LECTURE 1: LINEAR SYMPLECTIC GEOMETRY

Contents

1. Linear symplectic structure 3
2. Distinguished subspaces 5
3. Linear complex structure 7
4. The symplectic group 10

*********************************************************************************

Information:

Course Name: Symplectic Geometry
Instructor: Zuoqin Wang
Time/Room: Wed. 2:00pm-6:00pm @ 1318
Reference books:

- Lectures on Symplectic Geometry by A. Canas de Silver
- Symplectic Techniques in Physics by V. Guillemin and S. Sternberg
- Lectures on Symplectic Manifolds by A. Weinstein
- Introduction to Symplectic Topology by D. McDuff and D. Salamon
- Foundations of Mechanics by R. Abraham and J. Marsden
- Geometric Quantization by Woodhouse

Course webpage: http://staff.ustc.edu.cn/~wangzuoq/Symp15/SympGeom.html

*********************************************************************************

Introduction:

The word *symplectic* was invented by Hermann Weyl in 1939: he replaced the Latin roots in the word *complex*, com-plexus, by the corresponding Greek roots, sym-plektikos.

What is symplectic geometry?

- *Geometry* = background space (smooth manifold) + extra structure (tensor).
  - Riemannian geometry = smooth manifold + metric structure.
    * metric structure = positive-definite symmetric 2-tensor
  - Complex geometry = smooth manifold + complex structure.
    * complex structure = involutive endomorphism ((1,1)-tensor)
  - *Symplectic geometry* = smooth manifold + symplectic structure
LECTURE 1: LINEAR SYMPLECTIC GEOMETRY

* Symplectic structure = closed non-degenerate 2-form
  * 2-form = anti-symmetric 2-tensor
– Contact geometry = smooth manifold + contact structure
* contact structure = local contact 1-form

• Symplectic geometry v.s. Riemannian geometry
  – Very different (although the definitions look similar)
    * All smooth manifolds admit a Riemannian structure, but only some of them admit symplectic structures.
    * Riemannian geometry is very rigid (isometry group is small), while symplectic geometry is quite soft (the group of symplectomorphisms is large)
    * Riemannian manifolds have rich local geometry (curvature etc), while symplectic manifolds have no local geometry (Darboux theorem)
  – Still closely related
    * Each cotangent bundle is a symplectic manifold.
    * Many Riemannian geometry objects have their symplectic interpretations, e.g. geodesics on Riemannian manifolds lifts to geodesic flow on their cotangent bundles

• Symplectic geometry v.s. complex geometry
  – Many similarities. For example, in complex geometry one combine pairs of real coordinates \((x, y)\) into complex coordinates \(z = x + iy\). In symplectic geometry one has Darboux coordinates that play a similar role.

• Symplectic geometry v.s. contact geometry
  – contact geometry = the odd-dim analogue of symplectic geometry

• Symplectic geometry v.s. analysis
  – Symplectic geometry is a language which can facilitate communication between geometry and analysis (Alan Weinstein).
  – (LAST SEMESTER) Quantization: one can construct analytic objects (e.g. FIOs) from symplectic ones (e.g. Lagrangians).

• Symplectic geometry v.s. algebra
  – The orbit method (Kostant, Kirillov etc) in constructing Lie group representations uses symplectic geometry in an essential way: coadjoint orbits are naturally symplectic manifolds. \(\rightarrow\) geometric quantization

• Symplectic geometry v.s. physics
  – mathematics is created to solve specific problems in physics and provides the very language in which the laws of physics are formulated. (Victor Guillemin and Shlomo Sternberg)
    * Riemannian geometry \(\leftrightarrow\) general relativity
    * Symplectic geometry \(\leftrightarrow\) classical mechanics (and quantum mechanics via quantization), geometrical optics etc.
  – Symplectic geometry has its origin in physics
Lecture 1: Linear Symplectic Geometry

* Lagrange’s work (1808) on celestial mechanics, Hamilton, Jacobi, Liouville, Poisson, Poincare, Arnold etc.
– An old name of symplectic geometry: the theory of canonical transformations

In this course, we plan to cover

- Basic symplectic geometry
  - Linear symplectic geometry
  - Symplectic manifolds
  - Local normal forms
  - Lagrangian submanifolds v.s. symplectomorphisms
  - Related geometric structures
  - Hamiltonian geometry

- Symplectic group actions (= symmetry in classical mechanics)
  - The moment map
  - Symplectic reduction
  - The convexity theorem
  - Toric manifolds

- Geometric quantization
  - Prequantization
  - Polarization
  - Geometric quantization

*********************************************************************************

1. Linear Symplectic Structure

\[ \text{Definitions and examples.} \]

Let \( V \) be a (finite dimensional) real vector space and \( \Omega: V \times V \to \mathbb{R} \) a bilinear map. \( \Omega \) is called *anti-symmetric* if for all \( u, v \in V \),

\[ \Omega(u, v) = -\Omega(v, u). \]

It is called *non-degenerate* if the associated map

\[ \tilde{\Omega}: V \to V^*, \quad \tilde{\Omega}(u)(v) = \Omega(u, v) \]

is bijective. Obviously the non-degeneracy is equivalent to the condition

\[ \Omega(u, v) = 0, \forall v \in \Omega \implies u = 0. \]

Note that one can regard \( \Omega \) as a linear 2-form \( \Omega \in \Lambda^2(V^*) \) via

\[ \Omega(u, v) = \iota_v \iota_u \Omega. \]

**Definition 1.1.** A *symplectic vector space* is a pair \((V, \Omega)\), where \( V \) is a real vector space, and \( \Omega \) a non-degenerate anti-symmetric bilinear map. \( \Omega \) is called a *linear symplectic structure* or a *linear symplectic form* on \( V \).
Example. Let $V = \mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$ and define
$$\Omega_0((x, \xi), (y, \eta)) := \langle x, \eta \rangle - \langle \xi, y \rangle,$$
then $(V, \Omega_0)$ is a symplectic vector space. Let $\{e_1, \cdots, e_n, f_1, \cdots, f_n\}$ be the standard basis of $\mathbb{R}^n \times \mathbb{R}^n$, then $\Omega$ is determined by the relations
$$\Omega_0(e_i, e_j) = \Omega_0(f_i, f_j) = 0, \quad \Omega_0(e_i, f_j) = \delta_{ij}, \quad \forall i, j.$$
Denote by $\{e_1^*, \cdots, e_n^*, f_1^*, \cdots, f_n^*\}$ the dual basis of $(\mathbb{R}^n)^* \times (\mathbb{R}^n)^*$, then as a linear 2-form one has
$$\Omega_0 = \sum_{i=1}^{n} e_i^* \wedge f_i^*.$$

Example. More generally, for any finitely dimensional vector space $U$, the vector space $V = U \oplus U^*$ admits a canonical symplectic structure
$$\Omega((u, \alpha), (v, \beta)) = \beta(u) - \alpha(v).$$

Example. For any nondegenerate skew-symmetric $2n \times 2n$ matrix, the 2-form $\Omega_A$ on $\mathbb{R}^{2n}$ defined by
$$\Omega_A(X, Y) = \langle X, AY \rangle$$
is a symplectic form on $\mathbb{R}^{2n}$.

Linear Darboux theorem.

Definition 1.2. Let $(V_1, \Omega_1)$ and $(V_2, \Omega_2)$ be symplectic vector spaces. A linear map $F : V_1 \to V_2$ is called a linear symplectomorphism (or a linear canonical transformation) if it is a linear isomorphism and satisfies
$$F^* \Omega_2 = \Omega_1.$$

Example. Any linear isomorphism $L : U_1 \to U_2$ lifts to a linear symplectomorphism
$$F : U_1 \oplus U_1^* \to U_2 \oplus U_2^*, \quad F((u, \alpha)) = (L(u), (L^*)^{-1}(\alpha)).$$
It is not hard to check that $F$ is a linear symplectomorphism.

Theorem 1.3 (Linear Darboux theorem). For any linear symplectic vector space $(V, \Omega)$, there exists a basis $\{e_1, \cdots, e_n, f_1, \cdots, f_n\}$ of $V$ so that
$$\Omega(e_i, e_j) = \Omega(f_i, f_j) = 0, \quad \Omega(e_i, f_j) = \delta_{ij}, \quad \forall i, j.$$
The basis is called a Darboux basis of $(V, \Omega)$.

Remark. The theorem is equivalent to saying that given any symplectic vector space $(V, \Omega)$, there exists a dual basis $\{e_1^*, \cdots, e_n^*, f_1^*, \cdots, f_n^*\}$ of $V^*$ so that as a linear 2-form,
$$\Omega = \sum_{i=1}^{n} e_i^* \wedge f_i^*.$$
This is also equivalent to saying that there exists a linear symplectomorphism
\[ F : (V, \Omega) \to (\mathbb{R}^{2n}, \Omega_0). \]

In particular,
- Any symplectic vector space is even-dimensional.
- Any even dimensional vector space admits a linear symplectic form.
- Up to linear symplectomorphisms, there is a unique linear symplectic form on each even dimensional vector space.

**Proof of the linear Darboux theorem.** Apply the Gram-Schmidt process with respect to the linear symplectic form \( \Omega \). Details left as an exercise. \( \square \)

### Symplectic volume form.

Since a linear symplectic form is a linear 2-form, a natural question is: which 2-form in \( \Lambda^2(V^*) \) is a linear symplectic form on \( V \)?

**Proposition 1.4.** Let \( V \) be a \( 2n \) dimensional vector space. A linear 2-form \( \Omega \in \Lambda^2(V^*) \) is a linear symplectic form on \( V \) if and only if as a \( 2n \)-form,

\[ \Omega^n = \Omega \wedge \cdots \wedge \Omega \neq 0 \in \Lambda^{2n}(V^*). \]

*We will call \( \frac{\Omega^n}{n!} \) a symplectic volume form or a Liouville volume form on \( V \).*

**Proof.** If \( \Omega \) is symplectic, then according to the linear Darboux theorem, one can choose a dual basis of \( V^* \) so that \( \Omega \) is given by (5). It follows

\[ \Omega^n = n! e_1^* \wedge f_1^* \wedge \cdots \wedge e_n^* \wedge f_n^* \neq 0. \]

Conversely, if \( \Omega \) is degenerate, then there exists \( u \in V \) so that \( \Omega(u, v) = 0 \) for all \( v \in V \). Extend \( u \) into a basis \( \{u_1, \ldots, u_{2n}\} \) of \( V \) with \( u_1 = u \). Then since \( \dim \Lambda^{2n}(V) = 1 \), \( u_1 \wedge \cdots \wedge u_{2n} \) is a basis of \( \Lambda^{2n}(V) \). But \( \Omega^n(u_1 \wedge \cdots \wedge u_{2n}) = 0 \). So \( \Omega^n = 0 \). \( \square \)

### 2. Distinguished subspaces

#### Symplectic ortho-complement.

Now we turn to study interesting vector subspaces of a symplectic vector space \( (V, \Omega) \). A vector subspace \( W \) of \( V \) is called a *symplectic subspace* if \( \Omega \mid_{W \times W} \) is a linear symplectic form on \( W \). Symplectic subspaces are of course important. However, in symplectic vector spaces there are many other types of vector subspaces that are even more important.

**Definition 2.1.** The *symplectic ortho-complement* of a vector subspace \( W \subset V \) is

\[ W^\Omega = \{ v \in V \mid \Omega(v, w) = 0 \text{ for all } w \in W \}. \]

**Example.** If \( (V, \Omega) = (\mathbb{R}^{2n}, \Omega_0) \) and \( W = \text{span}\{e_1, e_2, f_1, f_3\} \), then

\[ W^\Omega = \text{span}\{e_2, f_3, e_4, \ldots, e_n, f_4, \ldots, f_3\}. \]
From the definition one immediately see that if \( W_1 \subset W_2 \), then \( W_1^\Omega \subset W_2^\Omega \), and, as a consequence,

**Lemma 2.2.** Let \( W_1, W_2 \) be subspaces of \((V, \Omega)\), then

1. \((W_1 + W_2)^\Omega = W_1^\Omega \cap W_2^\Omega\).
2. \((W_1 \cap W_2)^\Omega = W_1^\Omega + W_2^\Omega\).

One can easily observe the difference the symplectic ortho-complement and the standard ortho-complement \( W^\perp \) with respect to an inner product on \( V \). For example, one always have \( W \cap W^\perp = \{0\} \) while in most cases \( W \cap W^\Omega \neq \{0\} \). However, \( W^\Omega \) and \( W^\perp \) do have the same dimensions:

**Proposition 2.3.** \( \dim W^\Omega = 2n - \dim W \).

**Proof.** Let \( \tilde{W} = \text{Im}(\tilde{\Omega}|_W) \subset V^* \). Then \( \dim \tilde{W} = \dim W \) since \( \tilde{\Omega} \) is bijective. But we also have

\[
W^\Omega = (\tilde{W})^0 = \{u \in V : l(u) = 0 \text{ for all } l \in \tilde{W}\}.
\]

So the conclusion follows. \(\square\)

As an immediate consequence, we get

**Corollary 2.4.** \((W^\Omega)^\Omega = W\).

**Proof.** This follows from dimension counting and the fact \( W \subset (W^\Omega)^\Omega \). \(\square\)

Obviously \( W \cap W^\Omega \) is a subspace of \( W \), so one can form the quotient space \( W/W \cap W^\Omega \). The symplectic form \( \Omega \) is reduced to a 2-form \( \Omega' \) on \( W/W \cap W^\Omega \), since if \( w_1, w_2 \in W \) and \( w'_1, w'_2 \in W \cap W^\Omega \), then

\[
\Omega(w_1 + w'_1, w_2 + w'_2) = \Omega(w_1, w_2).
\]

Moreover, \( \Omega' \) is non-degenerate, since if \( w \in W \), and \( \Omega(v, w) = 0 \) for all \( v \in W \), then by definition \( w \in W^\Omega \). So we get

**Proposition 2.5.** \( \Omega' \) is a symplectic form on \( W/W \cap W^\Omega \).

Using this proposition one can extend the linear Darboux theorem to

**Theorem 2.6 (The Linear “Relative Darboux Theorem”).** Given any subspace \( W \subset V \), we can choose a symplectic basis \( \{e_1, \ldots, e_n, f_1, \ldots, f_n\} \) of \((V^{2n}, \Omega)\) such that \( W = \text{span}\{e_1, \ldots, e_{k+l}, f_1, \ldots, f_k\} \), \( W^\Omega = \text{span}\{e_{k+1}, \ldots, e_n, f_{k+l+1}, \ldots, f_n\} \) and thus \( W \cap W^\Omega = \text{span}\{e_{k+1}, \ldots, e_{k+l}\} \).

**Proof.** Exercise. \(\square\)

In particular,

**Corollary 2.7.** \( W \) is a symplectic subspace of \((V, \Omega) \iff W \cap W^\Omega = \{0\} \iff V = W \oplus W^\Omega. \)
Isotropic, coisotropic, and Lagrangian subspaces.

**Definition 2.8.** A vector subspace $W$ of a symplectic vector space $(V, \Omega)$ is called

- **isotropic** if $W \subset W^\Omega$.
  - Equivalently: $\Omega|_{W \times W} = 0$.
  - Equivalently: $\iota^*\Omega = 0 \in \Lambda^2(W^*)$, where $\iota : W \hookrightarrow V$ is the inclusion.
  - In particular dim $W \leq \text{dim } V/2$.
- **coisotropic** if $W \supset W^\Omega$.
  - Equivalently: $W^\Omega$ is isotropic.
  - In particular dim $W \geq \text{dim } V/2$.
- **Lagrangian** if $W = W^\Omega$.
  - Equivalently: $W$ is isotropic and dim $W = \text{dim } V/2$.
  - Equivalently: $W$ is coisotropic and dim $W = \text{dim } V/2$.
  - Equivalently: $W$ is both isotropic and coisotropic.
  - In particular dim $W = \text{dim } V/2$.

**Example.** If $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ is a Darboux basis of $(V, \Omega)$, then for any $0 \leq k \leq n$, $W_k = \text{span}\{e_1, \ldots, e_k, f_{k+1}, \ldots, f_n\}$ is a Lagrangian subspace of $(V, \Omega)$.

**Example.** Let $F : (V_1, \Omega_1) \to (V_2, \Omega_2)$ be any linear symplectomorphism. Note that $\Omega = \Omega_1 \oplus (-\Omega_2)$ is a symplectic structure on $V = V_1 \oplus V_2$. It is easy to check that the graph of $F$,

$$\Gamma = \{(v_1, F(v_1)) \mid v_1 \in V_1\},$$

is a Lagrangian subspace of $(V, \Omega)$.

**Linear symplectic reduction.**

**Theorem 2.9.** Let $W$ be a coisotropic subspace of $(V, \Omega)$, then

1. The induced 2-form $\Omega'$ is symplectic on the quotient $V' = W/\cap W^\Omega$.
2. If $\Lambda \subset V$ is a Lagrangian subspace, then

$$\Lambda' = ((\Lambda \cap W) + W^\Omega)/W^\Omega$$

is a Lagrangian subspace of $W/W^\Omega$.

**Proof.** (1) This is a special case of proposition 2.5.

(2) We first check that $\tilde{\Lambda} = \Lambda \cap W + W^\Omega$ is a Lagrangian subspace of $V$:

$$\tilde{\Lambda}^\Omega = (\Lambda \cap W)^\Omega \cap W = (\Lambda + W^\Omega) \cap W = \Lambda \cap W + W^\Omega = \tilde{\Lambda}. $$

It follows that $\Lambda'$ is isotropic in $V'$ and dim $\Lambda' = \frac{1}{2} \text{dim } V'$.

**Linear complex structure.**

**Definition 3.1.** A complex structure on a vector space $V$ is an automorphism $J : V \to V$ such that $J^2 = -\text{Id}$. Such a pair $(V, J)$ is called a complex vector space.
The basic example is of course $\mathbb{C}^n = \mathbb{R}^{2n}$, with standard complex structure $J_0$ corresponding to the map “multiplication by $i = \sqrt{-1}$:

$$J_0 x_i = y_0, J_0 y_i = -x_i.$$ 

Remarks. Complex structure is very similar to symplectic structure:

1. Since $\det J^2 \geq 0$, $\dim V$ must be even.
2. For any $2n$ dimensional vector space $V$ with basis $x_1, \cdots, x_n, y_1, \cdots, y_n$, the linear map $J$ defined by

$$J x_i = y_i, \quad J y_i = -x_i$$

is a complex structure on $V$. As in the symplectic case, $(\mathbb{R}^{2n}, J_0)$ is essentially the only complex vector space of dimension $2n$.

**Theorem 3.2.** Let $V$ be an $2n$ dimensional real vector space and let $J$ be a complex structure on $V$. Then there exists a vector space isomorphism $\Phi : \mathbb{R}^{2n} \rightarrow V$ such that $J \Phi = \Phi J_0$.

**Proof.** Exercise. $\square$

\[ \square \] **Compatible complex structure.**

Now suppose $(V, \Omega)$ is symplectic vector space which admits with a complex structure $J$.

**Definition 3.3.** Let $(V, \Omega)$ be a symplectic vector space, and $J$ a complex structure on $V$.

1. We say that $J$ is **tamed** by $\Omega$ if the quadratic form $\Omega(v, Jv)$ is positive definite.
2. We say that $J$ is **compatible** with $\Omega$ if it is tamed by $\Omega$ and $J$ is a symplectomorphism, i.e.

$$\Omega(Jv, Jw) = \Omega(v, w).$$

An equivalent condition for $J$ compatible with $\Omega$ is that

$$G(v, w) = \Omega(v, Jw)$$

defines a positive definite inner product on $V$. One can easily check that $J_0$ is compatible with $\Omega_0$ on $\mathbb{R}^{2n}$.

The space of $\Omega$ compatible complex structures is denoted by $\mathcal{J}(V, \Omega)$. It is a subset of $\text{End}(V)$. We will see later that it is in fact a smooth submanifold.

**Proposition 3.4.** Every symplectic vector space admits a compatible complex structure. Moreover, given any inner product $g(\cdot, \cdot)$ on $V$, one can canonically construct such a $J$. 
Proof. Take an inner product $g$ on $V$. Since both $g$ and $\Omega$ are nondegenerate, there exists a $A \in \text{End}(V)$ such that $\Omega(v, w) = g(Av, w)$ for all $v, w \in V$. In other words, $A$ is the transpose matrix of $\Omega$ in an orthogonal basis. Since $\Omega$ is skew-symmetric and nondegenerate, we conclude that $A$ is skew-symmetric and invertible. Moreover, $AA^* = -A^2$ is symmetric and positive definite, which has a square root $\sqrt{AA^*}$. It is easy to see that $A$ preserves the eigenspace of $AA^*$, thus preserves the eigenspace of $\sqrt{AA^*}$. So $A$ commutes with $\sqrt{AA^*}$. Define

$$J = \left(\sqrt{AA^*}\right)^{-1} A.$$  

Then $J, \sqrt{AA^*}$ and $A$ commutes with each other. But $A$ is skew-symmetric and $\sqrt{AA^*}$ is symmetric, so $J$ is skew-symmetric. Moreover, $J$ is an orthogonal matrix

$$J^*J = A^*(\sqrt{AA^*})^{-1}(\sqrt{AA^*})^{-1}A = A^*(AA^*)^{-1}A = \text{Id}.$$

(This shows that the decomposition $A = \sqrt{AA^*}J$ is just the polar decomposition of $A$). As a corollary, $J^2 = -JJ^* = -\text{Id}$, i.e. $J$ is a almost complex structure. Now it is straightforward to check the compatibility:

$$\Omega(v, Jv) = g(Av, Jv) = g(-JA v, v) = g(\sqrt{AA^*}v, v) > 0,$$
$$\Omega(Jv, Jw) = g(AJv, Jw) = g(JAv, Jw) = g(Av, w) = \Omega(v, w).$$

This completes the proof. \qed

Remark. 1. In general the given inner product $g$ doesn’t equal the inner product $G$ constructed via $\Omega$ and $J$ above. In fact, there are related to each other via

$$G(v, w) = \Omega(v, Jw) = g(\sqrt{AA^*}v, w).$$

However, if the inner product $g$ was already compatible with $\Omega$, then $AA^* = \text{Id}$ and thus $g$ coincides with $G$.

2. If $(V_t, \Omega_t)$ is a smooth family of symplectic vector spaces, then we can choose a smooth family of inner products $g_t$ and get a smooth family of compatible complex structures $J_t$.  

Now we can prove

\textbf{Theorem 3.5.} The set $\mathcal{J}(V, \Omega)$ is contractible.

\textit{Proof.} Fix a $\Omega$-compatible complex structure $J$ on $V$. Define the contraction map $f : [0, 1] \times \mathcal{J}(V, \Omega) \to \mathcal{J}(V, \Omega)$ as follows: For any $J' \in \mathcal{J}(V, \Omega)$, we have a naturally defined inner product $g'$. Let $g_t = tg + (1-t)g'$, then $g_t$ is an inner product on $V$, which gives us a canonically defined continuous family of complex structure $J_t$, see remark 2 above. Moreover, by remark 1 we know that $J_0 = J', J_1 = J$. Thus $f$ is continuous with $f(0, J') = J'$ and $f(1, J') = J$. \qed
4. THE SYMPLECTIC GROUP

Student presentation after lecture 2: ZHANG Pei.
I will add more details later.