LECTURE 1: LINEAR SYMPLECTIC GEOMETRY

Contents

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Information:

Course Name: Symplectic Geometry

Instructor: Zuoqin Wang

Time/Room: Wed. 2:00pm-6:00pm @ 1318

Reference books:

- *Lectures on Symplectic Geometry* by A. Canas de Silver
- *Symplectic Techniques in Physics* by V. Guillemin and S. Sternberg
- *Lectures on Symplectic Manifolds* by A. Weinstein
- *Introduction to Symplectic Topology* by D. McDuff and D. Salamon
- *Foundations of Mechanics* by R. Abraham and J. Marsden
- *Geometric Quantization* by Woodhouse

Course webpage: http://staff.ustc.edu.cn/˜ wangzuoq/Symp15/SympGeom.html

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Introduction:

The word *symplectic* was invented by Hermann Weyl in 1939: he replaced the Latin roots in the word *complex*, com-plexus, by the corresponding Greek roots, sym-plektikos.

What is symplectic geometry?

- *Geometry* = background space (smooth manifold) + extra structure (tensor).
  - Riemannian geometry = smooth manifold + metric structure.
    - metric structure = positive-definite symmetric 2-tensor
  - Complex geometry = smooth manifold + complex structure.
    - complex structure = involutive endomorphism ((1,1)-tensor)
  - *Symplectic geometry* = smooth manifold + symplectic structure

        - Symplectic structure = closed non-degenerate 2-form
            - 2-form = anti-symmetric 2-tensor
    - Contact geometry = smooth manifold + contact structure
        - contact structure = local contact 1-form
- Symplectic geometry v.s. Riemannian geometry
    - Very different (although the definitions look similar)
        - All smooth manifolds admit a Riemannian structure, but only some of them admit symplectic structures.
        - Riemannian geometry is very rigid (isometry group is small), while symplectic geometry is quite soft (the group of symplectomorphisms is large)
        - Riemannian manifolds have rich local geometry (curvature etc), while symplectic manifolds have no local geometry (Darboux theorem)
    - Still closely related
        - Each cotangent bundle is a symplectic manifold.
        - Many Riemannian geometry objects have their symplectic interpretations, e.g. geodesics on Riemannian manifolds lifts to geodesic flow on their cotangent bundles
- Symplectic geometry v.s. complex geometry
    - Many similarities. For example, in complex geometry one combine pairs of real coordinates (x, y) into complex coordinates $z = x + iy$. In symplectic geometry one has Darboux coordinates that play a similar role.
- Symplectic geometry v.s. contact geometry
    - contact geometry = the odd-dim analogue of symplectic geometry
- Symplectic geometry v.s. analysis
    - Symplectic geometry is *a language which can facilitate communication between geometry and analysis* (Alan Weinstein).
    - (LAST SEMESTER) Quantization: one can construct analytic objects (e.g. FIOs) from symplectic ones (e.g. Lagrangians).
- Symplectic geometry v.s. algebra
    - The *orbit method* (Kostant, Kirillov etc) in constructing Lie group representations uses symplectic geometry in an essential way: coadjoint orbits are naturally symplectic manifolds. $\longrightarrow$ geometric quantization
- Symplectic geometry v.s. physics
    - mathematics *is created to solve specific problems in physics* and *provides the very language in which the laws of physics are formulated.* (Victor Guillemin and Shlomo Sternberg)
        - Riemannian geometry $\longleftrightarrow$ general relativity
        - Symplectic geometry $\longleftrightarrow$ classical mechanics (and quantum mechanics via quantization), geometrical optics etc.
    - Symplectic geometry has its origin in physics

    - ∗ Lagrange's work (1808) on celestial mechanics, Hamilton, Jacobi, Liouville, Poisson, Poincare, Arnold etc.
  - An old name of symplectic geometry: the theory of canonical transformations

In this course, we plan to cover

- Basic symplectic geometry
  - Linear symplectic geometry
  - Symplectic manifolds
  - Local normal forms
  - Lagrangian submanifolds v.s. symplectomorphisms
  - Related geometric structures
  - Hamiltonian geometry
- Symplectic group actions (= symmetry in classical mechanics)
  - The moment map
  - Symplectic reduction
  - The convexity theorem
  - Toric manifolds
- Geometric quantization
  - Prequantization
  - Polarization
  - Geometric quantization

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1. Linear symplectic structure

¶ Definitions and examples.

Let V be a (finite dimensional) real vector space and $\Omega : V \times V \to \mathbb{R}$ a bilinear map. Ω is called *anti-symmetric* if for all $u, v \in V$,

$$\Omega(u,v) = -\Omega(v,u). \tag{1}$$

It is called *non-degenerate* if the associated map

$$\widetilde{\Omega} : V \to V^*, \quad \widetilde{\Omega}(u)(v) = \Omega(u,v) \tag{2}$$

is bijective. Obviously the non-degeneracy is equivalent to the condition

$$\Omega(u,v) = 0, \forall v \in \Omega \Longrightarrow u = 0.$$

Note that one can regard Ω as a linear 2-form $\Omega \in \Lambda^2(V^*)$ via

$$\Omega(u,v) = \iota_v \iota_u \Omega.$$

Definition 1.1. A *symplectic vector space* is a pair (V, Ω), where V is a real vector space, and Ω a non-degenerate anti-symmetric bilinear map. Ω is called a *linear symplectic structure* or a *linear symplectic form* on V.

Example. Let $V = \mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$ and define

$$\Omega_0((x,\xi),(y,\eta)) := \langle x, \eta\rangle - \langle \xi, y\rangle,$$

then (V, Ω_0) is a symplectic vector space. Let $\{e_1, \cdots, e_n, f_1, \cdots, f_n\}$ be the standard basis of $\mathbb{R}^n \times \mathbb{R}^n$, then Ω is determined by the relations

$$\Omega_0(e_i, e_j) = \Omega_0(f_i, f_j) = 0, \quad \Omega_0(e_i, f_j) = \delta_{ij}, \qquad \forall i, j.$$

Denote by $\{e_1^*, \cdots, e_n^*, f_1^*, \cdots, f_n^*\}$ the dual basis of $(\mathbb{R}^n)^* \times (\mathbb{R}^n)^*$, then as a linear 2-form one has

$$\Omega_0 = \sum_{i=1}^n e_i^* \wedge f_i^*.$$

Example. More generally, for any finitely dimensional vector space U, the vector space $V = U \oplus U^*$ admits a canonical symplectic structure

$$\Omega((u,\alpha),(v,\beta)) = \beta(u) - \alpha(v).$$

Example. For any nondegenerate skew-symmetric $2n \times 2n$ matrix, the 2-form Ω_A on $\mathbb{R}^{2n}$ defined by

$$\Omega_A(X, Y) = \langle X, AY\rangle$$

is a symplectic form on $\mathbb{R}^{2n}$.

¶ Linear Darboux theorem.

Definition 1.2. Let (V_1, Ω_1) and (V_2, Ω_2) be symplectic vector spaces. A linear map $F : V_1 \to V_2$ is called a *linear symplectomorphism* (or a *linear canonical transformation*) if it is a linear isomorphism and satisfies

$$F^*\Omega_2 = \Omega_1. \tag{3}$$

Example. Any linear isomorphism $L : U_1 \to U_2$ lifts to a linear symplectomorphism

$$F : U_1 \oplus U_1^* \to U_2 \oplus U_2^*, \quad F((u,\alpha)) = (L(u), (L^*)^{-1}(\alpha)).$$

It is not hard to check that F is a linear symplectomorphism.

Theorem 1.3 (Linear Darboux theorem)**.** *For any linear symplectic vector space (V, Ω), there exists a basis $\{e_1, \cdots, e_n, f_1, \cdots, f_n\}$ of V so that*

$$\Omega(e_i, e_j) = \Omega(f_i, f_j) = 0, \quad \Omega(e_i, f_j) = \delta_{ij}, \qquad \forall i, j. \tag{4}$$

The basis is called a Darboux basis *of (V, Ω).*

Remark. The theorem is equivalent to saying that given any symplectic vector space (V, Ω), there exists a dual basis $\{e_1^*, \cdots, e_n^*, f_1^*, \cdots, f_n^*\}$ of V^* so that as a linear 2-form,

$$\Omega = \sum_{i=1}^n e_i^* \wedge f_i^*. \tag{5}$$

This is also equivalent to saying that there exists a linear symplectomorphism

$$F : (V, \Omega) \to (\mathbb{R}^{2n}, \Omega_0).$$

In particular,

- Any symplectic vector space is even-dimensional.
- Any even dimensional vector space admits a linear symplectic form.
- Up to linear symplectomorphisms, there is a unique linear symplectic form on each even dimensional vector space.

Proof of the linear Darboux theorem. Apply the Gram-Schmidt process with respect to the linear symplectic form Ω. Details left as an exercise. □

¶ Symplectic volume form.

Since a linear symplectic form is a linear 2-form, a natural question is: which 2-form in $\Lambda^2(V^*)$ is a linear symplectic form on V?

Proposition 1.4. *Let V be a $2n$ dimensional vector space. A linear 2-form $\Omega \in \Lambda^2(V^*)$ is a linear symplectic form on V if and only if as a $2n$-form,*

$$\Omega^n = \Omega \wedge \cdots \wedge \Omega \neq 0 \in \Lambda^{2n}(V^*). \tag{6}$$

[*We will call $\frac{\Omega^n}{n!}$ a* symplectic volume form *or a* Liouville volume form *on V.*]

Proof. If Ω is symplectic, then according to the linear Darboux theorem, one can choose a dual basis of V^* so that Ω is given by (5). It follows

$$\Omega^n = n! e_1^* \wedge f_1^* \wedge \cdots \wedge e_n^* \wedge f_n^* \neq 0.$$

Conversely, if Ω is degenerate, then there exists $u \in V$ so that $\Omega(u, v) = 0$ for all $v \in V$. Extend u into a basis $\{u_1, \cdots, u_{2n}\}$ of V with $u_1 = u$. Then since $\dim \Lambda^{2n}(V) = 1$, $u_1 \wedge \cdots \wedge u_{2n}$ is a basis of $\Lambda^{2n}(V)$. But $\Omega^n(u_1 \wedge \cdots \wedge u_{2n}) = 0$. So $\Omega^n = 0$. □

2. Distinguished subspaces

¶ Symplectic ortho-complement.

Now we turn to study interesting vector subspaces of a symplectic vector space (V, Ω). A vector subspace W of V is called a *symplectic subspace* if $\Omega|_{W \times W}$ is a linear symplectic form on W. Symplectic subspaces are of course important. However, in symplectic vector spaces there are many other types of vector subspaces that are even more important.

Definition 2.1. The *symplectic ortho-complement* of a vector subspace $W \subset V$ is

$$W^\Omega = \{v \in V \mid \Omega(v, w) = 0 \text{ for all } w \in W\}. \tag{7}$$

Example. If $(V, \Omega) = (\mathbb{R}^{2n}, \Omega_0)$ and $W = \text{span}\{e_1, e_2, f_1, f_3\}$, then

$$W^\Omega = \text{span}\{e_2, f_3, e_4, \cdots, e_n, f_4, \cdots, f_n\}.$$

From the definition one immediately see that if $W_1 \subset W_2$, then $W_2^\Omega \subset W_1^\Omega$, and, as a consequence,

Lemma 2.2. *Let W_1, W_2 be subspaces of (V, Ω), then*

(1) $(W_1 + W_2)^\Omega = W_1^\Omega \cap W_2^\Omega$.
(2) $(W_1 \cap W_2)^\Omega = W_1^\Omega + W_2^\Omega$.

One can easily observe the difference the symplectic ortho-complement and the standard ortho-complement $W^\perp$ with respect to an inner product on V. For example, one always have $W \cap W^\perp = \{0\}$ while in most cases $W \cap W^\Omega \neq \{0\}$. However, W^Ω and $W^\perp$ do have the same dimensions:

Proposition 2.3. $\dim W^\Omega = 2n - \dim W$.

Proof. Let $\widetilde{W} = \mathrm{Im}(\widetilde{\Omega}|_W) \subset V^*$. Then $\dim \widetilde{W} = \dim W$ since $\widetilde{\Omega}$ is bijective. But we also have

$$W^\Omega = (\widetilde{W})^0 = \{u \in V : l(u) = 0 \text{ for all } l \in \widetilde{W}\}.$$

So the conclusion follows. □

As an immediate consequence, we get

Corollary 2.4. $(W^\Omega)^\Omega = W$.

Proof. This follows from dimension counting and the fact $W \subset (W^\Omega)^\Omega$. □

Obviously $W \cap W^\Omega$ is a subspace of W, so one can form the quotient space $W/W \cap W^\Omega$. The symplectic form Ω is reduced to a 2-form Ω' on $W/W \cap W^\Omega$, since if $w_1, w_2 \in W$ and $w_1', w_2' \in W \cap W^\Omega$, then

$$\Omega(w_1 + w_1', w_2 + w_2') = \Omega(w_1, w_2).$$

Moreover, Ω' is non-degenerate, since if $w \in W$, and $\Omega(v, w) = 0$ for all $v \in W$, then by definition $w \in W^\Omega$. So we get

Proposition 2.5. *Ω' is a symplectic form on $W/W \cap W^\Omega$.*

Using this proposition one can extend the linear Darboux theorem to

Theorem 2.6 (The Linear "Relative Darboux Theorem")**.** *Given any subspace $W \subset V$, we can choose a symplectic basis $\{e_1, \cdots, e_n, f_1, \cdots, f_n\}$ of (V^{2n}, Ω) such that $W = \mathrm{span}\{e_1, \cdots, e_{k+l}, f_1, \cdots, f_k\}$, $W^\Omega = \mathrm{span}\{e_{k+1}, \cdots, e_n, f_{k+l+1}, \cdots, f_n\}$ and thus $W \cap W^\Omega = \mathrm{span}\{e_{k+1}, \cdots, e_{k+l}\}$.*

Proof. Exercise. □

In particular,

Corollary 2.7. *W is a symplectic subspace of (V, Ω) $\iff$ $W \cap W^\Omega = \{0\}$ $\iff$ $V = W \oplus W^\Omega$.*

¶ Isotropic, coisotropic, and Lagrangian subspaces.

Definition 2.8. A vector subspace W of a symplectic vector space (V,Ω) is called

- *isotropic* if $W \subset W^\Omega$.
  - Equivalently: $\Omega|_{W\times W} = 0$.
  - Equivalently: $\iota^*\Omega = 0 \in \Lambda^2(W^*)$, where $\iota : W \hookrightarrow V$ is the inclusion.
  - In particular $\dim W \le \dim V/2$.
- *coisotropic* if $W \supset W^\Omega$.
  - Equivalently: W^Ω is isotropic.
  - In particular $\dim W \ge \dim V/2$.
- *Lagrangian* if $W = W^\Omega$.
  - Equivalently: W is isotropic and $\dim W = \dim V/2$.
  - Equivalently: W is coisotropic and $\dim W = \dim V/2$.
  - Equivalently: W is both isotropic and coisotropic.
  - In particular $\dim W = \dim V/2$.

Example. If $\{e_1,\cdots,e_n,f_1,\cdots,f_n\}$ is a Darboux basis of (V,Ω), then for any $0 \le k \le n$, $W_k = \mathrm{span}\{e_1,\cdots,e_k,f_{k+1},\cdots,f_n\}$ is a Lagrangian subspace of (V,Ω).

Example. Let $F : (V_1,\Omega_1) \to (V_2,\Omega_2)$ be any linear symplectomorphism. Note that $\Omega = \Omega_1 \oplus (-\Omega_2)$ is a symplectic structure on $V = V_1 \oplus V_2$. It is easy to check that the graph of F,

$$\Gamma = \{(v_1, F(v_1)) \mid v_1 \in V_1\},$$

is a Lagrangian subspace of (V,Ω).

¶ Linear symplectic reduction.

Theorem 2.9. *Let W be a coisotropic subspace of (V,Ω), then*

(1) *The induced 2-form Ω' is symplectic on the quotient $V' = W/\cap W^\Omega$.*
(2) *If $\Lambda \subset V$ is a Lagrangian subspace, then*

$$\Lambda' = ((\Lambda \cap W) + W^\Omega)/W^\Omega$$

is a Lagrangian subspace of W/W^Ω.

Proof. (1) This is a special case of proposition 2.5.

(2) We first check that $\tilde\Lambda = \Lambda \cap W + W^\Omega$ is a Lagrangian subspace of V:

$$\tilde\Lambda^\Omega = (\Lambda\cap W)^\Omega \cap W = (\Lambda + W^\Omega)\cap W = \Lambda \cap W + W^\Omega = \tilde\Lambda.$$

It follows that Λ' is isotropic in V' and $\dim \Lambda' = \frac{1}{2}\dim V'$. □

3. Linear complex structure

¶ Linear complex structure.

Definition 3.1. A *complex structure* on a vector space V is an automorphism $J : V \to V$ such that $J^2 = -\mathrm{Id}$. Such a pair (V,J) is called a *complex vector space.*

The basic example is of course $\mathbb{C}^n = \mathbb{R}^{2n}$, with standard complex structure J_0 corresponding to the map "multiplication by $i = \sqrt{-1}$:

$$J_0 x_i = y_0, J_0 y_i = -x_i.$$

Remarks. Complex structure is very similar to symplectic structure:

(1) Since $\det J^2 \geq 0$, $\dim V$ must be even.
(2) For any $2n$ dimensional vector space V with basis $x_1, \cdots, x_n, y_1, \cdots, y_n$, the linear map J defined by

$$Jx_i = y_i, \quad Jy_i = -x_i$$

is a complex structure on V. As in the symplectic case, $(\mathbb{R}^{2n}, J_0)$ is essentially the *only* complex vector space of dimension $2n$.

Theorem 3.2. *Let V be an $2n$ dimensional real vector space and let J be a compex structure on V. Then there exists a vector space isomorphism $\Phi : \mathbb{R}^{2n} \to V$ such that $J\Phi = \Phi J_0$.*

Proof. Exercise. □

¶ Compatible complex structure.

Now suppose (V, Ω) is symplectic vector space which admits with a complex structure J.

Definition 3.3. Let (V, Ω) be a symplectic vector space, and J a complex structure on V.

(1) We say that J is *tamed* by Ω if the quadratic form $\Omega(v, Jv)$ is positive definite.
(2) We say that J is *compatible* with Ω if it is tamed by Ω and J is a symplectomorphism, i.e.

$$\Omega(Jv, Jw) = \Omega(v, w).$$

An equivalent condition for J compatible with Ω is that

$$G(v, w) = \Omega(v, Jw)$$

defines a positive definite inner product on V. One can easily check that J_0 is compatible with Ω_0 on $\mathbb{R}^{2n}$.

The space of Ω compatible complex structures is denoted by $\mathcal{J}(V, \Omega)$. It is a subset of $End(V)$. We will see later that it is in fact a smooth submanifold.

Proposition 3.4. *Every symplectic vector space admits a compatible complex structure. Moreover, given any inner product $g(\cdot, \cdot)$ on V, one can canonically construct such a J.*

Proof. Take an inner product g on V. Since both g and Ω are nondegenerate, there exists a $A \in End(V)$ such that $\Omega(v, w) = g(Av, w)$ for all $v, w \in V$. In other words, A is the transpose matrix of Ω in an orthogonal basis. Since Ω is skew-symmetric and nondegenerate, we conclude that A is skew-symmetric and invertible. Moreover, $AA^* = -A^2$ is symmetric and positive definite, which has a square root $\sqrt{AA^*}$. It is easy to see that A preserves the eigenspace of AA^*, thus preserves the eigenspace of $\sqrt{AA^*}$. So A commutes with $\sqrt{AA^*}$. Define

$$J = \left(\sqrt{AA^*}\right)^{-1} A.$$

Then $J, \sqrt{AA^*}$ and A commutes with each other. But A is skew-symmetric and $\sqrt{AA^*}$ is symmetric, so J is skew-symmetric. Moreover, J is an orthogonal matrix

$$J^*J = A^*(\sqrt{AA^*})^{-1}(\sqrt{AA^*})^{-1}A = A^*(AA^*)^{-1}A = \mathrm{Id}.$$

(This shows that the decomposition $A = \sqrt{AA^*}J$ is just the polar decomposition of A). As a corollary, $J^2 = -JJ^* = -Id$, i.e. J is a almost complex structure. Now it is straightforward to check the compatibility:

$$\Omega(v, Jv) = g(Av, Jv) = g(-JAv, v) = g(\sqrt{AA^*}v, v) > 0,$$
$$\Omega(Jv, Jw) = g(AJv, Jw) = g(JAv, Jw) = g(Av, w) = \Omega(v, w).$$

This completes the proof. □

Remark. 1. In general the given inner product g doesn't equal the inner product G constructed via Ω and J above. In fact, there are related to each other via

$$G(v, w) = \Omega(v, Jw) = g(\sqrt{AA^*}v, w).$$

However, if the inner product g was already compatible with Ω, then $AA^* = \mathrm{Id}$ and thus g coincides with G.

2. If (V_t, Ω_t) is a smooth family of symplectic vector spaces, then we can choose a smooth family of inner products g_t and get a smooth family of compatible complex structures J_t.

Now we can prove

Theorem 3.5. *The set $\mathcal{J}(V, \Omega)$ is contractible.*

Proof. Fix a Ω-compatible complex structure J on V. Define the contraction map $f : [0, 1] \times \mathcal{J}(V, \Omega) \to \mathcal{J}(V, \Omega)$ as follows: For any $J' \in \mathcal{J}(V, \Omega)$, we have a naturally defined inner product g'. Let $g_t = tg + (1 - t)g'$, then g_t is an inner product on V, which gives us a canonically defined continuous family of complex structure J_t, see remark 2 above. Moreover, by remark 1 we know that $J_0 = J', J_1 = J$. Thus f is continuous with $f(0, J') = J'$ and $f(1, J') = J$. □

4. The symplectic group

Student presentation after lecture 2: ZHANG Pei.

I will add more details later.