LECTURE 4: SYMPLECTOMORPHISMS

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1. THE GROUP OF SYMPLECTOMORPHISMS

¶ Diffeomorphisms v.s. vector fields.

Let $M$ be a smooth manifold and $\text{Vect}(M)$ the set of all smooth vectors on $M$. It is well known that for any $X, Y \in \text{Vect}(M)$, the Lie bracket

$$[X,Y] = XY - YX \in \text{Vect}(M)$$

and is bilinear, anti-symmetric and satisfies the Jacobi identity

$$[[X,Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0.$$

As a consequence, $(\text{Vect}(M), [\cdot, \cdot])$ is an (infinitely dimensional) Lie algebra. What is the corresponding “Lie group”?

Well, any smooth vector field $X \in \text{Vect}(M)$ generates (at least locally) a one-parameter subgroup of diffeomorphisms of $M$

$$\rho_t = \exp(tX) : M \to M, \quad \rho_t(x) = \gamma^X_x(t),$$

where $\gamma^X_x$ is the integral curve of $X$ starting at $x$. Conversely, given any one parameter subgroup $\rho_t$ of diffeomorphisms on $M$, one gets a smooth vector field via

$$X(x) = \left. \frac{d}{dt} \right|_{t=0} \rho_t(x).$$

So, the group of diffeomorphisms,

$$\text{Diff}(M) = \{ \varphi : M \to M \mid \varphi \text{ is a diffeomorphism} \},$$

is the “Lie group” whose Lie algebra is $\text{Vect}(M)$. 
The group of symplectomorphisms.

Now let \((M, \omega)\) be a symplectic manifold and 
\[
\text{Symp}(M, \omega) = \{ \varphi : M \to M \mid \varphi \text{ is a symplectomorphism} \}
\]
the group of symplectomorphisms of \((M, \omega)\). This is a “closed” subgroup of \(\text{Diff}(M)\).
Both \(\text{Symp}(M, \omega)\) and \(\text{Diff}(M)\) are large in the sense that they are infinitely dimensional.

Example. We have seen in lecture 2 that any diffeomorphism \(\varphi : X \to X\) lifts to a symplectomorphism \(\tilde{\varphi} : T^*X \to T^*X\) via
\[
\tilde{\varphi}(x, \xi) = (\varphi(x), (d\varphi^*)_x^{-1}(\xi)).
\]
It is easy to check \(\tilde{\varphi}_2 \circ \tilde{\varphi}_1 = \tilde{\varphi}_2 \circ \tilde{\varphi}_1\). So we get a natural group monomorphism
\[
\text{Diff}(X) \to \text{Symp}(T^*X, \omega_{\text{can}}).
\]

Example. Here is another subgroup of \(\text{Symp}(T^*X, \omega_{\text{can}})\): For any \(\beta \in \Omega^1(X)\), let 
\[
G_{\beta} : T^*X \to T^*X \text{ be the diffeomorphism}
\]
\[
G_{\beta}(x, \xi) = (x, \xi + \beta_x).
\]

Lemma 1.1. \(G_{\beta}\) is a symplectomorphism if and only if \(\beta\) is closed.

Proof. We shall prove \(G_{\beta}^*\alpha_{\text{can}} - \alpha_{\text{can}} = \pi^*\beta\), which implies the conclusion. Recall the reproducing property of \(\alpha_{\text{can}}\): \(\alpha_{\text{can}}\) is the unique 1-form on \(T^*X\) such that for any \(\mu \in \Omega^1(X)\), 
\[
s_{\mu}^*\alpha = \mu.
\]
It follows
\[
s_{\mu}^*G_{\beta}^*\alpha_{\text{can}} = (G_{\beta} \circ s_{\mu})^*\alpha_{\text{can}} = s_{\mu + \beta}^*\alpha_{\text{can}} = \mu + \beta = s_{\mu}^*\pi^*\beta + \mu.
\]
So by reproducing property again we see 
\[
G_{\beta}^*\alpha_{\text{can}} - \pi^*\beta = \alpha_{\text{can}}.
\]

As a consequence, we get another group monomorphism
\[
Z^1(X) \to \text{Symp}(T^*X, \omega_{\text{can}}),
\]
where \(Z^1(X)\) is the space of closed 1-forms on \(X\).

Symplectic vector fields.

Question: What is the “Lie algebra” of \(\text{Symp}(M, \omega)\)? Well, since \(\text{Symp}(M, \omega)\) is a “closed” subgroup of \(\text{Diff}(M)\), its Lie algebra should be a Lie subalgebra of \((\text{Vect}(M), [\cdot, \cdot])\). Let’s try to find the condition for a smooth vector to be “symplectic”.

A symplectic vector field \(X\) should be a vector field whose flow \(\{ \rho_t \}\) consists of symplectomorphisms. In other words \(\rho_t^*\omega = \omega\) for all \(t\). It follows
\[
0 = \frac{d}{dt}\rho_t^*\omega = \rho_t^*\mathcal{L}_X\omega.
\]

Definition 1.2. A smooth vector field \(X\) is called symplectic if \(\mathcal{L}_X\omega = 0\).

Since \(\omega\) is closed, the Cartan’s magic formula implies
Lemma 1.3. A vector field $X$ on $(M, \omega)$ is symplectic if and only if $\iota_X \omega$ is closed.

The set of all symplectic vector fields on $(M, \omega)$ is denoted by $\text{Vect}(M, \omega)$. We need to check that if $X, Y \in \text{Vect}(M, \omega)$, so is $[X, Y]$. In other words, $(\text{Vect}(M, \omega), [\cdot, \cdot])$ is a Lie sub algebra of $(\text{Vect}(M), [\cdot, \cdot])$. This follows from

Lemma 1.4. If $X, Y$ are symplectic, $\iota_{[X,Y]} \omega = d(-\omega(X,Y))$.

Proof. By the definition of exterior differential, for any $Z \in \text{Vect}(M)$,

$0 = (d\omega)(X,Y,Z) = X(\omega(Y,Z)) - Y(\omega(X,Z)) + Z(\omega(X,Y)) - \omega([X,Y], Z) + \omega([X, Z], Y) - \omega([Y, Z], X)$

$0 = (d\iota_X \omega)(Y,Z) = Y(\omega(X,Z)) - Z(\omega(X,Y)) - \omega(X, [Y,Z]),$

$0 = (d\iota_Y \omega)(X,Z) = X(\omega(Y,Z)) - Z(\omega(Y,X)) - \omega(Y, [X,Z]).$

Comparing the three equations we conclude

$-Z(\omega(X,Y)) - \omega([X,Y], Z) = 0,$

or in other words,

$\iota_{[X,Y]} \omega = d(-\omega(X,Y)).$

$\square$

Remark. So the bracket $[X,Y]$ of two symplectic vector fields is better than being a symplectic vector field: $\iota_{[X,Y]} \omega$ is not only closed, but in fact exact!.

¶ Hamiltonian vector fields.

Definition 1.5. A vector field $X$ is Hamiltonian if $\iota_X \omega$ is exact.

Remark. According to the lemma above, the space of all Hamiltonian vector fields is an ideal of $\text{Vect}(M, \omega)$.

So if $X$ is hamiltonian, then there exists a smooth function $f \in C^\infty(M)$ so that $\iota_X \omega = df$. Conversely, since $\omega$ is non-degenerate, for any $f \in C^\infty(M)$, there is a unique vector field $X_f$ on $M$ so that

$\iota_{X_f} \omega = df.$

We will call $X_f$ the Hamiltonian vector field associated to the Hamiltonian function $f$. The flow generated by $X_f$ is called the Hamiltonian flow associated to $f$.

Remark. Assume $M$ is connected. Then two smooth functions define the same Hamiltonian vector field if and only if they differ by a constant. So as a vector space the space of Hamiltonian vector fields is isomorphic to $C^\infty(M)/\mathbb{R}$, which can also be identified with

$C^\infty(M)_0 = \{f \in C^\infty(M) \mid \int_M f(x) dx = 0.\}$

if $M$ is compact.
Example. On the 2-sphere $S^2$ with symplectic form $\omega = d\theta \wedge dz$, if we take $f(z, \theta) = z$ the height function, then $X_f = \frac{\partial}{\partial \theta}$, and the Hamiltonian flow is the rotation about the vertical axis:
$$\rho_t(z, \theta) = (z, \theta + t).$$

Example. On $(\mathbb{T}^2, d\theta_1 \wedge d\theta_2)$ the vector fields $\frac{\partial}{\partial \theta_1}$ and $\frac{\partial}{\partial \theta_2}$ are both symplectic but not Hamiltonian.

We will return to Hamiltonian vector fields later.

### Hamiltonian symplectomorphisms.

Recall that an isotopy is a family of diffeomorphisms $\rho_t$ so that $\rho_0 = \text{Id}$. If each $\rho_t$ is a symplectomorphism, we call the isotopy a *symplectic isotopy*. It is easy to see that a symplectic isotopy is generated by a family of symplectic vector fields $X_t$ with
$$\frac{d}{dt}\rho_t = X_t(\rho_t).$$
If each $X_t$ is not only symplectic, but in fact Hamiltonian, then we call the isotopy a *Hamiltonian isotopy*.

**Definition 1.6.** A symplectomorphism $\varphi$ is called *Hamiltonian symplectomorphism* if there exists a Hamiltonian isotopy $\rho_t$ such that $\rho_1 = \varphi$.

The space of Hamiltonian symplectomorphisms is denoted by $\text{Ham}(M, \omega)$. It turns out that $\text{Ham}(M, \omega)$ is a normal subgroup of $\text{Symp}(M, \omega)$ whose Lie algebra is the algebra of all Hamiltonian vector fields.

Since any function (modulo constants) gives a family of Hamiltonian symplectomorphisms, we see that the group of symplectomorphisms is huge.

### 2. Symplectomorphisms as Lagrangian submanifolds

**Lagrangian submanifolds v.s. symplectomorphisms.**

Weinstein’s symplectic creed:

**EVERYTHING IS A LAGRANGIAN SUBMANIFOLD!**

In what follows we shall study symplectomorphisms according to this creed.

Let $(M_1, \omega_1)$ and $(M_2, \omega_2)$ be $2n$ dimensional symplectic manifolds and let $\text{pr}_i : M_1 \times M_2 \to M_i$ be the projection. We have seen in lecture 2 that for any nonzero real numbers $\lambda_1$ and $\lambda_2$, $\lambda_1 \text{pr}_1^* \omega_1 + \lambda_2 \text{pr}_2^* \omega_2$ is a symplectic form on the product $M = M_1 \times M_2$. In particular, one has two important symplectic forms:

- the *product symplectic form* $\omega = \text{pr}_1^* \omega_1 + \text{pr}_2^* \omega_2$,
- the *twisted product form* $\bar{\omega} = \text{pr}_1^* \omega_1 - \text{pr}_2^* \omega_2$. 

Now let \( f : M_1 \to M_2 \) be a diffeomorphism, then its graph
\[
\Gamma_f = \{(x, f(x)) \mid x \in M_1\}
\]
is a \( 2n \) dimensional submanifold of the \( 4n \) dimensional manifold \( M_1 \times M_2 \).

**Theorem 2.1.** \( f \) is a symplectomorphism if and only if \( \Gamma_f \) is Lagrangian with respect to \( \tilde{\omega} \).

**Proof.** Let \( \iota : \Gamma_f \hookrightarrow M \) be the inclusion and \( \gamma : M_1 \to \Gamma_f \) be the obvious diffeomorphism, then
\[
\Gamma_f \text{ is Lagrangian } \iff \iota^* \tilde{\omega} = 0 \\
\iff \gamma^* \iota^* \tilde{\omega} = 0 \\
\iff \gamma^* \iota^* (\text{pr}_1^* \omega_1 - \text{pr}_2^* \omega_2) = 0 \\
\iff \omega_1 - f^* \omega_2 = 0 \\
\iff f \text{ is a symplectomorphism.}
\]
\[\square\]

**Lift of smooth maps as Lagrangian submanifolds.**

In particular, suppose \( M_1 = T^* X_1 \) and \( M_2 = T^* X_2 \) be cotangent bundles and \( \omega_1 = -d\alpha_1, \omega_2 = -d\alpha_2 \) the canonical symplectic forms. Then \( M = M_1 \times M_2 = T^* X \), where \( X = X_1 \times X_2 \). Moreover, the canonical 1-form on \( M = T^* X \) is \( \alpha = \alpha_1 \oplus \alpha_2 \), so the product symplectic form \( \omega = \omega_1 \oplus \omega_2 \) on \( M = T^* X \) is the canonical symplectic form. Let
\[
\sigma_2 : M_2 \to M_2, (x, \xi) \mapsto (x, -\xi).
\]
Then \( \sigma_2^* \alpha_2 = -\alpha_2 \), and thus \( \sigma_2^* \omega_2 = -\omega_2 \). It follows theorem that

**Proposition 2.2.** If \( f : M_1 \to M_2 \) is a diffeomorphism, then \( f \) is a symplectomorphism if and only if \( \Gamma_{\sigma_2 \circ f} \) is a Lagrangian submanifold of \( (M, \omega) \).

Now suppose \( g : X_1 \to X_2 \) is a diffeomorphism. As we have seen, \( g \) lifts to a symplectomorphism \( \tilde{g} : M_1 \to M_2 \). Recall that
\[
\tilde{g}(x, \xi) = (y, \eta) \iff y = g(x), \xi = (dg_x)^* \eta.
\]
As a consequence,
\[
\Gamma_{\tilde{g}} = \{(x, \xi, y, \eta) \mid y = g(x), \xi = -(dg_x)^* \eta\}
\]
is a Lagrangian submanifold of \( M = T^* X \). Here is another way to see this: Let
\[
X_g = \{(x, g(x)) \mid x \in X_1\} \subset X = X_1 \times X_2
\]
be the graph of \( g \), then \( \Gamma_{\tilde{g}} = N^* X_g \). In fact, we have a more general theorem:

**Theorem 2.3.** Let \( X_1, X_2 \) be arbitrary smooth manifolds and \( g : X_1 \to X_2 \) a smooth map, then the set \( \Gamma_{\tilde{g}} \) defined as above is exactly \( N^* X_g \), and thus a Lagrangian submanifold of \( M = T^* X \).
Proof. For any \((x, g(x)) \in X_g\),

\[ T_{(x,g(x))}X_g = \{(v, dg_x(v)) \mid v \in T_x X_1\} \]

By definition, \(N^*_{(x,g(x))}X_g\) is the subspace of \(T^*_{(x,g(x))}X_g\) that annihilates \(T_{(x,g(x))}X_g\), so

\[(\xi, \eta) \in N^*_{(x,g(x))}X_g \quad \iff \quad \langle \xi, v \rangle + \langle \eta, dg_x(v) \rangle = 0, \forall v \in T_x X_1\]

\[ \iff \langle \xi + (dg_x)^* \eta, v \rangle = 0, \forall v \in T_x X_1 \]

\[ \iff \xi = -(dg_x)^* \eta. \]

\(\Box\)

### Fixed points of symplectomorphisms.

The identity map \(\text{Id} : M \to M\) is a symplectomorphism, and the corresponding Lagrangian submanifold of \((M \times M, \tilde{\omega})\) is the diagonal \(\Delta = \{(x, x) \mid x \in M\}\). We can canonically identify \(\Delta\) with \(M\). According to the Weinstein’s Lagrangian neighborhood theorem, there exists a neighborhood \(U\) of \(\Delta \simeq M\) in \((M \times M, \tilde{\omega})\), a neighborhood \(U_0\) of \(M\) in \(T^*M\) and a symplectomorphism \(\varphi : U_0 \to U\).

**Definition 2.4.** We say a diffeomorphism \(f : M \to M\) is \(C^1\) close to \(\text{Id}\) if its graph \(\Gamma_f\) lies in \(U\), which, under the map \(\varphi^{-1}\), is the graph of a 1-form \(\mu\) on \(M\).

Now let \((M, \omega)\) be a compact symplectic manifold and \(f \in \text{Symp}(M, \omega)\) a symplectomorphism that is \(C^1\) close to the identity symplectomorphism. Since \(\Gamma_f\) is a Lagrangian submanifold of \((M \times M, \tilde{\omega})\) and \(\varphi\) is a symplectomorphism, \(\varphi^{-1}(\Gamma_f)\) is a Lagrangian submanifold of \((T^*M, \omega_{can})\). So if it is the graph of a one-form \(\mu\), \(\mu\) must be closed.

**Remark.** By this way we get an identification of a \(C^1\) neighborhood of \(\text{Id}\) with a neighborhood of 0 in the space of closed 1-forms. So the tangent space of \(\text{Id}\) in \(\text{Symp}(M, \omega)\) can be identified with the space of closed 1-forms on \(M\). This coincides with our earlier “Lie group-Lie algebra” observation, since symplectic vector fields are in one-to-one correspondence with closed 1-forms under \(\omega\).

**Theorem 2.5.** Let \((M, \omega)\) be a compact symplectic manifold with \(H^1(M) = 0\). Then any symplectomorphism of \(M\) which is sufficiently \(C^1\) close to \(\text{Id}\) has at least two fixed points.

**Proof.** Let \(f \in \text{Symp}(M, \omega)\) is \(C^1\) close to \(\text{Id}\). Then under the map \(\varphi\), the graph of \(f\) is identified with a closed one-form \(\mu\) on \(M\). Since \(H^1(M) = 0\), one can find a smooth function \(h \in C^\infty(M)\) so that \(\mu = dh\). Since \(M\) is compact, \(h\) admits at least two critical points (the maximum and the minimum). Obviously any critical point of \(h\) gives an intersection point of \(\Gamma_f\) with \(\Delta\), which yields a fixed point of \(f\). \(\Box\)
The Arnold conjecture.

**Conjecture 2.6** (Arnold, symplectomorphism version). Let \((M, \omega)\) be a compact symplectic manifold and \(f : M \to M\) a Hamiltonian symplectomorphism. Then

\[\#\text{(fixed points of } f) \geq \text{minimal number of critical points of a Morse function on } M.\]

**Conjecture 2.7** (Arnold, Lagrangian version). Let \((M, \omega)\) be a compact symplectic manifold, \(L \subset M\) a Lagrangian submanifold, and \(f : M \to M\) a Hamiltonian symplectomorphism. Then

\[\#(L \cap f(L)) \geq \text{minimal number of critical points of a Morse function on } L.\]

Note that by Morse theory, the minimal number of critical points of a Morse function on \(M\) is at least \(\sum_i \dim H^i(M)\).

The conjecture is only proven in special cases via the theory of Floer homology.

### 3. Generating functions

**Generating function for horizontal Lagrangian submanifolds.**

Let \(M = T^*X\) be the cotangent bundle of any smooth manifold \(X\) and \(\omega\) the canonical symplectic form. We have seen that a horizontal submanifold

\[X_\mu = \{(x, \mu_x) \mid x \in X\},\]

is Lagrangian if and only if \(d\mu = 0\).

**Definition 3.1.** If \(\mu\) is exact, i.e. \(\mu = d\varphi\) for some smooth function \(\varphi \in C^\infty(X)\), then we call \(\varphi\) a generating function of the Lagrangian submanifold \(\Lambda_\mu\).

Note that proposition 2.2 is equivalent to

The graph of \(f\) is a Lagrangian \(\Leftrightarrow \sigma_2 \circ f\) is a symplectomorphism.

From this correspondence it is natural to define

**Definition 3.2.** If \(\Gamma_f = \Lambda_{d\varphi}\) for some \(\varphi \in C^\infty(X_1 \times X_2)\), we say \(\varphi\) a generating function for the symplectomorphism \(\sigma_2 \circ f\).

*Remark.* Usually one only need to find generating functions *locally.*

**Constructing symplectomorphisms.**

Now suppose we have a Lagrangian submanifold \(\Lambda_{d\varphi}\) generated by function \(\varphi\). When will it generate a symplectomorphism? In other words, we want \(\Lambda_{d\varphi}\) to be the graph of some diffeomorphism \(f : M_1 \to M_2\). We denote \(pr_i : M = M_1 \times M_2 \to M_i\) be the projection maps. We choose local coordinate patches \((U_1, x_1, \cdots, x_n)\) and \((U_2, y_1, \cdots, y_n)\) on \(X_1\) and \(X_2\) respectively. Then \(\Lambda_{d\varphi}\) is described locally by the
equations $\xi_i = \frac{\partial \varphi}{\partial x_i}(x, y), \eta_i = -\frac{\partial \varphi}{\partial y_i}(x, y)$. Therefore, given any point $(x, \xi) \in M_1$, to find its image $(y, \eta) = f(x, \xi)$ we need to solve the equations

\begin{equation}
\begin{aligned}
\xi_i &= \frac{\partial \varphi}{\partial x_i}(x, y), \\
\eta_i &= -\frac{\partial \varphi}{\partial y_i}(x, y).
\end{aligned}
\end{equation}

According to the implicit function theorem, to solve the first equation $\xi_i = \frac{\partial \varphi}{\partial x_i}(x, y)$ for $y$ locally, we need the condition

\begin{equation}
\det \left[ \frac{\partial^2 \varphi}{\partial x_i \partial y_j} \right] \neq 0.
\end{equation}

Of course after solving $y$ we may feed it into the second equation to get $\eta$.

### Examples of generating functions.

**Example.** Let $X_1 = X_2 = \mathbb{R}^n$ and $B = (b_{ij})$ a non-singular $n \times n$ matrix. Then the function $\varphi(x, y) = \sum b_{ij} x_i y_j$ generates a linear symplectomorphism $T_B : T^*\mathbb{R}^n \to T^*\mathbb{R}^n$ which maps $(x, \xi)$ to $(B^{-1} \xi, -B^T x)$.

In particular, if $B = I$, i.e. $\varphi(x, y) = \sum x_i y_i$, then $T_B$ maps $(x, \xi)$ to $(\xi, -x)$.

**Example.** Let $X_1 = X_2 = \mathbb{R}^n$ and $\varphi(x, y) = -\frac{|x-y|^2}{2}$. Then equation (1) becomes

\begin{equation}
\begin{aligned}
\xi_i &= \frac{\partial \varphi}{\partial x_i}(x, y) = y_i - x_i \\
\eta_i &= -\frac{\partial \varphi}{\partial y_i}(x, y) = y_i - x_i \iff \begin{cases} 
\ y_i = x_i + \xi_i, \\
\ "\eta_i = \xi_i.
\end{cases}
\end{aligned}
\end{equation}

So the symplectomorphism generated by $\varphi$ is $f(x, \xi) = (x + \xi, \xi)$.

More generally, if $X$ is a Riemannian manifold and $\varphi(x, y) = -\frac{d(x, y)^2}{2}$, where $d(x, y)$ is the Riemannian distance from $x$ to $y$, then the symplectomorphism generated by $\varphi$ is the geodesic flow.

**Example.** Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n \times (\mathbb{R}^n)^*$ and $\varphi = \varphi(x, \eta) \in C^\infty(\mathcal{O})$ be a twisted generating function. Suppose $\det(\frac{\partial^2 \varphi}{\partial x_i \partial y_j}) \neq 0$. Then by the same argument above or by composing the symplectomorphism we solved from (1) with the symplectomorphism $(x, \xi) \to (\xi, -x)$, we see that locally the set defined by

$\xi_i = \frac{\partial \varphi}{\partial x_i}(x, \eta), \ y_i = \frac{\partial \varphi}{\partial \eta_i}(x, \eta)$

is the graph of a symplectomorphism.

**Example.** The identity symplectomorphism $\text{Id} : T^*\mathbb{R}^n \to T^*\mathbb{R}^n$ cannot be generated by functions of the usual form. However, if we take a twisted generating function $\varphi(x, \eta) = \sum x_i \eta_i$, then it generates the identity symplectomorphism.

**Example.** More generally, if $\mathcal{U}_1$ is an open subset of $\mathbb{R}^n$ and $f : \mathcal{U}_1 \to \mathcal{U}_2$ a diffeomorphism, then we have seen that its canonical lifting $\tilde{f} : T^*\mathcal{U}_1 \to T^*\mathcal{U}_2$ is a symplectomorphism. One can check that this is generated by $\varphi(x, \eta) = \sum f_i(x) \eta_i$. 


4. THE BILLIARDS

Student presentation.