\|_0 = \sup_{L \in \Omega} |\phi(x)(L)| = \sup_{L \in \Omega} |Lx| \le \sup_{L \in \Omega} \|L\|_{X^*} \cdot \|x\|_X = \|x\|_X.$$

  On the other hand, by the last corollary of the Hahn-Banach theorem that we learned at the end of lecture 12, for any $x \in X$ one can find $L \in X^*$ such that $\|L\|_{X^*} = 1$ and $Lx = \|x\|_X$. So

  $$\|\phi(x)\|_0 = \sup_{L \in \Omega} |Lx| \ge \|x\|_X.$$

Note that the norm-preserving property also implies ϕ is injective. So ϕ is a linear isometry from X to its image.

Finally since X is a Banach space, the image $\phi(X)$ has to be closed in $C(\Omega)$. □

Using the weak $*$-topology, we can also prove the following property for the weak topology.

Theorem 1.7. *Let X be a locally convex topological vector space, and $E \subset X$ a subset. Then E is bounded in the original topology iff it is bounded in the weak topology.*

Proof. Since each weak neighborhood of 0 in X is an neighborhood of 0 in the original topology, by definition of boundedness (For any neighborhood U of 0 and all sufficiently large α, $E \subset \alpha U$), we see if E is bounded in X, then it is weakly bounded.

Conversely suppose $E \subset X$ is weakly bounded, and U is any original neighborhood of 0 in X. Since X is locally convex, one can find a convex balanced original neighborhood V of 0 in X such that

$$\overline{V} \subset U.$$

Denote

$$K = \{L \in X^* : |Lx| \leq 1, \forall x \in V\} \tag{7}$$

Claim: $\overline{V} = \{x \in X : |Lx| \leq 1, \forall L \in K\}$.

We first prove the claim in two steps:

- By the definition of K,
$$V \subset \{x \in X : |Lx| \leq 1, \forall L \in K\} = \bigcap_{L \in K} L^{-1}(\overline{B(0,1)}),$$
which is closed in the original topology. So
$$\overline{V} \subset \{x \in X : |Lx| \leq 1, \forall L \in K\}.$$
- Conversely suppose $x_0 \notin \overline{V}$. Then by the geometric Hahn-Banach theorem, one can separate x_0 and $\overline{V}$, i.e., there exists $L \in X^*$ such that Lx_0 lies outside the closure of $L(\overline{V})$. Since $\overline{V}$ is balanced, $L(\overline{V})$ is a balanced closed set in $\mathbb{F}$, i.e. $L(\overline{V}) = \overline{B(0,R)}$ for some $R > 0$. By rescaling it is easy to construct $L \in X^*$ such that $|L(\overline{V})| \leq 1$ but $|Lx_0| > 1$. So
$$x_0 \notin \{x \in X : |Lx| \leq 1, \forall L \in K\}.$$

Back to the proof. Since E is weakly bounded, for each $L \in X^*$, there exists $\alpha(L) < \infty$, such that (c.f. PSet 8-1 problem 1)

$$|Lx| \leq \alpha(L), \quad \forall x \in E \tag{8}$$

Since K is obviously convex (by the triangle inequality) and weak $*$-compact (by the Banach-Alaoglu theorem), we apply the following variation of the Banach-Steinhaus theorem (see the proof at page 47 of Rudin's book)

Theorem 1.8. *Let X, Y be topological vector spaces, $K \subset X$ is compact and convex. Let Γ be a family of continuous linear mappings from X to Y. Suppose for all $x \in K$, the orbit*

$$\Gamma_x = \{Lx : L \in \Gamma\}$$

is bounded in Y. Then there exists a bounded set $B \subset Y$ such that

$$L(K) \subset B, \quad \forall L \in \Gamma.$$

to conclude (where we take X to be X^*, Y to be $\mathbb{R}$, Γ be the family of maps $\{ev_x \mid x \in E\}$, and K be K) that there exists $\alpha < \infty$ such that for any $L \in K$,

$$|Lx| < \alpha, \quad \forall x \in E.$$

As a consequence of this fact and the claim above, for any $x \in E$,

$$\frac{1}{\alpha} x \in \overline{V} \subset U.$$

Since V is balanced, for any $t > \alpha$, we have

$$E \subset t\overline{V} \subset tU.$$

So E is bounded in the original topology. □

An immediate consequence is the following criteria for boundedness:

Corollary 1.9. *Let X be a normed vector space, $E \subset X$. If for any $L \in X^*$, we have*

$$\sup_{x \in E} |Lx| < \infty,$$

then E is bounded in the original topology.

Finally we shall state a variation of the Banach-Alaoglu theorem. But before that, we need a definition:

Definition 1.10. We say X is *separable* if it contains a countable dense subset.

Example 1.1. $\mathbb{R}^n$ is separable since $\mathbb{Q}^n \subset \mathbb{R}^n$ is dense.

Example 1.2. l^∞ is NOT separable. In fact, let $X \subset l^\infty$ be the set

$$X = \{x = (a_1, a_2, \cdots) \mid a_i = 0 \text{ or } 1\}.$$

Then X is uncountable, and any pair of points $x_1 \neq x_2 \in X$ satisfy $\|x_1 - x_2\|_\infty = 1$.

Now let $A \subset l^\infty$ be any dense subset, then for any $x \in X$, there is some $y \in A$ such that $\|y - x\|_\infty < \frac{1}{2}$. So the cardinality of A is at least the cardinality of X. This implies that A must be uncountable.

Theorem 1.11. *Let X be a separable topological vector space. If $K \subset X^*$ is weakly $*$-compact, then K is metrizable in the weak $*$-topology. (Although X^* may be not metrizable in the weak $*$-topology)*

Proof. Let x_n be a countable dense set in X. For each n, let

$$f_n : X^* \to \mathbb{F}, \quad f_n(L) = Lx_n.$$

Then f_n is continuous with respect to the weak $*$-topology. Moreover, the family $\{f_n\}$ separates points in X^*:

$$f_n(L) = f_n(L'), \forall n \Longrightarrow Lx_n = L'x_n, \forall n \Longrightarrow Lx = L'x, \forall x \Longrightarrow L = L'.$$

Since each f_n is bounded on K (by compactness of K), we can find c_n such that

$$|f_n(L)| \leq c_n, \quad \forall L \in K.$$

One can check that

$$d(L_1, L_2) = \sum_n \frac{1}{2^n c_n} |f_n(L_1) - f_n(L_2)| \tag{9}$$

defines a metric on K which is compatible with the weak-$*$ topology. □

Corollary 1.12. *Let X be a separable topological vector space, $V \subset X$ be a neighborhood of 0. If $\{L_n\}$ is a sequence in X^* such that*

$$|L_n x| \leq 1, \quad \forall x \in V, \ \forall n = 1, 2, \cdots,$$

then there exists a subsequence $\{L_{n_k}\}$ and $L \in X^*$ such that

$$Lx = \lim_{k \to \infty} L_{n_k} x, \quad \forall x \in X.$$

Proof. Let $K = \{L \in X^* : |Lx| \leq 1, \forall x \in V\}$. Then K is weak $*$-compact and metrizable. So it is sequentially compact. □