LECTURE 2: SYMPLECTIC MANIFOLDS

Contents

1. Symplectic Manifolds

¶ Definitions.

Let M be a smooth manifold, and

$$\omega \in \Omega^2(M) = \Gamma^\infty(\Lambda^2 T^* M)$$

a smooth 2-form on M. Recall that by definition this means that for any $p \in M$,

$$\omega_p : T_pM \times T_pM \to \mathbb{R}$$

is a skew-symmetric bilinear map, and ω_p depends smoothly on p.

Definition 1.1. We call ω a *symplectic form* on M if

(1) (closeness) ω is a closed 2-form, i.e. $d\omega = 0$
(2) (non-degeneracy) for $\forall p \in M$, ω_p is a linear symplectic form on T_pM.

We will call the pair (M, ω) a *symplectic manifold*.

Remarks. Let (M, ω) be a symplectic manifold. According to the linear theory,

- $\dim M = \dim T_pM$ must be even.
- If we denote $\dim M = 2n$, then

  $$\omega_p^n \neq 0 \in \Lambda^{2n}(T_p^* M),$$

  i.e. ω^n is a non-vanishing $2n$ form, thus a *volume form*, on M. As a consequence of the existence of a volume form, we see M must be orientable.

  Definition 1.2. We will call $\frac{\omega^n}{n!}$ the *symplectic volume form* or the *Liouville volume form* of (M, g).

- If ω is not only closed but also exact, i.e. there exists a 1-form α on M so that $\omega = d\alpha$, then we say (M, ω) is an *exact symplectic manifold*.

¶ Examples.

Example. $(\mathbb{R}^{2n}, \Omega_0)$ is of course the simplest symplectic manifold.

Example. Let S be any oriented surface and ω any volume form on S. Then

- ω is non-degenerate since it is a volume form;
- ω is closed since it is already a top form.

In other words, (S, ω) is symplectic.

Example. If (M, ω) is a symplectic manifold, and $U \subset M$ an open subset, then (U, ω) is a symplectic manifold.

Example. If (M, ω) is a symplectic manifold, then for any $\lambda \neq 0$, $(M, \lambda\omega)$ is a symplectic manifold.

Example. If (M_1, ω_1) and (M_2, ω_2) are symplectic manifolds, then for any nonzero real numbers λ_1, λ_2, the 2-form $\omega = \lambda_1\omega_1 \oplus \lambda_2\omega_2$ is a symplectic form on the product manifold $M = M_1 \times M_2$.

Example. The $2n$-dimensional torus $\mathbb{T}^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$, with its standard symplectic form $\omega_0 = \sum dx_j \wedge dy_j$, is a symplectic manifold.

Example. We will see more examples later, including cotangent bundles, coadjoint orbits, Kähler manifolds etc. (So, yes, we have "as many" symplectic manifolds as smooth manifolds!)

¶ Distinguished submanifolds.

As in the linear case we can define

Definition 1.3. Let (M, ω) be a $2n$-dimensional symplectic manifold and $Z \subset M$ a smooth submanifold. We call Z

- *isotropic* if for all $p \in Z$, T_pZ is an isotropic subspace of (T_pM, ω_p).
- *coisotropic* if for all $p \in Z$, T_pZ is a coisotropic subspace of (T_pM, ω_p).
- *Lagrangian* if for all $p \in Z$, T_pZ is a Lagrangian subspace of (T_pM, ω_p).
- *symplectic* if for all $p \in Z$, T_pZ is a symplectic subspace of (T_pM, ω_p).

The following properties follows immediately from the corresponding linear ones:

(1) $\iota : Z \hookrightarrow M$ is a symplectic submanifold if and only if $\iota^*\omega$ is symplectic.
(2) $\iota : Z \hookrightarrow M$ is an isotropic submanifold if and only if $\iota^*\omega = 0$.
(3) $\iota : Z \hookrightarrow M$ is a Lagrangian submanifold if and only if $\iota^*\omega = 0$ and $\dim Z = \frac{1}{2}\dim M$.
(4) If Z is an isotropic submanifold of M, then $\dim Z \leq \frac{1}{2}\dim M$.
(5) If Z is an coisotropic submanifold of M, then $\dim Z \geq \frac{1}{2}\dim M$.
(6) Any one dimensional submanifold is isotropic, any co-dimension one submanifold is coisotropic.

Lagrangian submanifolds are in some sense the most important object (according to Weinstein) in symplectic manifolds. We will study more details of Lagrangian submanifolds later.

¶ More topological restrictions.

As we have seen, to be a symplectic manifold, M must be even dimensional and oriented. There are more obstructions:

Proposition 1.4. *Let (M, ω) be a closed (=compact without boundary) symplectic manifold of dimension $2n$. Then for any $1 \le k \le n$, the de Rham cohomology group $H^{2k}(M) \neq \{0\}$.*

Proof. Since ω is closed, $[\omega] \in H^2(M)$ is a de Rham cohomology class. So for any $1 \le k \le n$, $[\omega^k] \in H^{2k}(M)$. If $H^{2k}(M) = \{0\}$, then $\omega^k = d\alpha$ for some $\alpha \in \Omega^{2k-1}(M)$. It follows

$$\omega^n = \omega^k \wedge \omega^{n-k} = d(\alpha \wedge \omega^{n-k}),$$

which is impossible since ω^n is a volume form, and a volume form on a closed manifold cannot be exact. $\square$

As an immediate consequence, we see that S^2 is the only sphere that admits a symplectic structure:

Corollary 1.5. *For any $n > 1$, the sphere S^{2n} admits no symplectic structure.*

Proof. In algebraic topology one can prove $H^2(S^{2n}) = 0$ for any $n > 1$. $\square$

What we really proved is that $[\omega] \neq 0 \in H^2(M)$ if M is compact. So

Corollary 1.6. *Any exact symplectic manifold is non-compact.*

Using a similar argument one gets

Proposition 1.7. *Any symplectic submanifold of $(\mathbb{R}^{2n}, \omega)$ is non-compact, where ω is any symplectic form on $\mathbb{R}^{2n}$.*

Proof. Since $\mathbb{R}^{2n}$ is contractible, any closed 2-form is exact. It follows that there exists a one-form α on M so that $\omega = d\alpha$. As a consequence, for any m, the $2m$ form $\omega^m = d(\alpha \wedge \omega^{m-1})$ is exact on $\mathbb{R}^{2n}$, and thus exact on any submanifold.

On the other hand, if M^{2m} is a compact symplectic submanifold of $\mathbb{R}^{2n}$, then ω^m is a volume form on M, which is a contradiction since on a compact manifold the volume form can never be exact. $\square$

Remark. On the other hand, One can easily construct many symplectic submanifolds. For example, one can identify the symplectic manifold $\mathbb{R}^{2n}$ with the complex manifold $\mathbb{C}^n$. Then any complex submanifold is a symplectic submanifold.

Remark. There exists more subtle obstructions for the existence of symplectic structure. For example, the existence of a symplectic structure implies the existence of an almost complex structure, which gives an obstruction in the Pontryagin class.

¶ Symplectomorphisms.

As in the linear case, we can define

Definition 1.8. Let (M_1, ω_1) and (M_2, ω_2) be symplectic manifolds. A smooth map $f : M_1 \to M_2$ is called a *symplectomorphism* (or a *canonical transformation*) if it is a diffeomorphism and

$$f^*\omega_2 = \omega_1. \tag{1}$$

We say (M_1, ω_1) is *symplectomorphic* to (M_2, ω_2) if such a symplectomorphism exists.

Next time we will prove :

Theorem 1.9 (Darboux theorem)**.** *Let (M, ω) be a symplectic manifold of dimension $2n$. Then for any $p \in M$, there exists a neighborhood U of p in M and a neighborhood U_0 of 0 in $\mathbb{R}^{2n}$ so that (U, ω) is symplectomorphic to (U_0, Ω_0), where Ω_0 is the standard (linear) symplectic form on $\mathbb{R}^{2n}$.*

Remark.

(1) In the language of local coordinates, this means that near any $p \in M$, one can find coordinate patch $(\mathcal{U}, x_1, \cdots, x_n, \xi_1, \cdots, \xi_n)$ centered at p such that on $\mathcal{U}$,
$$\omega = \sum dx_i \wedge d\xi_i.$$

Definition 1.10. This coordinate patch is called a *Darboux coordinate patch.*

(2) As a consequence we see that for symplectic manifolds there is no *local geometry*: locally all symplectic manifolds of the same dimension look the same. So symplectic geometry is very different from Riemannian geometry where one has many local geometric quantities (like curvature). (However, there are much to say about the global geometry/topology of symplectic manifolds!)

2. The cotangent bundle

¶ The canonical symplectic structure on cotangent bundles.

Let X be an n-dimensional smooth manifold and denote by $M = T^*X$ its cotangent bundle. Let
$$\pi : T^*X \to X, \quad (x, \xi) \mapsto x$$
be the bundle projection map. In manifold theory we learned that T^*X is a smooth manifold of dimension $2n$, and any local coordinate system on X induces an coordinate system in T^*X, as follows: From any coordinate patch $(\mathcal{U}, x_1, \cdots, x_n)$ of X one can construct, for x in U, a basis $\{\partial_1, \cdots, \partial_n\}$ of T_xX, which depends smoothly in x. We denote the dual basis of T_x^*X by $\{dx_1, \cdots, dx_n\}$. This gives the induced coordinate system, $(x_1, \cdots, x_n, \xi_1, \cdots, \xi_n)$, on $M_{\mathcal{U}} = \pi^{-1}(\mathcal{U})$. Namely, if $\xi \in T_x^*X$, then
$$\xi = \sum \xi_i (dx_i)_x.$$

For any $p = (x, \xi) \in M$, we let

$$\alpha_p = (d\pi_p)^* \xi. \tag{2}$$

Note that by definition $\xi \in T_x^* X$, so for any $p \in T^*X$,

$$\alpha_p = (d\pi_p)^* \xi \in T_p^*(T^*X).$$

In other words, we get a smooth 1-form

$$\alpha \in \Omega^1(M) = \Gamma^\infty(T^*(T^*X)).$$

Note that for any $v_p \in T_pM$,

$$\langle \alpha_p, v_p \rangle_p = \langle (d\pi_p)^* \xi, v_p \rangle_p = \langle \xi, d\pi_p(v_p) \rangle_x.$$

Definition 2.1. We call α the *canonical 1-form* (or *tautological 1-form*) on T^*X.

Proposition 2.2. *In local coordinates described above,*

$$\alpha = \sum_{i=1}^n \xi_i dx_i. \tag{3}$$

Proof. Let $v_p = \sum_{i=1}^n (a_i \frac{\partial}{\partial x_i} + b_i \frac{\partial}{\partial \xi_i}) \in T_pM$ be an arbitrary tangent vector. Then

$$\langle \alpha_p, v_p \rangle_p = \langle \xi, (d\pi_p) v_p \rangle_x = \langle \sum \xi_i (dx_i)_x, \sum a_i \frac{\partial}{\partial x_i} \rangle_x = \sum a_i \xi_i = \langle \sum \xi_i dx_i, v_p \rangle_p.$$

The equation follows. $\square$

As a consequence, if we let

$$\omega = -d\alpha, \tag{4}$$

then ω is closed, and from the local expression of α,

$$\omega = \sum dx_i \wedge d\xi_i \tag{5}$$

we see that ω_p is also non-degenerate for all p. It follows that ω is a symplectic form on M.

Definition 2.3. We call $\omega = -d\alpha$ the *canonical symplectic form* on $M = T^*X$.

Remark. In the future we will denote the canonical 1-form α_{can}, and the canonical symplectic form ω by ω_{can}. Obviously (T^*X, ω_{can}) is an exact symplectic manifold, and any coordinate patch on X induces a Darboux coordinate patch on (T^*X, ω_{can}).

¶ The reproducing property.

Let $\mu \in \Omega^1(X)$ be an arbitrary smooth 1-form. By definition we can identify μ with a smooth section $s_\mu : X \to M = T^*X$ via

$$s_\mu(x) = (x, \mu_x).$$

Let $\beta \in \Omega^1(M)$ be a 1-form on M, then $s_\mu^*\beta$ is a 1-form on X. It is not hard to check by definition that

$$(s_\mu^*\beta)_x = (ds_\mu)_x^*\beta_p.$$

A crucial property for the canonical 1-form $\alpha \in \Omega^1(M)$ is the following

Theorem 2.4 (Reproducing property)**.** *For any 1-form $\mu \in \Omega^1(X)$, we have*

$$s_\mu^*\alpha = \mu. \tag{6}$$

Conversely, if $\alpha \in \Omega^1(M)$ is a 1-form such that (6) hold for all 1-form $\mu \in \Omega^1(X)$, then α is the canonical 1-form.

Proof. At any point $p = (x, \xi)$ we have $\alpha_p = (d\pi_p)^*\xi$. So at $p = s_\mu(x) = (x, \mu_x)$ we have $\alpha_p = (d\pi_p)^*\mu_x$. It follows

$$(s_\mu^*\alpha)_x = (ds_\mu)_x^*\alpha_p = (ds_\mu)_x^*(d\pi_p)^*\mu_x = (d(\pi \circ s_\mu))_x^*\mu_x = \mu_x.$$

Conversely, suppose $\alpha_0 \in \Omega^1(M)$ is another 1-form on M satisfying the reproducing property above, then for any 1-form $\mu \in \Omega^1(X)$, we have $s_\mu^*(\alpha - \alpha_0) = 0$. So for any $v \in T_xX$,

$$0 = \langle (ds_\mu)_x^*(\alpha - \alpha_0)_p, v\rangle = \langle (\alpha - \alpha_0)_p, (ds_\mu)_x(v)\rangle.$$

For each $p = (x, \xi)$, the set of all vectors of the this form,

$$\{(ds_\mu)_x v \mid \mu \in \Omega^1(X), \mu_x = \xi, v \in T_xX, \}$$

span T_pM, so we conclude that $\alpha = \alpha_0$. $\square$

¶ Naturality.

The construction of the canonical symplectic form on cotangent bundles is natural in the following sense: Suppose X and Y are smooth manifolds of dimension n and $f : X \to Y$ a diffeomorphism. We can "lift" f to a smooth map $\tilde{f} : T^*X \to T^*Y$ by

$$\tilde{f}(x, \xi) = (f(x), (df_x^*)^{-1}(\xi)). \tag{7}$$

Theorem 2.5 (Naturality)**.** *The map $\tilde{f}$ is a diffeomorphism and preserves the canonical 1-forms:*

$$\tilde{f}^*\alpha_{T^*Y} = \alpha_{T^*X}.$$

Proof. It is not hard to check that $\tilde{f}$ is a diffeomorphism. Denote the projections by $\pi_1 : T^*X \to X$ and $\pi_2 : T^*Y \to Y$. By definition

$$\pi_2 \circ \tilde{f} = f \circ \pi_1.$$

So if we denote $\tilde{f}(x, \xi) = (y, \eta)$, then

$$\tilde{f}^* \alpha_{T^*Y} = d\tilde{f}^* \circ (d\pi_2^*)\eta = (d\pi_1^* \circ df^*)\eta = (d\pi_1^*)\xi = \alpha_{T^*X}.$$

□

As a consequence, we get immediately

Theorem 2.6. *The map $\tilde{f} : T^*X \to T^*Y$ constructed above is a symplectomorphism with respect to the canonical symplectic forms.*

¶ Lagrangian submanifolds of T^*X.

By definition, a Lagrangian submanifold Λ of (T^*X, ω_{can}) is a smooth submanifold of dimension $\dim \Lambda = \dim X$ and satisfies $\iota^*\omega_{can} = 0$, where $\iota : \Lambda \hookrightarrow T^*X$ is the inclusion map. From the local expression of ω, it is easy to see

Example. The zero section of T^*X,

$$X_0 = \{(x, \xi) \in T^*X \mid x \in X, \xi = 0 \in T_x^*X\}$$

is a Lagrangian submanifold of T^*X.

Example. For each $x \in X$, the cotangent fiber

$$T_x^*X = \{(x, \xi) \in T^*X \mid \xi \in T_x^*X\}$$

is a (vertical) Lagrangian submanifold of T^*X.

The first example is the graph of the zero 1-form on X. Suppose $\mu \in \Omega^1(M)$ is any smooth 1-form on X. Then its graph,

$$X_\mu = \{(x, \mu_x) \mid x \in X\},$$

is a n dimensional submanifold of T^*X. [Such a submanifold is called *horizontal* since $\pi \circ \iota$ is a diffeomorphism from the submanifold onto (its image in) the base manifold X.]

A natural question is: when will X_μ a Lagrangian submanifolds?

Proposition 2.7. *X_μ is Lagrangian if and only if $d\mu = 0$.*

Proof. Let $\iota : X_\mu \hookrightarrow T^*X$ be the inclusion map. Note that $\pi \circ \iota : X_\mu \to X$ is a diffeomorphism. Let $\gamma : X \to X_\mu$ be its inverse. Then by definition

$$s_\mu = \iota \circ \gamma.$$

So from the reproducing property,

$$X_\mu \text{ is Lagrangian } \Leftrightarrow \iota^* d\alpha = 0 \Leftrightarrow \gamma^*\iota^* d\alpha = 0 \Leftrightarrow d(s_\mu^*\alpha) = 0 \Leftrightarrow d\mu = 0.$$

□

The proposition characterize the horizontal Lagrangian submanifolds of T^*X. Of course there are many non-horizontal Lagrangian submanifolds of T^*X, the simplest one being the vertical fiber we described just now. More generally, for any submanifold S of X, we define its *conormal bundle* in T^*X to be

$$N^*S = \{(x, \xi) \in T^*X \mid x \in S, \xi \in T_x^*X \text{ so that } \xi(v) = 0 \text{ for all } v \in T_xS\}.$$

For example, the zero section is the conormal bundle of X itself, while the vertical fiber T_x^*X is the conormal bundle of the zero dimensional submanifold $\{x\}$.

Obviously if the dimension of S is k, then at any $x \in S$, the fiber has dimension $n - k$. So N^*S has dimension n, half the dimension of T^*X.

Proposition 2.8. *For any smooth submanifold S of X, the conormal bundle N^*S is a Lagrangian submanifold.*

Proof. Suppose S is described in local coordinates by $x_{k+1} = \cdots = x_n = 0$, then at each $x \in S$, the conormal fiber N_x^*S is described by $\xi_1 = \cdots = \xi_k = 0$. It follows that the symplectic form $\omega = \sum dx_i \wedge d\xi_i$ vanishes on N_x^*S. $\square$

3. Coadjoint orbits

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More details will be added.