Notes for Functional Analysis

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1 Lecture 18

1.1 The convex hull

Let X be any vector space, and $E \subset X$ a subset.

Definition 1.1. The *convex hull* of E is the set

$$co(E) = \{x = t_1x_1 + \cdots + t_nx_n \in X \mid x_1, \cdots, x_n \in E, t_i \geq 0, t_1 + \cdots + t_n = 1\} \tag{1}$$

(Such an element x is called a *convex combination* of $x_1, \cdots, x_n$.)

From the definition it is easy to see that

- $E \subset co(E)$, $co(E)$ is convex.
- If $A \subset X$ is convex and $E \subset A$, then $co(E) \subset A$.

Since any intersection of convex sets is still convex, we immediately get the following equivalent description of the convex hull (which is also widely used as definition, e.g., PSET 3-1):

Proposition 1.2. (Alternate definition) *The convex hull of a set E is*

$$co(E) = \bigcap_{E \subset A, A \text{ convex}} A.$$

Now let X be a topological vector space.

Definition 1.3. We call the closure $\overline{co(E)}$ the *closed convex hull* of E.

Obviously $\overline{co(E)}$ is a closed convex set, and it is the smallest closed convex set containing E. It is also the intersection of all closed convex sets that contains E.

Example 1.1. In PSET 2-2, Problem 4(2) (c.f. the arguments on page 37 of Rudin's book) we have seen that if $X = L^p([0,1])$ for $0 < p < 1$, and E is any set containing at least one interior point, then $co(E) = X$.

Example 1.2. For any $x, y \in X$, $co(x, y)$ is the line segment connecting x and y.

Example 1.3. Let $X = \mathbb{R}^n$ and $E = S(0,1) = \{x \in \mathbb{R}^n : \|x\| = 1\}$. Then

$$co(E) = \overline{B(0,1)} = \{x \in \mathbb{R}^n : \|x\| \leq 1\}.$$

The following proposition is very useful. It says that if a sequence x_n converges weakly to x, then one can find a new sequence y_n, each is a convex combination of x_n's, so that y_n converges to x in the original topology!

Proposition 1.4. *If $x_n \xrightarrow{\omega} x$ in a locally convex topological vector space X, then*

$$x \in \overline{co(\{x_n\})}.$$

Proof. $x_n \xrightarrow{\omega} x$ implies $x \in \overline{co(\{x_n\})}^{\omega}$. But in a locally convex topological vector space,

$$\overline{co(\{x_n\})}^{\omega} = \overline{co(\{x_n\})}$$

since $co(\{x_n\})$ is convex. □

We have profound interest in compact convex sets.

Proposition 1.5. *If $A_1, \cdots, A_n$ are compact convex subsets in a topological vector space X, then $co(A_1 \cup \cdots \cup A_n)$ is compact.*

Proof. Let S be the simplex

$$S = \{(t_1, \cdots, t_n) \in \mathbb{R}^n : t_i \geq 0, t_1 + \cdots + t_n = 1\}.$$

Consider the map

$$f : S \times A_1 \times \cdots \times A_n \to X, \quad ((t_1, \cdots, t_n), x_1, \cdots, x_n) \mapsto t_1 x_1 + \cdots + t_n x_n.$$

Then f is continuous. Since $S \times A_1 \times \cdots \times A_n$ is compact, the image of

$$K = f(S \times A_1 \times \cdots \times A_n)$$

is compact in X. In what follows we will prove $K = co(A_1 \cup \cdots \cup A_n)$.

- By definition, $K \subset co(A_1 \bigcup \cdots \bigcup A_n)$.
- K is convex since we have

$$\begin{aligned} &t(t_1 x_1 + \cdots + t_n x_n) + (1-t)(t_1' x_1' + \cdots + t_n' x_n') \\ =&(t t_1 x_1 + (1-t) t_1' x_1') + \cdots + (t t_n x_n + (1-t) t_n' x_n') \\ =&(t t_1 + (1-t) t_1') x_1'' + \cdots + (t t_n + (1-t) t_n') x_n'' \\ \in& K, \end{aligned}$$

  where

$$x_i'' = \begin{cases} \frac{t t_i x_i + (1-t) t_i' x_i'}{(t t_1 + (1-t) t_i')} \in A_i, & t_i \neq 0 \text{ or } t_i' \neq 0 \\ \text{any element in } A_i, & \text{otherwise} \end{cases}.$$

- $\bigcup_{i=1}^{n} A_i \subset K$ since $A_i \subset K$ for all i.

So $co(A_1 \cup \cdots \cup A_n) = K$ is compact. □

In particular,

Corollary 1.6. *$co(\{x_1, \cdots, x_n\})$ is compact.*

In general, even if K is compact, it is possible that $co(K)$ is not closed or compact.

Definition 1.7. A set E in a topological vector space X is called *totally bounded* if for any open neighborhood U of 0, one can find $x_1, \cdots, x_n \in X$ such that

$$E \subset \bigcup_{i=1}^{n} (x_i + U).$$

Remark. If X is metrizable and the metric is translation invariant, then E is totally bounded iff for any $\varepsilon > 0$, there exists $x_1, \cdots, x_n$ such that

$$E \subset \bigcup_{k=1}^{n} B(x_k, \varepsilon).$$

In other words, for any ε one can find an ε-net $\{x_1, \cdots, x_n\}$ of E. In particular, any compact set in X is totally bounded.

Proposition 1.8. *If X is a locally convex topological vector space, $E \subset X$ is totally bounded, then $co(E)$ is totally bounded.*

Proof. Let U be any neighborhood of 0 in X. Choose a convex neighborhood V of 0 such that $V + V \subset U$. Since E is totally bounded, one can find $x_1, \cdots, x_n$ such that

$$E \subset \bigcup_{k=1}^{n} (x_k + V).$$

It follows

$$E \subset co(\{x_1, \cdots, x_n\}) + V.$$

Note that the right hand side is convex, so

$$co(E) \subset co(\{x_1, \cdots, x_n\}) + V.$$

Since $co(\{x_1, \cdots, x_n\})$ is compact, and

$$\{y + V \mid y \in co(\{x_1, \cdots, x_n\})\}$$

is an open covering of $co(\{x_1, \cdots, x_n\})$, one can find $y_1, \cdots, y_m \in co(\{x_1, \cdots, x_n\})$ such that

$$co(\{x_1, \cdots, x_n\}) \subset \bigcup_{k=1}^{m} (y_i + V).$$

It follows that

$$co(E) \subset \bigcup_{k=1}^{m} (y_i + V) + V \subset \bigcup_{k=1}^{m} (y_i + U).$$

So $co(E)$ is totally bounded. □

As a consequence, we have

Corollary 1.9. *If X is Frechét, and $K \subset X$ is compact, then $\overline{co(K)}$ is compact.*

Proof. Since K compact, it is totally bounded. So $co(K)$ is totally bounded. Now use the following fact whose proof is left as an exercise:
Fact: The closure of a totally bounded set in a complete metric space is compact. □

1.2 Extreme points

First suppose X is a vector space, and $E \subset X$ a subset.

Definition 1.10. A non-empty subset S in E is called an *extreme set* of E if for any $x \in S$,

$$x = tx_1 + (1-t)x_2 \text{ for some } 0 < t < 1, x_1, x_2 \in E \implies x_1 \in S, x_2 \in S.$$

In other words, any point in S cannot be written as a convex combination of two points in E that are not both in S.

Definition 1.11. A point x is called an *extreme point* of E if $\{x\}$ is an extreme set of E.

So a point is an extreme point of E if and only if it cannot be written as a convex combination of other elements in E.

We will denote the set of all extreme points of E by $Ex(E)$, i.e.

$$Ex(E) = \text{ the set of extreme points of } E.$$

Example 1.4. E is an extreme set of E.

Example 1.5. Let $X = \mathbb{R}^n$ and

$$E = \overline{B(0,1)} = \{x \in X : \|x\| \leq 1\}.$$

Then the unit sphere $S(0,1)$ is an extreme set.

Example 1.6. Let $X = C([0,1])$ and

$$K = \overline{B(0,1)} = \{f \in X : \|f\|_\infty \le 1\}.$$

Then the extreme set of K contains only two "points":

$$Ex(K) = \{\chi_{[0,1]}, -\chi_{[0,1]}\}.$$

In fact, if $f \in C([0,1])$ and $|f(t_0)| < 1$ for some $t_0 \in [0,1]$, then one can always write f as a linear combination of two other elements in K, whose values at t_0 are $f(t_0) \pm \varepsilon$ for some small ε.

Example 1.7. Let

$$X = c_0 = \{(a_1, a_2, \cdots) \in l^\infty : \lim_{n\to\infty} a_n = 0\} \subset l^\infty$$

and let $K = \overline{B(0,1)} \subset X$. Then

$$Ex(K) = \emptyset.$$

The proof is similar to the previous example and is left as an exercise.

The following theorem is very useful in many applications.

Theorem 1.12 (Krein-Milman)**.** *Let X be a topological vector space such that X^* separates points, and K is any nonempty compact convex subset of X. Then*

$$K = \overline{co(Ex(K))}.$$

Before we prove the Krein-Milman theorem, we first prove

Proposition 1.13. *Let X be a topological vector space such that X^* separates points, $K \subset X$ is compact and non-empty. Then any compact extreme set of K contains at least one extreme point of K.*

(In particular, since K is a compact extreme set of K, $Ex(K) \neq \emptyset$.)

Proof. We shall use Zorn's lemma. Let K_0 be any compact extreme set of K and let

$$\mathcal{P} = \{\text{all compact extreme set of } K \text{ that is contained in } K_0\}.$$

We define a partial order relation on $\mathcal{P}$ by

$$A \preceq B \Longleftrightarrow A \subset B.$$

For any totally ordered subset $\mathcal{Q}$ of $\mathcal{P}$, we let

$$S_{\mathcal{Q}} = \bigcap_{S\in\mathcal{Q}} S.$$

Then

- $S_{\mathcal{Q}}$ is compact and $S_{\mathcal{Q}} \subset K_0$, since it is a closed subset of the compact set K_0.
- $S_{\mathcal{Q}} \neq \emptyset$. This is a consequence of the so-called *finite intersection property*, but we will give a direct proof here: Suppose $S_{\mathcal{Q}} = \emptyset$, then

  $$K_0 = K_0 \setminus S_{\mathcal{Q}} = \bigcup_{S \in \mathcal{Q}} K_0 \setminus S.$$

  But K_0 is compact, and each $K_0 \setminus S$ is open in K_0. So there exists $S_1, \cdots, S_N \in \mathcal{Q}$ so that

  $$K_0 = \bigcup_{k=1}^{N} K_0 \setminus S_k = K_0 \setminus \bigcap_{k=1}^{N} S_k.$$

  It follows that

  $$\bigcap_{k=1}^{N} S_k = \emptyset,$$

  which is impossible since S_k's are in Q and Q is a totally ordered set (with respect to set inclusion), so that the intersection $S_1 \cap \cdots \cap S_N$ must equals one of the S_k's.
- $S_{\mathcal{Q}}$ is an extreme set of K: Suppose $x \in S_{\mathcal{Q}}$, and $x = tx_1 + (1-t)x_2$ for some $0 < t < 1$ and $x_1, x_2 \in K$. Then for any $S \in \mathcal{Q}$, one has $x \in S$. Since S is an extremal set, we must have $x_1, x_2 \in S$ for all S. It follows that $x_1, x_2 \in S_{\mathcal{Q}}$.

It follows that $S_{\mathcal{Q}}$ is a lower bound of $\mathcal{Q}$.

So we can apply Zorn's lemma to conclude that $\mathcal{P}$ has a minimal element S_0. On the other hand, by the next two lemmas, any set containing at least two elements cannot be a minimal element of $\mathcal{P}$. So S_0 contains only one point, which is an extreme point by definition. □

The two lemmas needed in the last paragraph is listed as follows, whose proofs are left as exercises.

Lemma 1.14. *If X is a topological vector space such that X^* separates points, A, B are disjoint nonempty compact convex sets in X, then there exists $L \in X^*$ such that*

$$\sup_{x \in A} \mathrm{Re} Lx < \inf_{y \in B} \mathrm{Re} Ly.$$

Lemma 1.15. *Suppose X is a topological vector space such that that X^* separates points, and $K \subset X$ is compact. If S is an extreme set of K and $L \in X^*$, then*

$$S_L = \{x \in S \mid \mathrm{Re} Lx = \sup_{y \in K} \mathrm{Re} Ly\}$$

is an extreme set of K.

Now we are ready to prove the Krein-Milman theorem.

Proof of the Krein-Milman theorem.
Since K is a convex compact set and $K \supset Ex(K)$, we have

$$K \supset \overline{co(Ex(K))}.$$

In particular, $\overline{co(Ex(K))}$ is compact.
If $K \neq \overline{co(Ex(K))}$, then there exists $x_0 \in K, x_0 \neq \overline{co(Ex(K))}$. Applying lemma 1.14, we can find $L \in X^*$ such that

$$\mathrm{Re}Lx < \mathrm{Re}Lx_0, \forall x \in \overline{co(Ex(K))}.$$

Applying lemma 1.15, we conclude that the set

$$K_L = \{x \in K : \mathrm{Re}Lx = \sup_{y \in K} \mathrm{Re}Ly\}$$

is a compact extreme set of K. But

$$K_L \cap \overline{co(Ex(K))} = \emptyset,$$

so K_L contains no extreme points of K, which contradicts with proposition 1.13. □

Remark. Using theorem 1.6 in lecture 13 (the "closed v.s. compact" version of the geometric Hahn-Banach theorem in locally convex topological vector space) instead of lemma 1.14, one can prove

> *Theorem* 1.16. *If K is a compact subset in a locally convex topological vector space, then $K \subset \overline{co(Ex(K))}$.*

Remark. In general, $\overline{co(K)}$ may have extreme points that are not in K. But if $\overline{co(K)}$ is compact, then one has the following theorem of Milman:

> *Theorem* 1.17. *If X is a local convex topological vector space and K is compact, and if also $\overline{co(K)}$ is compact, then $Ex(\overline{co(K)}) \subset K$.*

For a proof, c.f. page 76-77 of Rudin's book.

The Krein-Milman theorem has various applications to many different problems, c.f., for example, chapters 13 and 14 in Lax's book. Here we just mention a couple of them.

- **Application 1:** The Stone-Weierstrass Theorem and its generalizations.

  ZHAN Haofeng will report this in the Hua Seminar.

- **Application 2:** Some Banach spaces are not the dual space of any Banach space.

Corollary 1.18. *Let X be a Banach space, and $B = \{L \in X^* : \|L\|_{X^*} \le 1\}$. Then*

$$B = \overline{co(Ex(B))}^{weak*}.$$

Proof. Acoording to the Banach-Alaoglu theorem (corollary 1.5 in lecture 15), B is compact if we equip X with the weak-* topology. Now apply the Krein-Milman theorem. □

So the closed unit ball in the dual of any Bananch space must contain enough extreme points to generate the ball itself. In particular,

Corollary 1.19. *Let X be the subspace c_0 of l^∞ that contains all sequences that converges to 0:*

$$X = \{(a_1, a_2, \cdots) : \lim_{i\to\infty} a_i = 0\} \subset l^\infty.$$

Then X is not the dual of any Banach space.

Proof. Let B be the unit ball in c_0. Then we have seen $Ex(B) = \emptyset$. □

- **Application 3:** Birkhoff's theorem on double stochastic matrices.

  An $n \times n$ matrix (a_{ij}) is called a *double stochastic matrix* if each entry $a_{ij} \ge 0$, and

$$\sum_{i=1}^n a_{ij} = \sum_{i=1}^n a_{ji} = 1, \quad \forall 1 \le j \le n.$$

  For example, permutation matrices are special double stochastic matrices whose entries are integers: $a_{ij} = 0$ or 1.

Theorem 1.20 (Birkhoff)**.** *Any double stochastic matrix is a convex combination of the permutation matrices.*

Idea of Proof: Let K be the set of all double stochastic matrices. Then one can prove

(1) K is compact and convex,

(2) $Ex(K)$ equals the set of all permutation matrices.