LECTURE 5: COMPLEX AND KÄHLER MANIFOLDS

Contents

1. Almost complex manifolds

¶ Almost complex structures.

Recall that a complex structure on a (real) vector space V is automorphism $J : V \to V$ such that

$$J^2 = -\mathrm{Id}.$$

Roughly speaking, a complex structure on V enable us to "multiply $\sqrt{-1}$" on V and thus convert V into a complex vector space.

Definition 1.1. An *almost complex structure* J on a (real) manifold M is an assignment of complex structures J_p on the tangent spaces T_pM which depend smoothly on p. The pair (M, J) is called an *almost complex manifold*.

In other words, an almost complex structure on M is a $(1,1)$ tensor field $J : TM \to TM$ so that $J^2 = -\mathrm{Id}$.

Remark. As in the symplectic case, an almost complex manifold must be $2n$ dimensional. Moreover, it is not hard to prove that any almost complex manifold must be orientable. On the other hand, there does exists even dimensional orientable manifolds which admit no almost complex structure. There exists subtle topological obstructions in the Pontryagin class. For example, there is no almost complex structure on S^4 (Ehresmann and Hopf).

Example. As in the symplectic case, any oriented surface Σ admits an almost complex structure: Let

$$\nu : \Sigma \to S^2$$

be the Gauss map which associates to every point $x \in \Sigma$ the outward unit normal vector $\nu(x)$. Define $J_x : T_x\Sigma \to T_x\Sigma$ by

$$J_x u = \nu(x) \times u,$$

where $\times$ is the cross product between vectors in $\mathbb{R}^3$. It is quite obvious that J_x is an almost complex structure on Σ.

Example. We have seen that on S^6 there is no symplectic structure since $H^2(S^6) = 0$. However, there exists an almost complex structure on S^6. More generally, every oriented hypersurface $M \subset \mathbb{R}^7$ admits an almost complex structure. The construction is almost the same as the previous example: first of all, there exists a notion of "cross product" for vectors in $\mathbb{R}^7$: we identify $\mathbb{R}^7$ as the imaginary Cayley numbers, and define the vector product $u \times v$ as the imaginary part of the product of u and v as Cayley numbers. Again we define

$$J_x u = \nu(x) \times u,$$

where $\nu : M \to S^6$ is the Gauss map that maps every point to its unit out normal. Then J is an almost complex structure. Details left as an exercise.

Remark. S^2 and S^6 are the only spheres that admit almost complex structures.

¶ Compatible triple.

Now let (M, ω) be a symplectic manifold, and J an almost complex structure on M. Then at each tangent space T_pM we have the linear symplectic structure ω_p and the linear complex structure J_p. Recall from lecture 1 that J_p is *tamed* by ω_p if the quadratic form $\omega_p(v, J_p v)$ is positive definite, and J_p is *compatible* with ω_p if it is tamed by ω_p and is a linear symplectomorphism on (T_pM, ω_p), or equivalently,

$$g_p(v, w) := \omega_p(v, J_p w)$$

is an inner product on T_pM.

Definition 1.2. We say an almost complex structure J on M is *compatible* with a symplectic structure ω on M if at each p, J_p is compatible with ω_p.

Equivalently, J is compatible with ω if and only if the assignment

$$g_p : T_pM \times T_pM \to \mathbb{R}, \quad g_p(u, v) := \omega_p(u, Jv)$$

defines a Riemannian structure on M. So on M we get three structures: a symplectic structure ω, an almost complex structure J and a Riemannian structure g, and they are related by

$$\begin{aligned} g(u, v) &= \omega(u, Jv), \\ \omega(u, v) &= g(Ju, v), \\ J(u) &= \tilde{g}^{-1}(\tilde{\omega}(u)), \end{aligned}$$

where $\tilde{g}$ and $\tilde{\omega}$ are the linear isomorphisms from TM to T^*M that is induced by g and ω respectively. Such a triple (ω, g, J) is called a *compatible triple*.

¶ Almost complex = almost symplectic.

According to proposition 3.4 and its corollary in lecture 1 we get immediately

Proposition 1.3. *For any symplectic manifold (M, ω), there exists an almost complex structure J which is compatible with ω. Moreover, the space of such almost complex structures is contractible.*

Remark. Obviously the proposition holds for any non-degenerate 2-form ω on M which does not have to be closed. Such a pair (M, ω) is called an *almost symplectic manifold.*

Conversely, one can prove (exercise)

Proposition 1.4. *Given any almost complex structure on M, there exists an almost symplectic structure ω which is compatible with J. Moreover, the space of such almost symplectic structures is contractible.*

So the set of almost symplectic manifolds coincides with the set of almost complex manifolds.

Example. For the almost complex structures on surfaces (or hypersurfaces in $\mathbb{R}^7$) that we described above,

$$\omega_x(v, w) = \langle \nu(x), v \times w \rangle$$

defines a compatible almost symplectic structure (which is symplectic for surfaces but not symplectic for S^6).

The following question is still open:

Donaldson's question: Let M be a compact 4-manifold and J an almost complex structure on M which is tamed by some symplectic structure ω. Is there a symplectic form on M that is compatible with J?

An important progress was made by Taubes who answered the problem affirmatively for generically almost complex structures with $b^+ = 1$.

¶ Almost complex submanifolds.

Almost complex structure provides a method to construct symplectic submanifolds.

Definition 1.5. A submanifold X of an almost complex manifold (M, J) is an *almost complex submanifold* if $J(TX) \subset TX$.

Proposition 1.6. *Let (M, ω) be a symplectic manifold and J a compatible almost complex structure on M. Then any almost complex submanifold of (M, J) is a symplectic submanifold of (M, ω).*

Proof. Let $\iota : X \to M$ be the inclusion. Then $\iota^*\omega$ is a closed 2-form on X. It is non-degenerate since

$$\omega_x(u, v) = g_x(J_x u, v)$$

and $g_x|_{T_xX}$ is nondegenerate. □

¶ The splitting of tangent vectors.

Let (M, J) be an almost complex manifold. Denote by $T_{\mathbb{C}}M = TM \otimes \mathbb{C}$ the complexified tangent bundle. We extend J linearly to $T_{\mathbb{C}}M$ by

$$J(v \otimes z) = Jv \otimes z, \quad v \in TM, z \in \mathbb{C}.$$

Then again $J^2 = -\mathrm{Id}$, but now on a complex vector space $T_pM \otimes \mathbb{C}$ instead of on a real vector space. So for each $p \in M$ the map J_p has eigenvalues $\pm i$, and we have an eigenspace decomposition

$$TM \otimes \mathbb{C} = T_{1,0} \oplus T_{0,1},$$

where

$$T_{1,0} = \{v \in TM \otimes \mathbb{C} \mid Jv = iv\}$$

is the $+i$-eigenspace of J and

$$T_{0,1} = \{v \in TM \otimes \mathbb{C} \mid Jv = -iv\}$$

is the $-i$-eigenspace of J. We will call vectors in $T_{1,0}$ the *J-holomorphic tangent vectors* and vectors in $T_{0,1}$ the *J-anti-holomorphic tangent vectors*.

Lemma 1.7. *J-holomorphic tangent vectors are of the form $v \otimes 1 - Jv \otimes i$ for some $v \in TM$, while J-anti-holomorphic tangent vectors are of the form $v \otimes 1 + Jv \otimes i$ for some $v \in TM$.*

Proof. Obviously for any $v \in TM$,

$$J(v \otimes 1 - Jv \otimes i) = Jv \otimes 1 + v \otimes i = i(v \otimes 1 - Jv \otimes i)$$

while

$$J(v \otimes 1 + Jv \otimes i) = Jv \otimes 1 - v \otimes i = -i(v \otimes 1 + Jv \otimes i).$$

The conclusion follows from dimension counting. □

As a consequence, we see

Corollary 1.8. *If we write $v = v_{1,0} + v_{0,1}$ according to the splitting above, then*

$$v_{1,0} = \frac{1}{2}(v - iJv), \quad v_{0,1} = \frac{1}{2}(v + iJv).$$

¶ The splitting of differential forms.

Similarly one can split the complexified cotangent space $T^*M \otimes \mathbb{C}$ as

$$T^*M \otimes \mathbb{C} = T^{1,0} \oplus T^{0,1},$$

where

$$\begin{aligned} T^{1,0} = (T_{1,0})^* &= \{\eta \in T^*M \otimes \mathbb{C} \mid \eta(Jw) = i\eta(w), \forall w \in TM \otimes \mathbb{C}\} \\ &= \{\xi \otimes 1 - (\xi \circ J) \otimes i \mid \xi \in T^*M\} \end{aligned}$$

is the dual space of $T_{1,0}$, and

$$\begin{aligned} T^{0,1} = (T_{0,1})^* &= \{\eta \in T^*M \otimes \mathbb{C} \mid \eta(Jw) = -i\eta(w), \forall w \in TM \otimes \mathbb{C}\} \\ &= \{\xi \otimes 1 + (\xi \circ J) \otimes i \mid \xi \in T^*M\} \end{aligned}$$

is the dual space of $T_{0,1}$. More over, any covector η has a splitting

$$\eta = \eta^{1,0} + \eta^{0,1},$$

where

$$\eta^{1,0} = \frac{1}{2}(\eta - i\eta \circ J), \quad \eta^{0,1} = \frac{1}{2}(\eta + i\eta \circ J).$$

The splitting of covectors gives us a splitting of k-forms

$$\Omega^k(M, \mathbb{C}) = \oplus_{l+m=k}\Omega^{l,m}(M, \mathbb{C}),$$

where $\Omega^{l,m}(M, \mathbb{C}) = \Gamma^\infty(\Lambda^l T^{1,0} \wedge \Lambda^m T^{0,1})$ is the space of (l,m)-*forms* on M.

For $\beta \in \Omega^{l,m}(M, \mathbb{C}) \subset \Omega^k(M, \mathbb{C})$, we have $d\beta \in \Omega^{k+1}(M, \mathbb{C})$. So we have a splitting

$$d\beta = (d\beta)^{k+1,0} + (d\beta)^{k,1} + \cdots + (d\beta)^{1,k} + (d\beta)^{0,k+1}.$$

Definition 1.9. For $\beta \in \Omega^{l,m}(M, \mathbb{C})$,

$$\partial\beta = (d\beta)^{l+1,m}, \quad \bar{\partial}\beta = (d\beta)^{l,m+1}.$$

Note that for functions we always have

$$df = \partial f + \bar{\partial} f,$$

while for more general differential forms we don't have $d = \partial + \bar{\partial}$.

2. Complex manifolds

¶ Complex manifolds.

Recall that a smooth manifold is a topological space that locally looks like $\mathbb{R}^n$, with diffeomorphic transition maps.

Definition 2.1. A *complex manifold* of complex dimension n is a manifold that locally homeomorphic to open subsets in $\mathbb{C}^n$, with biholomorphic transition maps.

Obviously any complex manifold is a real manifold, but the converse is not true. As in the symplectic case, a complex manifold must be of even dimensional if view as a real manifold, and must be orientable. In fact, we have

Proposition 2.2. *Any complex manifold has a canonical almost complex structure.*

Proof. Let M be a complex manifold and (U, V, φ) be a complex chart for M, where U is an open set in M, and V an open set in $\mathbb{C}^n$. We denote $\varphi = (z_1, \cdots, z_n)$, with $z_i = x_i + \sqrt{-1}y_i$. Then $(x_1, \cdots, x_n, y_1, \cdots, y_n)$ is a coordinate system on U when we view M as a real manifold. So

$$T_pM = \mathbb{R}\text{-span of } \left\{ \frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i} \;\middle|\; i = 1, \cdots, n \right\}.$$

We define J on U by the recipe

$$J(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial y_i}, \quad J(\frac{\partial}{\partial y_i}) = -\frac{\partial}{\partial x_i}$$

for $i = 1, \cdots, n$, and extends to T_pM by linearity. Obviously $J^2 = -\mathrm{Id}$. It remains to prove that J is globally well-defined, i.e. it is independent of the choice of complex coordinate charts.

Suppose (U', V', φ') is another coordinate chart, with $\varphi' = (w_1, \cdots, w_n)$ and $w_i = u_i + \sqrt{-1}v_i$. Then on the overlap $U \cap U'$ the transition map

$$\psi : \varphi(U \cap U') \to \varphi'(U \cap U'), \quad z \mapsto w = \psi(z)$$

is biholomorphic. If we write the map as

$$u_i = u_i(x, y), \quad v_i = v_i(x, y)$$

in real coordinates, then the real tangent vectors are related by

$$\frac{\partial}{\partial x_k} = \sum_j \frac{\partial u_j}{\partial x_k}\frac{\partial}{\partial u_j} + \frac{\partial v_j}{\partial x_k}\frac{\partial}{\partial v_j}$$
$$\frac{\partial}{\partial y_k} = \sum_j \frac{\partial u_j}{\partial y_k}\frac{\partial}{\partial u_j} + \frac{\partial v_j}{\partial y_k}\frac{\partial}{\partial v_j},$$

while the Cauchy-Riemann equation gives

$$\frac{\partial u_j}{\partial x_k} = \frac{\partial v_j}{\partial y_k}, \quad \frac{\partial u_j}{\partial y_k} = -\frac{\partial v_j}{\partial x_k}.$$

It follows that

$$J'(\frac{\partial}{\partial x_k}) = J'(\sum_j \frac{\partial u_j}{\partial x_k}\frac{\partial}{\partial u_j} + \frac{\partial v_j}{\partial x_k}\frac{\partial}{\partial v_j}) = \sum_j \frac{\partial v_j}{\partial y_k}\frac{\partial}{\partial v_j} + \frac{\partial u_j}{\partial y_k}\frac{\partial}{\partial u_j} = \frac{\partial}{\partial y_k}.$$

Since $J' = -\mathrm{Id}$, we must also have $J'(\frac{\partial}{\partial y_i}) = -\frac{\partial}{\partial x_i}$. It follows $J' = J$. □

Conversely, not every almost complex manifold admits a complex structure. One example is $\mathbb{CP}^2\#\mathbb{CP}^2\#\mathbb{CP}^2$. We have seen that S^2 and S^6 are the only spheres that admits almost complex structure. We will see below that S^2 admits a complex structure. One of the major open question in complex geometry is

Open problem: Is there a complex structure on S^6?

¶ Differential forms on complex manifolds.

Now suppose M is a complex manifold and J its canonical almost complex structure. Then in local coordinates

$$T_pM \otimes \mathbb{C} = \mathbb{C}\text{-span of } \left\{ \frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i} \;\middle|\; i = 1, \cdots, n \right\}.$$

and the two eigenspaces of J are

$$T_{1,0} = \mathbb{C}\text{-span of } \left\{ \frac{1}{2}\left(\frac{\partial}{\partial x_i} - i\frac{\partial}{\partial y_i}\right) \;\middle|\; i = 1, \cdots, n \right\},$$

$$T_{0,1} = \mathbb{C}\text{-span of } \left\{ \frac{1}{2}\left(\frac{\partial}{\partial x_i} + i\frac{\partial}{\partial y_i}\right) \;\middle|\; i = 1, \cdots, n \right\},$$

We define

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right), \quad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right),$$

then

$$T_{1,0} = \mathbb{C}\text{-span of } \left\{ \frac{\partial}{\partial z_j} \;\middle|\; j = 1, \cdots, n \right\}, \quad T_{0,1} = \mathbb{C}\text{-span of } \left\{ \frac{\partial}{\partial \bar{z}_j} \;\middle|\; j = 1, \cdots, n \right\},$$

Similarly if we put

$$dz_j = dx_j + idy_j, \quad d\bar{z}_j = dx_j - idy_j,$$

then

$$T^{1,0} = \mathbb{C}\text{-span of } \{dz_j \mid j = 1, \cdots, n\}, \quad T^{0,1} = \mathbb{C}\text{-span of } \{d\bar{z}_j \mid j = 1, \cdots, n\},$$

Note that under these notions, the fact $df = \partial f + \partial \bar{f}$ is given explicitly as

$$df = \sum_j \left(\frac{\partial f}{\partial x_j} dx_j + \frac{\partial f}{\partial y_j} dy_j \right) = \sum_j \left(\frac{\partial f}{\partial z_j} dz_j + \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j \right).$$

As a consequence, any (l, m)-form $\beta \in \Omega^{l,m}(M, \mathbb{C})$ can be expressed locally as

$$\beta = \sum_{|J|=l, |K|=m} b_{J,K} dz_J \wedge d\bar{z}_K$$

for some smooth functions $b_{J,K} \in C^\infty(U, \mathbb{C})$, where we use the notion dz_J to represent $dz_{j_1} \wedge \cdots \wedge dz_{j_l}$ for a multi-index $J = (j_1, \cdots, j_l)$, and likewise for $d\bar{z}_K$. This nice local expression implies

Theorem 2.3. *On complex manifolds $d = \partial + \bar{\partial}$ for any (l, m)-forms.*

Proof. The local expression above for $\beta \in \Omega^{l,m}(M,\mathbb{C})$ gives

$$d\beta = \sum_{|J|=l,|K|=m} db_{J,K} \wedge dz_J \wedge d\bar{z}_K.$$

The conclusion follows from the facts $db_{J,K} = \partial b_{J,K} + \bar{\partial} b_{J,K}$ and

$$\partial b_{J,K} = \sum \frac{\partial b_{J,K}}{\partial z_j} dz_j, \quad \bar{\partial} b_{J,K} = \sum \frac{\partial b_{J,K}}{\partial \bar{z}_j} d\bar{z}_j.$$

□

¶ Integrability.

The canonical almost complex structure on a complex manifold that we constructed above is just the map "multiplication by $\sqrt{-1}$" in coordinate charts.

Definition 2.4. An almost complex structure J on M is called *integrable* if it is a complex structure, i.e. there exists local complex coordinates on M so that $(M, J) = (\mathbb{C}^n, \sqrt{-1})$.

So a complex structure is an integrable almost complex structure.

It is hard to use the definition above to detect whether an almost complex structure is complex or not. However, we do have a very useful criteria.

Definition 2.5. The *Nijenhuis tensor* N_J is

$$N_J(u, v) = [Ju, Jv] - J[Ju, v] - J[u, Jv] - [u, v].$$

We will leave the proof of the next two exercises as an exercise.

Proposition 2.6. *N_J is a tensor, i.e. $N_j(fu, gv) = fgN_j(u, v)$ for vector fields u, v and smooth functions f, g.*

Proposition 2.7. *One has $N_J(u, v) = -8Re([u_{1,0}, v_{1,0}])_{0,1})$.*

As a consequence, we get

Corollary 2.8. $N_J = 0 \iff [T_{1,0}, T_{1,0}] \subset T_{1,0}$.

On a complex manifold $[\frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k}] = 0$. It follows

Corollary 2.9. *$N_J = 0$ for the canonical almost complex structure J on complex manifolds.*

The following theorem is a hard theorem which gives an easy-to-use characterization of integrability of almost complex structures:

Theorem 2.10 (Newlander-Nirenberg)**.** *Let J be an almost complex structure on M. Then*

$$J \text{ is integrable} \iff N_J = 0 \iff d = \partial + \bar{\partial}.$$

An an example, we immediately get

Theorem 2.11. *Any almost complex structure on a surface is integrable.*

Proof. A direct computation gives $N_J(v, v) = 0$ and $N_J(v, Jv) = 0$. Details left as an exercise. □

3. Kähler manifolds

¶ Kähler manifolds.

Definition 3.1. A *Kähler manifold* is a triple (M, ω, J), where ω is a symplectic form on M, and J an integrable complex structure on M which is compatible with ω. In this case we will call ω a *Kähler form*.

Example. $(\mathbb{C}^n, \Omega_0, J_0)$ is a Kähler manifold.

Example. Any oriented surface Σ carries a Kähler structure: one just choose ω to be the area form and choose J to be an almost complex structure that is compatible with ω.

Example. Complex tori $M = \mathbb{C}^n/\mathbb{Z}^n$: Since both the symplectic structure and the complex structure on $\mathbb{C}^n$ are invariant under *translations along real directions*, the standard symplectic and complex structures on $\mathbb{C}^n$ give us a Kähler structure on M.

Remark. By definition a Kähler manifold is both a symplectic manifold and a complex manifold. In 1976 Thurston constructed an example that is both symplectic and complex, but admits no Kähler structure.

People also constructed symplectic manifolds which do not admit any complex structure (Fernandez-Gotay-Gray 1988), and complex manifolds that admits no symplectic structure (Hopf surface $S^1 \times S^3 \simeq C^2 - \{0\}/\{(z_1, z_2) \sim (2z_1, 2z_2)\}$).

¶ The Kähler form.

Now let ω be a Kähler form on M. Then ω is a real-valued non-degenerate closed 2-form on M. Let's see what does these conditions give us:

- Since M is complex, locally
$$\omega = \sum a_{jk} dz_j \wedge dz_k + \sum b_{jk} dz_j \wedge d\bar{z}_k + \sum c_{jk} d\bar{z}_j \wedge d\bar{z}_k.$$
- Since J is a symplectomorphism,
$$J^*\omega = \omega.$$
On the other hand it is easy to check $J^* dz_j = i dz_j$ and $J^* d\bar{z}_j = -i d\bar{z}_j$. So
$$J^*\omega = \sum -a_{jk} dz_j \wedge dz_k + \sum b_{jk} dz_j \wedge d\bar{z}_k - \sum c_{jk} d\bar{z}_j \wedge d\bar{z}_k.$$
It follows
$$\omega = \sum b_{jk} dz_j \wedge d\bar{z}_k,$$

i.e. $\omega \in \Omega^{1,1}(M,\mathbb{C}) \cap \Omega^2(M)$. We will write $b_{jk} = \frac{i}{2}h_{jk}$, so that

$$\omega = \frac{i}{2}\sum h_{jk} dz_j \wedge d\bar{z}_k, \qquad h_{jk} \in C^\infty(U).$$

- Since ω is real-valued, $\bar{\omega} = \omega$. But
$$\bar{\omega} = -\frac{i}{2}\sum \overline{h_{jk}} d\bar{z}_j \wedge dz_k = \frac{i}{2}\sum \overline{h_{kj}} dz_j \wedge d\bar{z}_k,$$
so at each point $p \in M$ the matrix $(h_{jk}(p))$ is Hermitian.
- Moreover, one can check
$$\omega^n = n!(\frac{i}{2})^n \det(h_{jk}) dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n,$$
so the non-degeneracy condition of ω is equivalent to the fact that the matrix (h_{jk}) is non-singular.
- The *tamed* condition $\omega(v, Jv) > 0$ for each $v \neq 0$ implies that at each p, the matrix (h_{jk}) is positive definite.
- Finally, since $0 = d\omega = \partial\omega + \bar{\partial}\omega$, and $\partial\omega \in \Omega^{2,1}(M,\mathbb{C})$ and $\bar{\partial}\omega \in \Omega^{1,2}(M,\mathbb{C})$, we get
$$\partial\omega = 0, \quad \bar{\partial}\omega = 0.$$

In conclusion, we get

Theorem 3.2. *Kähler forms are ∂- and $\bar{\partial}$-closed $(1,1)$ forms which are given locally by*
$$\omega = \frac{i}{2}\sum h_{jk} dz_j \wedge d\bar{z}_k, \qquad h_{jk} \in C^\infty(U),$$
where at each p, the matrix (h_{jk}) is a positive-definite Hermitian matrix.

In general one cannot hope that two symplectic forms in the same cohomology class are symplectomorphic, unless they are connected by a path of symplectic structures. As an application of the previous theorem, we have

Corollary 3.3. *Let M be compact and ω_1, ω_2 be Kähler forms on M with $[\omega_1] = [\omega_2] \in H^2(M)$, then (M, ω_1) and (M, ω_2) are symplectomorphic.*

Proof. On a local chart
$$\omega_i = \frac{i}{2}\sum h^i_{jk} dz_j \wedge d\bar{z}_k, \qquad h^i_{jk} \in C^\infty(U).$$
We let
$$\omega_t = \frac{i}{2}\sum((1-t)h^1_{jk} + th^0_{jk}) dz_j \wedge d\bar{z}_k, \qquad h^i_{jk} \in C^\infty(U).$$
Then $((1-t)h^1_{jk} + th^0_{jk})$ is a positive definite Hermitian matrix, so ω_t's are all symplectic. Now apply Moser's trick. □

4. Dolbeault cohomology

Student presentation.