LECTURE 3: LOCAL NORMAL FORMS

Contents

1. Isotopy

¶ Some backgrounds on the Lie derivative.

Let M be a smooth manifold, and X a smooth vector field on M. If X is complete, i.e. the integral curve γ_p generated by X starting from p is defined over $\mathbb{R}$ for all p. Then X generates a *flow* $\{\phi_t\}$ on M, i.e.

- For each t, $\phi_t : M \to M$ is a diffeomorphism,
- For each t, s, $\phi_t \circ \phi_s = \phi_{t+s}$.

where ϕ_t is defined explicitly by $\phi_t(p) = \gamma_p(t)$. We remark that if X is not complete, one can still define a *local flow* near any given point on M, which is enough for most of what follows.

Now let α be any k-form on M. Then the *Lie derivative* of α with respect to X is defined to be the k-form

$$\mathcal{L}_X\alpha = \left.\frac{d}{dt}\right|_{t=0} \phi_t^*\alpha. \tag{1}$$

Note that according the group law,

$$\frac{d}{dt}\phi_t^*\alpha = \left.\frac{d}{ds}\right|_{s=0} \phi_t^*\phi_s^*\alpha = \phi_t^*\mathcal{L}_X\alpha. \tag{2}$$

For the Lie derivative, the following formula is very useful and is known as the **Cartan's magic formula**:

$$\mathcal{L}_X\alpha = d\iota_X\alpha + \iota_X d\alpha, \tag{3}$$

where ι_X is the contraction operator. This can be proved via three steps: (1) check the formula holds for functions, (2) check both sides commutes with the differential d, (3) check both sides are derivatives for the algebra $(\Omega^*(M), \wedge)$, e,g, $\mathcal{L}_X(\alpha\wedge\beta) = (\mathcal{L}_X\alpha)\wedge\beta + \alpha\wedge\mathcal{L}_X\beta$ etc.

¶ Isotopies.

Definition 1.1. A smooth family $\phi_t : M \to M$ of diffeomorphisms with $\phi_0 = \mathrm{Id}$ is called an *isotopy*.

For each isotopy ϕ_t one can construct a time-dependent vector field X_t via

$$\frac{d}{dt}\phi_t = X_t(\phi_t),$$

or in other words,

$$X_t(p) = \frac{d}{dt}\bigg|_{s=t} \rho_s(\rho_{-t}(p)).$$

Conversely, given any compactly-supported time dependent vector field X_t, one can construct an isotopy ϕ_t so that the previous relation holds. We will call this isotopy the *flow* generated by X_t. When X_t is not compactly-supported, such a "flow" still exists locally near each point.

One can extend the equation (2) to isotopies:

Lemma 1.2. *Let X_t be a time-dependent vector field with flow ϕ_t. Then for $\forall \alpha \in \Omega^k(M)$,*

$$\frac{d}{dt}\phi_t^*\alpha = \phi_t^*\mathcal{L}_{X_t}\alpha.$$

Sketch. One need to (1) check the formula holds for functions, (2) check both sides commutes with the differential d, (3) check both sides are derivatives for the algebra $(\Omega^*(M), \wedge)$. □

It follows

Proposition 1.3. *Let α_t be a smooth family of k-forms. Then*

$$\frac{d}{dt}\phi_t^*\alpha_t = \phi_t^*\left(\mathcal{L}_{X_t}\alpha_t + \frac{d\alpha_t}{dt}\right).$$

Proof. According to chain rule, for any smooth function $f(x, y)$ of two variables,

$$\frac{d}{dt}f(t,t) = \frac{d}{dx}|_{x=t} f(x,t) + \frac{d}{dy}|_{y=t} f(t,y).$$

Apply this to our case, we get

$$\frac{d}{dt}\phi_t^*\alpha_t = \frac{d}{dx}\bigg|_{x=t}\phi_x^*\alpha_t + \frac{d}{dy}\bigg|_{y=t}\phi_t^*\alpha_y = \phi_t^*\left(\mathcal{L}_{X_t}\alpha_t + \frac{d\alpha_t}{dt}\right).$$

□

Remark. In what follows we will denote $\frac{d\alpha_t}{dt}$ by $\dot{\alpha}(t)$.

¶ Homotopy formula.

Let X_t be a time-dependent vector field whose flow ϕ_t exists for $0 \le t \le 1$. For any $\alpha \in \Omega^k(M)$ one defines $Q_t(\alpha) = \iota_{X_t}(\phi_t^*\alpha)$ and let

$$Q(\alpha) = \int_0^1 Q_t(\alpha)dt.$$

Then Q is a map from $\Omega^k(M)$ to $\Omega^{k-1}(M)$ which is obviously linear.

Theorem 1.4 (Homotopy formula)**.** *$\phi_1^*\alpha - \alpha = dQ(\alpha) + Q(d\alpha)$.*

Proof. One has

$$\frac{d}{dt}\phi_t^*\alpha = \mathcal{L}_{X_t}(\phi_t^*\alpha) = d\iota_{X_t}(\phi_t^*\alpha) + \iota_{X_t}d(\phi_t^*\alpha) = d(Q_t(\alpha)) + Q_t(d\alpha),$$

which implies

$$\phi_t^*\alpha - \alpha = \int_0^1 \left(\frac{d}{dt}\phi_t^*\alpha\right) dt = dQ(\alpha) + Q(d\alpha).$$

□

More generally, let $f, g : M_1 \to M_2$ be two smooth maps that are homotopic, i.e. there exists a smooth map $F : M_1 \times \mathbb{R} \to M_2$ so that $F(p, 0) = f(p)$ and $F(p, 1) = g(p)$ for all $p \in M_1$. Then there exists a homotopy operator $\tilde{Q} : \Omega^k(M_2) \to \Omega^{k-1}(M_1)$ so that

$$g^* - f^* = d\tilde{Q} + \tilde{Q}d. \tag{4}$$

To see this, one just apply the previous theorem to the manifold $W = M_1 \times \mathbb{R}$. In this case the vector field $\frac{\partial}{\partial t}$ is complete with flow $\phi_t(p, a) = (p, a + t)$. So one gets a linear homotopy map $Q : \Omega^k(W) \to \Omega^{k-1}(W)$ such that

$$\phi_1^* - \phi_0^* = dQ + Qd.$$

On the other hand we have $f = F \circ \iota$ and $g = F \circ \phi_1 \circ \iota$, where $\iota : M_1 \hookrightarrow W$ is the inclusion. So one get

$$g^* - f^* = \iota^*\phi_1^*F^* - \iota^*F^* = \iota^*(dQ + Qd)F^* = d\iota^*QF^* + \iota^*QdF^*.$$

The conclusion follows if one take $\tilde{Q} = \iota^*QF^*$.

As a consequence, we see

Corollary 1.5. *Let $\iota : N \hookrightarrow M$ be a submanifold, $\alpha \in \Omega^k(M)$ a closed k-form on M such that $\iota^*\alpha = 0$. Then one can find and a neighborhood $\mathcal{U}$ of N in M and a $(k-1)$-form $\beta \in \Omega^{k-1}(\mathcal{U})$ with $\beta = 0$ on N such that $\alpha = d\beta$ on $\mathcal{U}$.*

Remark. The fact "$\beta = 0$ on N" is in the sense of $(k-1)$-form on M, thus is much stronger than $\iota^*\beta = 0$..

Proof. By choice of Riemannian metric and its exponential map, one can find a neighborhood $\mathcal{U}$ of X in M and a smooth retract of ι onto X, that is, a one-parameter family of smooth maps $r_t : \mathcal{U} \to \mathcal{U}$ and a smooth map $\pi : \mathcal{U} \to X$ such that $r_1 = Id, r_0 = \iota \circ \pi$ and $r_t \circ \iota = \iota$. Applying the homotopy formula (4) one gets

$$\alpha - \pi^*\iota^*\alpha = (d\tilde{Q} + \tilde{Q}d)\alpha.$$

Since α is closed and $\iota^*\alpha = 0$, we see $\alpha = d\tilde{Q}\alpha$. So one only need to take $\beta = \tilde{Q}\alpha$. It remains to check $\beta = 0$ on N. $\square$

2. Moser's trick

¶ Moser's theorem.

This following method was first used by J. Moser in a very short paper in 1965, which turned out to be very useful in many situations, and thus is widely known as *Moser's trick* now.

Suppose we have two k-forms α_0 and α_1 on a smooth manifold M and we are trying to find a diffeomorphism $\phi : M \to M$ such that $\phi^*\alpha_1 = \alpha_0$. Moser's trick is to construct ϕ as the time-1 flow map of a time-dependent vector field X_t on M. In fact, Moser's trick does much more: for a smooth family of k-forms, α_t, connecting α_0 and α_1, one try to find a time-dependent vector field X_t on M so that its flow $\phi_t : M \to M$ satisfies, for all $0 \leq t \leq 1$,

$$\phi_t^*\alpha_t = \alpha_0. \tag{5}$$

To solve the equation (5), one only need to solve

$$0 = \frac{d}{dt}\phi_t^*\alpha_t = \phi_t^*\left(\dot{\alpha}_t + \mathcal{L}_{X_t}\alpha_t\right).$$

Inserting the Cartan's magic formula, the equation to be solved becomes

$$\dot{\alpha}_t + d\iota_{X_t}\alpha_t + \iota_{X_t}d\alpha_t = 0. \tag{6}$$

The last equation is much easier to solve in many cases. As an illustration of this method, we prove

Theorem 2.1 (Moser)**.** *Let M be compact and α_0, α_1 two volume forms on M. Then there exists a diffeomorphism $\phi : M \to M$ such that $\phi^*\alpha_1 = \alpha_0$ if and only if $\int_M \alpha_0 = \int_M \alpha_1$.*

Proof. If such a diffeomorphism exists, then obviously

$$\int_M \alpha_0 = \int_M \phi^*\alpha_1 = \int_{\phi(M)} \alpha_1 = \int_M \alpha_1.$$

Conversely suppose $\int_M \alpha_0 = \int_M \alpha_1$, i.e. $\int_M (\alpha_1 - \alpha_0) = 0$. Then

$$[\alpha_1 - \alpha_0] = 0 \in H^n_{deRham}(M),$$

i.e. there exists $\beta \in \Omega^{n-1}(M)$ so that

$$\alpha_1 - \alpha_0 = d\beta.$$

Now let

$$\alpha_t = (1-t)\alpha_0 + t\alpha_1.$$

Then α_t is a family of volume forms connecting α_0 and α_1, and $\dot{\alpha}_t = \alpha_1 - \alpha_0$. We want to find an isotopy ϕ_t so that $\phi_t^*\alpha_t = \alpha_0$, which implies the theorem. According to Moser's trick, it is enough to solve the equation (6), which, in our case, becomes

$$0 = \dot{\alpha}_t + d\iota_{X_t}\alpha_t + \iota_{X_t}d\alpha_t = d(\beta + \iota_{X_t}\alpha_t).$$

This is always solvable, because one can always find a vector field X_t solving the equation

$$\beta + \iota_{X_t}\alpha_t = 0,$$

since α_t's are volume forms. □

Remark. In fact Moser proved more: there exists a smooth family of diffeomorphisms ϕ_t and a smooth family of α_t such that $\phi_t^*\alpha_t = \alpha_0$.

¶ Classification of 2-dimensional compact symplectic manifolds.

As an application of Moser't theorem, we have

Theorem 2.2 (Classification of compact symplectic surfaces)**.** *Let (M_1, ω_1) and (M_2, ω_2) be two closed 2-dimensional symplectic manifolds. Then they are symplectomorphic if and only if they have the same genus and the same symplectic area.*

Proof. This follows from the fact that *two smooth compact surfaces are diffeomorphic if and only if they have the same genus* together with Moser's theorem. □

Remark. This is no such classification theorem for dimensions ≥ 4.

¶ Deformation of symplectic structure in the same cohomology class.

As another application of Moser's trick, one can prove that a deformation in the same de Rham cohomology class will not give us any *new* symplectic structure, i.e.

Theorem 2.3. *Let M be compact and $\omega_t = \omega_0 + d\beta_t$ a smooth family of symplectic forms on M. Then there exists a smooth family of diffeomorphisms $\phi_t : M \to M$ so that $\phi_t^*\omega_t = \omega_0$.*

Proof. Repeat Moser's argument as before. Now Moser's equation (6) becomes

$$d(\dot{\beta}_t + \iota_{X_t}\omega_t) = 0,$$

and thus it is enough to find find a vector field X_t solving

$$\dot{\beta}_t + \iota_{X_t}\omega_t = 0,$$

which is always solvable because of the non-degeneracy of the symplectic form. □

3. Darboux style theorems

¶ Weinstein's proof of Darboux's theorem.

Before we prove Darboux theorem, we first prove

Theorem 3.1 (Weinstein). *Let M be a smooth manifold and $i : N \hookrightarrow M$ a compact submanifold. Let ω_0 and ω_1 be two symplectic forms on M such that $\omega_0|_N = \omega_1|_X$. Then there exist neighborhoods $\mathcal{U}_0$ and $\mathcal{U}_1$ of N in M and a smooth map $\phi : \mathcal{U}_0 \to \mathcal{U}_1$ such that $\phi|_N = \mathrm{Id}$ and $\phi^*\omega_1 = \omega_0$.*

Proof. Let $\omega_t = (1-t)\omega_0 + t\omega_1$. Since $\omega_t = \omega_0$ on N and N is compact, one can find a tubular neighborhood $\mathcal{U}$ of N so that ω_t is symplectic on $\mathcal{U}$ for all $0 \le t \le 1$. According to the corollary of the homotopy formula above, there exists a 1-form α on $\mathcal{U}$ with $\alpha|_N = 0$ such that $\dot{\omega}_t = \omega_1 - \omega_0 = d\alpha$ on $\mathcal{U}$. Again we solve the equation

$$\iota_{X_t}\omega_t + \alpha = 0$$

to get a vector field X_t on $\mathcal{U}$. Since $\alpha = 0$ on N, we see $X_t = 0$ on N. So we may shrink $\mathcal{U}$ to a neighborhood $\mathcal{U}_0$ of N so that the flow of X_t is defined for $0 \le t \le 1$ on $\mathcal{U}_0$. Now set ϕ to be the time-1 map of the flow of X_t on $\mathcal{U}_0$ and set $\mathcal{U}_1 = \phi(\mathcal{U}_0)$. ☐

As a consequence, we get

Theorem 3.2 (Darboux's theorem). *Let (M, ω) be a symplectic manifold of dimension $2n$. Then for any $p \in M$, there exists a coordinate patch $(\mathcal{U}, x_1, \cdots, x_n, \xi_1, \cdots, \xi_n)$ centered at p such that on $\mathcal{U}$,*

$$\omega = \sum dx_i \wedge d\xi_i.$$

Proof. Pick any symplectic basis $\{x_1'. \cdots, x_n', \xi_1', \cdots, \xi_n'\}$ for the symplectic vector space (T_pM, ω_p), and extend it to a coordinate system in a neighborhood $\mathcal{U}'$ of p. On $\mathcal{U}'$ one has two symplectic forms: the given one $\omega_0 = \omega$, and a new one $\omega_1 = \sum dx_i' \wedge d\xi_i'$. Now apply the previous theorem with $X = \{p\}$ and $M = \mathcal{U}'$, we can find neighborhood $\mathcal{U}_0$ and $\mathcal{U}_1$ of p in $\mathcal{U}'$ and a diffeomorphism $\varphi : \mathcal{U}_0 \to \mathcal{U}_1$ so that $\varphi(p) = p$ and

$$\varphi^*(\sum dx_i' \wedge d\xi_i') = \omega.$$

To complete the proof we only need to set $x_i = \varphi^*(x_i')$ and $\xi_i = \varphi^*(\xi_i')$. ☐

¶ Lagrangian neighborhood theorem.

Theorem 3.3 (Weinstein). *Let M be a smooth manifold of dimension $2n$ and ω_1, ω_2 two symplectic forms on M. Let $\iota : X \hookrightarrow M$ be a submanifold of M which is Lagrangian with respect to both ω_1 and ω_2. Then there exists neighborhoods $\mathcal{U}_0$ and $\mathcal{U}_1$ of X in M and a diffeomorphism $\varphi : \mathcal{U}_0 \to \mathcal{U}_1$ with $\varphi|_X = \mathrm{Id}$ such that $\phi^*\omega_1 = \omega_0$.*

Proof. Student presentation. ☐

As a consequence we can show that near a Lagrangian submanifold the symplectic manifold "looks like" the cotangent bundle!

Theorem 3.4 (Tubular neighborhood theorem for Lagrangian)**.** *Let* (M,ω) *be a symplectic manifold and* $\iota : X \hookrightarrow M$ *a Lagrangian submanifold. Then there exists a neighborhood* $\mathcal{U}_0$ *of* X *in* T^*X*, a neighborhood* $\mathcal{U}$ *of* X *in* M *and a diffeomorphism* $\varphi : \mathcal{U}_0 \to \mathcal{U}$ *with* $\varphi|_X = \mathrm{Id}$ *so that* $\varphi^*\omega = \omega_0$*.*

Proof. We will need

- Let M be a smooth manifold and $\iota : X \hookrightarrow M$ be a submanifold.

  Definition 3.5. The normal bundle NX of X in M is a vector bundle over X whose fiber at $x \in X$ is

  $$N_xX = T_xM/T_xX.$$

- If M is symplectic and X a Lagrangian submanifold, then NX is canonically identified with T^*X: At each point $x \in X$ the symplectic form gives a canonical non-degenerate pairing

  $$N_xX \times T_xX \to \mathbb{R}, ([v], u) \mapsto \omega_x(v, u),$$

  using which one gets a canonical identification of N_xX with T_x^*X.
- The standard tubular neighborhood theorem:

  Theorem 3.6. *Let* M *be a* n *dimensional manifold and* $\iota : X \hookrightarrow M$ *a* k*-dimensional submanifold. Then there exist a neighborhood* $\mathcal{U}_0$ *of* X *in* NX*, a neighborhood* $\mathcal{U}$ *of* X *in* M *and a diffeomorphism* $\psi : \mathcal{U}_0 \to \mathcal{U}$ *so that* $\psi|_X = \mathrm{Id}$.

Back to the the theorem. By the facts above, we can find a neighborhood $\mathcal{U}_0$ of X in T^*X, a neighborhood U of X in M and a diffeomorphism $\psi : \mathcal{U}_0 \to \mathcal{U}$ such that $\psi|_X = \mathrm{Id}$. Now on the manifold $\mathcal{U}_0$ one has two symplectic forms: $\omega_0 =$ the canonical symplectic form on T^*X, and $\omega_1 = \psi^*\omega$. Moreover, X is a Lagrangian submanifold with respect to both symplectic forms. Now the theorem follows from theorem 3.3. ☐

Remark. The theorem extends further to isotropic submanifolds.