LECTURE 4: SYMPLECTOMORPHISMS

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1. The group of Symplectomorphisms

¶ Diffeomorphisms v.s. vector fields.

Let M be a smooth manifold and $\mathrm{Vect}(M)$ the set of all smooth vectors on M. It is well known that for any $X, Y \in \mathrm{Vect}(M)$, the Lie bracket

$$[X, Y] = XY - YX \in \mathrm{Vect}(M)$$

and is bilinear, anti-symmetric and satisfies the Jacobi identity

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0.$$

As a consequence, $(\mathrm{Vect}(M), [\cdot, \cdot])$ is an (infinitely dimensional) Lie algebra. What is the corresponding "Lie group"?

Well, any smooth vector field $X \in \mathrm{Vect}(M)$ generates (at least locally) a one-parameter subgroup of diffeomorphisms of M

$$\rho_t = \exp(tX) : M \to M, \quad \rho_t(x) = \gamma_x^X(t),$$

where γ_x^X is the integral curve of X starting at x. Conversely, given any one parameter subgroup ρ_t of diffeomorphisms on M, one gets a smooth vector field via

$$X(x) = \left.\frac{d}{dt}\right|_{t=0} \rho_t(x).$$

So, the group of diffeomorphisms,

$$\mathrm{Diff}(M) = \{\varphi : M \to M \mid \varphi \text{ is a diffeomorphism}\},$$

is the "Lie group" whose Lie algebra is $\mathrm{Vect}(M)$.

¶ The group of symplectomorphisms.

Now let (M,ω) be a symplectic manifold and

$$\mathrm{Symp}(M,\omega)=\{\varphi : M\to M \mid \varphi \text{ is a symplectomorphism}\}$$

the group of symplectomorphisms of (M,ω). This is a "closed" subgroup of $\mathrm{Diff}(M)$. Both $\mathrm{Symp}(M,\omega)$ and $\mathrm{Diff}(M)$ are large in the sense that they are infinitely dimensional.

Example. We have seen in lecture 2 that any diffeomorphism $\varphi : X\to X$ lifts to a symplectomorphism $\tilde\varphi : T^*X\to T^*X$ via

$$\tilde\varphi(x,\xi)=(\varphi(x),(d\varphi_x^*)^{-1}(\xi)).$$

It is easy to check $\widetilde{\varphi_2\circ\varphi_1}=\tilde\varphi_2\circ\tilde\varphi_1$. So we get a natural group monomorphism

$$\mathrm{Diff}(X)\to\mathrm{Symp}(T^*X,\omega_{can}).$$

Example. Here is another subgroup of $\mathrm{Symp}(T^*X,\omega_{can})$: For any $\beta\in\Omega^1(X)$, let $G_\beta : T^*X\to T^*X$ be the diffeomorphism

$$G_\beta(x,\xi)=(x,\xi+\beta_x).$$

Lemma 1.1. *G_β is a symplectomorphism if and only if β is closed.*

Proof. We shall prove $G_\beta^*\alpha_{can}-\alpha_{can}=\pi^*\beta$, which implies the conclusion. Recall the reproducing property of α_{can}: α_{can} is the unique 1-form on T^*X such that for any $\mu\in\Omega^1(X)$, $s_\mu^*\alpha=\mu$. It follows

$$s_\mu^*G_\beta^*\alpha_{can}=(G_\beta\circ s_\mu)^*\alpha_{can}=s_{\mu+\beta}^*\alpha_{can}=\mu+\beta=s_\mu^*\pi^*\beta+\mu.$$

So by reproducing property again we see $G_\beta^*\alpha_{can}-\pi^*\beta=\alpha_{can}$. □

As a consequence, we get another group monomorphism

$$Z^1(X)\to\mathrm{Symp}(T^*X,\omega_{can}),$$

where $Z^1(X)$ is the space of closed 1-forms on X.

¶ Symplectic vector fields.

Question: What is the "Lie algebra" of $\mathrm{Symp}(M,\omega)$? Well, since $\mathrm{Symp}(M,\omega)$ is a "closed" subgroup of $\mathrm{Diff}(M)$, its Lie algebra should be a Lie subalgebra of $(\mathrm{Vect}(M),[\cdot,\cdot])$. Let's try to find the condition for a smooth vector to be "symplectic".

A symplectic vector field X should be a vector field whose flow $\{\rho_t\}$ consists of symplectomorphisms. In other words $\rho_t^*\omega=\omega$ for all t. It follows

$$0=\frac{d}{dt}\rho_t^*\omega=\rho_t^*\mathcal{L}_X\omega.$$

Definition 1.2. A smooth vector field X is called *symplectic* if $\mathcal{L}_X\omega=0$.

Since ω is closed, the Cartan's magic formula implies

Lemma 1.3. *A vector field X on (M, ω) is symplectic if and only if $\iota_X\omega$ is closed.*

The set of all symplectic vector fields on (M, ω) is denoted by $\mathrm{Vect}(M, \omega)$. We need to check that if $X, Y \in \mathrm{Vect}(M, \omega)$, so is $[X, Y]$. In other words, $(\mathrm{Vect}(M, \omega), [\cdot, \cdot])$ is a Lie sub algebra of $(\mathrm{Vect}(M), [\cdot, \cdot])$. This follows from

Lemma 1.4. *If X, Y are symplectic, $\iota_{[X,Y]}\omega = d(-\omega(X, Y))$..*

Proof. By the definition of exterior differential, for any $Z \in \mathrm{Vect}(M)$,

$$0 = (d\omega)(X, Y, Z) = X(\omega(Y, Z)) - Y(\omega(X, Z)) + Z(\omega(X, Y)) - \omega([X, Y], Z) + \omega([X, Z], Y) - \omega([Y, Z], X)$$

$$0 = (d\iota_X\omega)(Y, Z) = Y(\omega(X, Z)) - Z(\omega(X, Y)) - \omega(X, [Y, Z]),$$

$$0 = (d\iota_Y\omega)(X, Z) = X(\omega(Y, Z)) - Z(\omega(Y, X)) - \omega(Y, [X, Z]).$$

Comparing the three equations we conclude

$$-Z(\omega(X, Y)) - \omega([X, Y], Z) = 0,$$

or in other words,

$$\iota_{[X,Y]}\omega = d(-\omega(X, Y)).$$

□

Remark. So the bracket $[X, Y]$ of two symplectic vector fields is better than being a symplectic vector field: $\iota_{[X,Y]}\omega$ is not only closed, but in fact *exact*!.

¶ Hamiltonian vector fields.

Definition 1.5. A vector field X is *Hamiltonian* if $\iota_X\omega$ is exact.

Remark. According to the lemma above, the space of all Hamiltonian vector fields is an ideal of $\mathrm{Vect}(M, \omega)$.

So if X is hamiltonian, then there exists a smooth function $f \in C^\infty(M)$ so that $\iota_X\omega = df$. Conversely, since ω is non-degenerate, for any $f \in C^\infty(M)$, there is a unique vector field X_f on M so that

$$\iota_{X_f}\omega = df.$$

We will call X_f the *Hamiltonian vector field* associated to the *Hamiltonian function* f. The flow generated by X_f is called the *Hamiltonian flow* associated to f.

Remark. Assume M is connected. Then two smooth functions define the same Hamiltonian vector field if and only if they differ by a constant. So as a vector space the space of Hamiltonian vector fields is isomorphic to $C^\infty(M)/\mathbb{R}$, which can also be identified with

$$C^\infty(M)_0 = \{f \in C^\infty(M) \mid \int_M f(x)dx = 0.\}$$

if M is compact.

Example. On the 2-sphere S^2 with symplectic form $\omega = d\theta \wedge dz$, if we take $f(z,\theta) = z$ the height function, then $X_f = \frac{\partial}{\partial\theta}$, and the Hamiltonian flow is the rotation about the vertical axis:

$$\rho_t(z,\theta) = (z, \theta + t).$$

Example. On $(\mathbb{T}^2, d\theta_1 \wedge d\theta_2)$ the vector fields $\frac{\partial}{\partial\theta_1}$ and $\frac{\partial}{\partial\theta_1}$ are both symplectic but not Hamiltonian.

We will return to Hamiltonian vector fields later.

¶ Hamiltonian symplectomorphisms.

Recall that an isotopy is a family of diffeomorphisms ρ_t so that $\rho_0 = \mathrm{Id}$. If each ρ_t is a symplectomorphism, we call the isotopy a *symplectic isotopy*. It is easy to see that a symplectic isotopy is generated by a family of symplectic vector fields X_t with

$$\frac{d}{dt}\rho_t = X_t(\rho_t).$$

If each X_t is not only symplectic, but in fact Hamiltonian, then we call the isotopy a *Hamiltonian isotopy*.

Definition 1.6. A symplectomorphism φ is called *Hamiltonian symplectomorphism* if there exists a Hamiltonian isotopy ρ_t such that $\rho_1 = \varphi$.

The space of Hamiltonian symplectomorphisms is denoted by $\mathrm{Ham}(M,\omega)$. It turns out that $\mathrm{Ham}(M,\omega)$ is a normal subgroup of $\mathrm{Symp}(M,\omega)$ whose Lie algebra is the algebra of all Hamiltonian vector fields.

Since any function (modulo constants) gives a family of Hamiltonian symplectomorphisms, we see that the group of symplectomorphisms is huge.

2. Symplectomorphisms as Lagrangian submanifolds

¶ Lagrangian submanifolds v.s. symplectomorphisms.

Weinstein's symplectic creed:

EVERYTHING IS A LAGRANGIAN SUBMANIFOLD!

In what follows we shall study symplectomorphisms according to this creed.

Let (M_1,ω_1) and (M_2,ω_2) be $2n$ dimensional symplectic manifolds and let $\mathrm{pr}_i : M_1 \times M_2 \to M_i$ be the projection. We have seen in lecture 2 that for any nonzero real numbers λ_1 and λ_2, $\lambda_1 \mathrm{pr}_1^*\omega_1 + \lambda_2 \mathrm{pr}_2^*\omega_2$ is a symplectic form on the product $M = M_1 \times M_2$. In particular, one has two important symplectic forms:

- the *product symplectic form* $\omega = \mathrm{pr}_1^*\omega_1 + \mathrm{pr}_2^*\omega_2$,
- the *twisted product form* $\tilde{\omega} = \mathrm{pr}_1^*\omega_1 - \mathrm{pr}_2^*\omega_2$.

Now let $f : M_1 \to M_2$ be a diffeomorphism, then its graph

$$\Gamma_f = \{(x, f(x) \mid x \in M_1\}$$

is a $2n$ dimensional submanifold of the $4n$ dimensional manifold $M_1 \times M_2$.

Theorem 2.1. *f is a symplectomorphism if and only if Γ_f is Lagrangian with respect to $\tilde{\omega}$.*

Proof. Let $\iota : \Gamma_f \hookrightarrow M$ be the inclusion and $\gamma : M_1 \to \Gamma_f$ be the obvious diffeomorphism, then

$$\begin{aligned}
\Gamma_f \text{ is Lagrangian } &\iff \iota^*\tilde{\omega} = 0 \\
&\iff \gamma^*\iota^*\tilde{\omega} = 0 \\
&\iff \gamma^*\iota^*(\mathrm{pr}_1^*\omega_1 - \mathrm{pr}_2^*\omega_2) = 0 \\
&\iff \omega_1 - f^*\omega_2 = 0 \\
&\iff f \text{ is a symplectomorphism.}
\end{aligned}$$

□

¶ Lift of smooth maps as Lagrangian submanifolds.

In particular, suppose $M_1 = T^*X_1$ and $M_2 = T^*X_2$ be cotangent bundles and $\omega_1 = -d\alpha_1, \omega_2 = -d\alpha_2$ the canonical symplectic forms. Then $M = M_1 \times M_2 = T^*X$, where $X = X_1 \times X_2$. Moreover, the canonical 1-form on $M = T^*X$ is $\alpha = \alpha_1 \oplus \alpha_2$, so the product symplectic form $\omega = \omega_1 \oplus \omega_2$ on $M = T^*X$ is the canonical symplectic form. Let

$$\sigma_2 : M_2 \to M_2, (x, \xi) \mapsto (x, -\xi).$$

Then $\sigma_2^*\alpha_2 = -\alpha_2$, and thus $\sigma_2^*\omega_2 = -\omega_2$. It follows theorem that

Proposition 2.2. *If $f : M_1 \to M_2$ is a diffeomorphism, then f is a symplectomorphism if and only if $\Gamma_{\sigma_2 \circ f}$ is a Lagrangian submanifold of (M, ω).*

Now suppose $g : X_1 \to X_2$ is a diffeomorphism. As we have seen, g lifts to a symplectomorphism $\tilde{g} : M_1 \to M_2$. Recall that

$$\tilde{g}(x, \xi) = (y, \eta) \iff y = g(x), \xi = (dg_x)^*\eta.$$

As a consequence,

$$\Gamma_{\tilde{g}}^{\sigma} = \{(x, \xi, y, \eta) \mid y = g(x), \xi = -(dg_x)^*\eta\}$$

is a Lagrangian submanifold of $M = T^*X$. Here is another way to see this: Let

$$X_g = \{(x, g(x)) \mid x \in X_1\} \subset X = X_1 \times X_2$$

be the graph of g, then $\Gamma_{\tilde{g}}^{\sigma} = N^*X_g$. In fact, we have a more general theorem:

Theorem 2.3. *Let X_1, X_2 be arbitrary smooth manifolds and $g : X_1 \to X_2$ a smooth map, then the set $\Gamma_{\tilde{g}}^{\sigma}$ defined as above is exactly N^*X_g, and thus a Lagrangian submanifold of $M = T^*X$.*

Proof. For any $(x, g(x)) \in X_g$,

$$T_{(x,g(x))}X_g = \{(v, dg_x(v)) \mid v \in T_xX_1\}.$$

By definition, $N^*_{(x,g(x))}X_g$ is the subspace of $T^*_{(x,g(x))}X$ that annihilates $T_{(x,g(x))}X_g$. so

$$\begin{aligned}(\xi, \eta) \in N^*_{(x,g(x))}X_g &\iff \langle \xi, v\rangle + \langle \eta, dg_x(v)\rangle = 0, \forall v \in T_xX_1\\ &\iff \langle \xi + (dg_x)^*\eta, v\rangle = 0, \forall v \in T_xX_1\\ &\iff \xi = -(dg_x)^*\eta.\end{aligned}$$

□

¶ Fixed points of symplectomorphisms.

The identity map $\mathrm{Id} : M \to M$ is a symplectomorphism, and the corresponding Lagrangian submanifold of $(M \times M, \tilde{\omega})$ is the diagonal $\Delta = \{(x, x) \mid x \in M\}$. We can canonically identify Δ with M. According to the Weinstein's Lagrangian neighborhood theorem, there exists a neighborhood $\mathcal{U}$ of $\Delta \simeq M$ in $(M \times M, \tilde{\omega})$, a neighborhood $\mathcal{U}_0$ of M in T^*M and a symplectomorphism $\varphi : \mathcal{U}_0 \to \mathcal{U}$.

Definition 2.4. We say a diffeomorphism $f : M \to M$ is C^1 close to Id if its graph Γ_f lies in $\mathcal{U}$, which, under the map φ^{-1}, is the graph of a 1-form μ on M.

Now let (M, ω) be a compact symplectic manifold and $f \in \mathrm{Symp}(M, \omega)$ a symplectomorphism that is C^1 closed to the identity symplectomorphism. Since Γ_f is a Lagrangian submanifold of $(M \times M, \tilde{\omega})$ and φ is a symplectomorphism, $\varphi^{-1}(\Gamma_f)$ is a Lagrangian submanifold of (T^*M, ω_{can}). So if it is the graph of a one-form μ, μ must be closed.

Remark. By this way we get an identification of a C^1 neighborhood of Id with a neighborhood of 0 in the space of closed 1-forms. So the tangent space of Id in $\mathrm{Symp}(M, \omega)$ can be identified with the space of closed 1-forms on M. This coincides with our earlier "Lie group-Lie algebra" observation, since symplectic vector fields are in one-to-one correspondence with closed 1-forms under ω.

Theorem 2.5. *Let (M, ω) be a compact symplectic manifold with $H^1(M) = 0$. Then any symplectomorphism of M which is sufficiently C^1 close to* Id *has at least two fixed points.*

Proof. Let $f \in \mathrm{Symp}(M, \omega)$ is C^1 close to Id. Then under the map φ, the graph of f is identified with a closed one-form μ on M. Since $H^1(M) = 0$, one can find a smooth function $h \in C^\infty(M)$ so that $\mu = dh$. Since M is compact, h admits at least two critical points (the maximum and the minimum). Obviously any critical point of h gives an intersection point of Γ_f with Δ, which yields a fixed point of f. □

¶ The Arnold conjecture.

Conjecture 2.6 (Arnold, symplectomorphism version)**.** *Let (M, ω) be a compact symplectic manifold and $f : M \to M$ a Hamiltonian symplectomorphism. Then*

$$\#(\textit{fixed points of } f) \geq \textit{ minimal number of critical points of a Morse function on } M.$$

Conjecture 2.7 (Arnold, Lagrangian version)**.** *Let (M, ω) be a compact symplectic manifold, $L \subset M$ a Lagrangian submanifold, and $f : M \to M$ a Hamiltonian symplectomorphism. Then*

$$\#(L \cap f(L)) \geq \textit{ minimal number of critical points of a Morse function on } L.$$

Note that by Morse theory, the minimal number of critical points of a Morse function on M is at least $\sum_i \dim H^i(M)$.

The conjecture is only proven in special cases via the theory of Floer homology.

3. Generating functions

¶ Generating function for horizontal Lagrangian submanifolds.

Let $M = T^*X$ be the cotangent bundle of any smooth manifold X and ω the canonical symplectic form. We have seen that a horizontal submanifold

$$X_\mu = \{(x, \mu_x) \mid x \in X\},$$

is Lagrangian if and only if $d\mu = 0$.

Definition 3.1. If μ is exact, i.e. $\mu = d\varphi$ for some smooth function $\varphi \in C^\infty(X)$, then we call φ a *generating function* of the Lagrangian submanifold Λ_μ.

Note that proposition 2.2 is equivalent to

The graph of f is a Lagrangian $\Leftrightarrow$ $\sigma_2 \circ f$ is a symplectomorphism.

From this correspondence it is natural to define

Definition 3.2. If $\Gamma_f = \Lambda_{d\varphi}$ for some $\varphi \in C^\infty(X_1 \times X_2)$, we say φ a generating function for the symplectomorphism $\sigma_2 \circ f$.

Remark. Usually one only need to find generating functions *locally*.

¶ Constructing symplectomorphisms.

Now suppose we have a Lagrangian submanifold $\Lambda_{d\varphi}$ generated by function φ. When will it generate a symplectomorphism? In other words, we want $\Lambda_{d\varphi}$ to be the graph of some diffeomorphism $f : M_1 \to M_2$. We denote $pr_i : M = M_1 \times M_2 \to M_i$ be the projection maps. We choose local coordinate patches $(\mathcal{U}_1, x_1, \cdots, x_n)$ and $(\mathcal{U}_2, y_1, \cdots, y_n)$ on X_1 and X_2 respectively. Then $\Lambda_{d\varphi}$ is described locally by the equations $\xi_i = \frac{\partial\varphi}{\partial x_i}, \eta_i = \frac{\partial\varphi}{\partial y_i}$. Therefore, given any point $(x,\xi) \in M_1$, to find its image $(y,\eta) = f(x,\xi)$ we need to solve the equations

$$\left\{\begin{array}{l} \xi_i = \frac{\partial\varphi}{\partial x_i}(x,y), \\ \eta_i = -\frac{\partial\varphi}{\partial y_i}(x,y). \end{array}\right. \tag{1}$$

According to the implicit function theorem, to solve the first equation $\xi_i = \frac{\partial\varphi}{\partial x_i}(x,y)$ for y locally, we need the condition

$$\det\left[\frac{\partial^2\varphi}{\partial x_i \partial y_j}\right] \neq 0. \tag{2}$$

Of course after solving y we may feed it into the second equation to get η.

¶ Examples of generating functions.

Example. Let $X_1 = X_2 = \mathbb{R}^n$ and $B = (b_{ij})$ a non-singular $n \times n$ matrix. Then the function $\varphi(x,y) = \sum b_{ij}x_iy_j$ generates a linear symplectomorphism $T_B : T^*\mathbb{R}^n \to T^*\mathbb{R}^n$ which maps (x,ξ) to $(B^{-1}\xi, -B^Tx)$.

In particular, if $B = I$, i.e. $\varphi(x,y) = \sum x_iy_i$, then T_B maps (x,ξ) to $(\xi,-x)$.

Example. Let $X_1 = X_2 = \mathbb{R}^n$ and $\varphi(x,y) = -\frac{|x-y|^2}{2}$. Then equation (1) becomes

$$\left\{\begin{array}{l} \xi_i = \frac{\partial\varphi}{\partial x_i}(x,y) = y_i - x_i \\ \eta_i = -\frac{\partial\varphi}{\partial y_i}(x,y) = y_i - x_i \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} y_i = x_i + \xi_i, \\ \eta_i = \xi_i. \end{array}\right.$$

So the symplectomorphism generated by φ is $f(x,\xi) = (x+\xi,\xi)$.

More generally, if X is a Riemannian manifold and $\varphi(x,y) = -\frac{d(x,y)^2}{2}$, where $d(x,y)$ is the Riemannian distance from x to y, then the symplectomorphism generated by φ is the geodesic flow.

Example. Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n \times (R^n)^*$ and $\varphi = \varphi(x,\eta) \in C^\infty(\mathcal{O})$ be a *twisted* generating function. Suppose $\det(\frac{\partial^2\varphi}{\partial x_i\partial\eta_j}) \neq 0$. Then by the same argument above or by composing the symplectomorphism we solved from (1) with the symplectomorphism $(x,\xi) \to (\xi,-x)$, we see that locally the set defined by

$$\xi_i = \frac{\partial\varphi}{\partial x_i}(x,\eta), \quad y_i = \frac{\partial\varphi}{\partial\eta_i}(x,\eta)$$

is the graph of a symplectomorphism.

Example. The identity symplectomorphism $\mathrm{Id} : T^*\mathbb{R}^n \to T^*\mathbb{R}^n$ cannot be generated by functions of the usual form. However, if we take a *twisted* generating function $\varphi(x,\eta) = \sum x_i\eta_i$, then it generates the identity symplectomorphism.

Example. More generally, if $\mathcal{U}_1$ is an open subset of $\mathbb{R}^n$ and $f : \mathcal{U}_1 \to \mathcal{U}_2$ a diffeomorphism, then we have seen that its canonical lifting $\tilde{f} : T^*\mathcal{U}_1 \to T^*\mathcal{U}_2$ is a symplectomorphism. One can check that this is generated by $\varphi(x,\eta) = \sum f_i(x)\eta_i$.

4. The billiards

Student presentation.