Notes for Functional Analysis

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November 13, 2015

1 Lecture 18

1.1 Characterizations of reflexive spaces

Recall that a Banach space X is reflexive if the inclusion $X \subset X^{**}$ is a Banach space isomorphism. The following theorem of Kakatani give us a very useful creteria of reflexive spaces:

Theorem 1.1. *A Banach space X is reflexive iff the closed unit ball*

$$B = \{x \in X : \|x\| \leq 1\}$$

is weakly compact.

Proof. First assume X is reflexive. Then B is the closed unit ball in X^{**}. By the Banach-Alaoglu theorem, B is compact with respect to the weak-$*$ topology of X^{**}. But by definition, in this case the weak-$*$ topology on X^{**} coincides with the weak topology on X (since both are the weakest topology on $X = X^{**}$ making elements in X^* continuous). So B is compact with respect to the weak topology on X.

Conversly suppose B is weakly compact. Let $\iota : X \hookrightarrow X^{**}$ be the canonical inclusion map that sends x to ev_x. Then we've seen that ι preserves the norm, and thus is continuous (with respect to the norm topologies). By PSet 8-1 Problem 3,

$$\iota : X_{(\text{weak})} \hookrightarrow X^{**}_{(\text{weak})}$$

is continuous. Since the weak-$*$ topology on X^{**} is weaker than the weak topology on X^{**} (since $X^{***} = (X^*)^{**} \supset X^*$. OH MY GOD!) we thus conclude that

$$\iota : X_{(\text{weak})} \hookrightarrow X^{**}_{(\text{weak-}*)}$$

is continuous. But by assumption $B \subset X_{(\text{weak})}$ is compact, so the image $\iota(B)$ is compact in $X^{**}_{(\text{weak-}*)}$. In particular, $\iota(B)$ is weak-$*$ closed. On the other hand, by PSet 10-1, Problem 3, $\iota(B)$ is weak-$*$ dense in B^{**}. So

$$\iota(B) = B^{**}.$$

As a consequence, $\iota(X) = X^{**}$, i.e. X is reflexive. $\square$

As a consequence we can prove the following property which enable us to construct many many reflexive spaces:

Proposition 1.2. *If M is a closed vector subspace of a reflexive Banach space X, then M is reflexive.*

Proof. According to the Hahn-Banach theorem, M^* is the restriction of elements in X^* to M. So the weak topology on M (defined using M^*) coincides with the topology on M that is induced by the weak topology on X (defined using X^*).

Now let B_X and B_M be the closed unit balls in X and M respectively. Since X is reflexive, B_X is weakly compact in X. On the other hand, since both B_X and M are closed and convex, they are weakly closed. So

$$B_M = B_X \cap M$$

is weakly closed in X, i.e. B_M is a weakly closed subset of the weakly compact set B_X. So B_M is weakly compact in X (with respect to the weak topology on X). By the first paragraph, B_M is weakly compact in M (with respect to the weak topology on M). So M is reflexive. □

The following proposition is natural (but not trivial). (Sometimes it can be used to justify that some Banach spaces are reflexive or not reflexive.)

Proposition 1.3. *A Banach space X is reflexive iff X^* is reflexive.*

Proof. First suppose X is reflexive, i.e., $\iota(X) = X^{**}$. Then the weak-$*$topology on X^* coincides with the weak topology on X^* (since both are the weakest topology making elements in X continuous). So the Banach-Alaoglu theorem implies B^* is weakly compact. By propositon 1.2, X^* is reflexive.

Conversely suppose X^* is reflexive. Then by the first part, X^{**} is reflexive. But X is a closed subspace of X^{**}. So X is reflexive. □

Remark. For the first part one can not argue as follows: "Since X is reflexive, so $X = X^{**}$. So $X^* = X^{***}$. So X^* is reflexive." In fact, this argument only shows that there is *some* Banach space isomorphism between X^{***} and X^* (which is induced by the Banach space isomorphism $\iota : X \to X^{**}$), but it does not implies that the canonical inclusion map $\iota : X^* \to X^{***}$ is a Banach space isomorphism.

1.2 Properties of reflexive spaces

We list several nice properties of reflexive spaces.

Corollary 1.4. *Let X be reflexive, $K \subset X$ be convex, bounded and closed. Then K is weakly compact.*

Proof. Since K is bounded, $K \subset tB$ for all large t. But X is reflexive, so tB is weakly compact. Since K is convex and closed, it is weakly closed. So K is a weakly closed subset of the weakly compact set tB. So K is weakly compact. $\square$

Remark. One can't remove convexity assumption above. For example,

$$S = \{x \in l^2 : \|x\| = 1\}$$

is a bounded closed subset in l^2 (which is reflexive since it is a Hilbert space), but S is not weakly compact since $e_n \in X$ but $e_n \rightharpoonup 0 \notin S$.

Recall in Lecture 7 we showed that in a Hilbert space, any non-empty closed convex subset contains a unique norm-minimizing element. The next proposition indicates that reflexive Banach spaces behaves like Hilbert spaces:

Proposition 1.5. *Let X be reflexive, and $C \subset X$ is closed, nonempty and convex. Then $\exists c \in C$ such that*

$$\|c\| = \inf_{x \in C} \|x\|.$$

(However, unlike the Hilbert space case, it may happen that such c is not unique.)

Proof. Let $t_0 = \inf_{x \in C} \|x\|$. For any $t > t_0$, we let

$$C_t = \{x \in C : \|x\| \le t\} = C \cap \overline{B(0,t)}.$$

Then C_t is closed, convex, nonempty and bounded, and thus is weakly compact.

Now consider the nested sequence of compact (with respect to weak topology) sets

$$C_{t+1} \supset C_{t_0+\frac{1}{2}} \supset C_{t_0+\frac{1}{3}} \supset \cdots$$

Then a standard arguments implies (c.f. page 6 of the typed notes for Lecture 16)

$$\bigcap_{n=1}^{\infty} C_{t_0+\frac{1}{n}} \neq \emptyset.$$

Pick any $c \in \bigcap_{n=1}^{\infty} C_{t_0+\frac{1}{n}}$, we see

$$\|c\| \le t_0 + \frac{1}{n}$$

for any n. So $\|c\| = \inf_{x \in C} \|x\|$. $\square$

Remark. The norm function $\|\cdot\|$ is NOT continuous with respect to the weak topology. (For example in l^2, we have $\|e_n\| = 1$ but $e_n \rightharpoonup 0$.) So one can not argue that the norm-minimizing element exists directly from the compactness of C_t.

For the next theorem we will need some properties of separable spaces. Recall X is separable iff X contains a countable dense subset. We have seen that in a separable space,

- Any weak-$*$ compact set is metrizable; (Lec 15)
- Any weak-$*$ bounded sequence has a weak-$*$ convergent subsequence. (PSet 9-1, 3(3))

Using the geometric Hahn-Banach theorem, one can prove

Theorem 1.6. *Let X be a normed vector space. If X^* is separable, then X is separable.*

We remark that the converse of the theorem is not true: l^1 is separable, but $l^\infty = (l^1)^*$ is not separable. The proofs are left as exercises. (Hints for the proof of theorem 1.6: Let $\{x_n^*\}$ be a countable dense subset of X^*. For each n, choose $x_n \in X$ such that $\|x_n\| \le 1, (x_n, x_n^*) \ge \frac{1}{2}\|x_n^*\|$. Show that E, the collection of all rational linear combinations of $\{x_n\}$'s, is dense in X.)

Now we are ready to prove the following remarkable property of reflexive spaces:

Theorem 1.7. *Any bounded sequence in a reflexive Banach space X has a weakly convergent subsequence.*

Proof. Let $\{x_n\}$ be bounded. Let

$$Y = \overline{span\{x_n\}}.$$

Then Y is separable. As a closed subspace of X, Y is reflexive. So $Y^{**} = Y$ is separable. By theorem 1.6, Y^* is separable.

Since $\{x_n\}$ is bounded in $X^{**} = X$, it is weakly bounded. So as elements in Y^{**}, $\{x_n\}$ is weakly bounded. But $Y^* = Y^{***}$ implies that the weak topology on Y^{**} coincides with the weak-$*$ topology on Y^{**}. So $\{x_n\}$ is weak-$*$ bounded in Y^{**}. By PSet9-1 problem 3(3) that we just mentioned, $\{x_n\}$ has a weak-$*$ convergent subsequence

$$x_{n_k} \xrightarrow{w*} x_0 \in Y^{**} = Y.$$

It follows

$$\langle x_{n_k}, y^* \rangle \to \langle x_0, y^* \rangle, \quad \forall y^* \in Y^* = Y^{***}.$$

So for any $x^* \in X^*$,

$$\langle x_{n_k}, x^* \rangle = \langle x_{n_k}, x^*|_Y \rangle \to \langle x_0, x^*|_Y \rangle = \langle x_0, x^* \rangle.$$

In other words, $x_{n_k} \xrightarrow{\omega} x_0$. □

Remark. Obviously the theorem is NOT true for the norm topology.

Remark. In fact, the converse is also true: X is reflexive iff any bounded sequence in X has a weakly convergent subsequence.