LECTURE 6: GEOMETRY OF HAMILTONIAN SYSTEMS

Contents

1. Geometry of Hamiltonian vector fields

¶ Symplectic vector field v.s. Hamiltonian vector field.

Let (M, ω) be a symplectic manifold. Then the non-degeneracy of ω gives us a linear isomorphism between vector fields and 1-forms on M:

$$\tilde{\omega} : \mathrm{Vect}(M) \to \Omega^1(M), \quad \Xi \mapsto \iota_\Xi \omega.$$

Recall that a vector field Ξ on M is called symplectic if $\iota_\Xi \omega$ is a closed 1-form on M, and it is called Hamiltonian if $\iota_\Xi \omega$ is an exact 1-form. So if we denote the set of symplectic vector fields by $\mathrm{Vect}(M, \omega)$ and the set of Hamiltonian vector fields by $\mathrm{Vect}_{Ham}(M, \omega)$, then the restriction of $\tilde{\omega}$ gives us linear isomorphisms

$$\tilde{\omega} : \mathrm{Vect}(M, \omega) \to Z^1(M)$$

and

$$\tilde{\omega} : \mathrm{Vect}_{Ham}(M, \omega) \to B^1(M),$$

where $Z^1(M)$ is the space of closed 1-forms on M, and $B^1(M)$ the space of exact 1-forms. As a consequence, the quotient $\mathrm{Vect}(M, \omega)/\mathrm{Vect}_{Ham}(M, \omega)$ is just the first deRham cohomology group $H^1(M)$, and we have an exact sequence of vector spaces

$$0 \to \mathrm{Vect}(M, \omega) \to \mathrm{Vect}_{Ham}(M, \omega) \to H^1(M) \to 0. \tag{1}$$

In particular, we see

Proposition 1.1. *If $H^1(M) = \{0\}$, then every symplectic vector field on M is Hamiltonian.*

¶ Smooth function v.s. Hamiltonian vector field.

Recall we showed in lecture 4 that if Ξ_1, Ξ_2 are two symplectic vector fields on M, then $[\Xi_1, \Xi_2]$ is a Hamiltonian vector field on M. In fact, one has

$$\iota_{[\Xi_1,\Xi_2]}\omega = -d(\omega(\Xi_1, \Xi_2)). \tag{2}$$

This implies

$$[\mathrm{Vect}(M,\omega), \mathrm{Vect}(M,\omega)] \subset \mathrm{Vect}_{Ham}(M,\omega).$$

In other words, as a Lie algebra, $\mathrm{Vect}_{Ham}(M,\omega)$ is an *ideal* of $\mathrm{Vect}(M,\omega)$. (This is of course related to the fact that $\mathrm{Ham}(M,\omega)$ is an ideal of the group $\mathrm{Symp}(M,\omega)$.) So the short exact sequence (1) is in fact a short exact sequence of Lie algebras, if we endowed with $H^1(M)$ the trivial Lie bracket.

Modulo locally constant functions, Hamiltonian vector fields are in one-to-one correspondence with smooth functions on M, in the following sense: So if Ξ is hamiltonian, then there exists a smooth function $f \in C^\infty(M)$ so that $\iota_\Xi\omega = df$; Conversely, since ω is non-degenerate, for any $f \in C^\infty(M)$, there is a unique vector field Ξ_f on M so that

$$\iota_{\Xi_f}\omega = df.$$

The space of locally constant functions on M is the 0th de Rham cohomology group $H^0(M)$. So we get another short exact sequence of vector spaces

$$0 \to H^0(M) \to C^\infty(M) \to \mathrm{Vect}_{Ham}(M,\omega) \to 0. \tag{3}$$

We shall see soon that this is again a short exact sequence of Lie algebras, provided we define a suitable Lie algebra structure on $C^\infty(M)$ (which is, however, different from the Poisson bracket we will define below by a negative sign).

Recall: The vector field Ξ_f is called the *Hamiltonian vector field* associated to the *Hamiltonian function* f. The flow generated by Ξ_f is called the *Hamiltonian flow* associated to f.

Example. Consider $M = \mathbb{R}^{2n}$ with canonical symplectic form $\omega = \sum dx_i \wedge d\xi_i$. Then

$$df = \sum \left(\frac{\partial f}{\partial x_i} dx_i + \frac{\partial f}{\partial \xi_i} d\xi_i \right).$$

It follows

$$\Xi_f = \sum \left(\frac{\partial f}{\partial \xi_i}\frac{\partial}{\partial x_i} - \frac{\partial f}{\partial x_i}\frac{\partial}{\partial \xi_i} \right) \tag{4}$$

So the integral curve of Ξ_f is a curve $(x(t), \xi(t))$ satisfying

$$\dot{x}(t) = \frac{\partial f}{\partial \xi_i}, \quad \dot{\xi}(t) = -\frac{\partial f}{\partial x_i}.$$

This set of equations is known as *Hamiltonian equations.*

Remark. Obviously the formula (4) also gives the local expression of Ξ_f on an arbitrary symplectic manifold if one uses the Darboux coordinates. The vector field Ξ_f is also called the *symplectic gradient* of f.

¶ Gradient vector field v.s. Hamiltonian vector field.

Back to the $\mathbb{R}^{2n}$ example. Note that the usual gradient vector field of f is

$$\nabla f = \sum \left(\frac{\partial f}{\partial x_i}\frac{\partial}{\partial x_i} + \frac{\partial f}{\partial \xi_i}\frac{\partial}{\partial \xi_i} \right).$$

So the symplectic gradient and the usual gradient of f are related by

$$J(\Xi_f) = \nabla f.$$

where J is the usual complex structure on $\mathbb{R}^{2n}$, i.e.

$$J(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial \xi_i}, J(\frac{\partial}{\partial \xi_i}) = -\frac{\partial}{\partial x_i}.$$

This observation is easily extended to Kähler manifolds, or more generally any symplectic manifold M with compatible triple (ω, J, g). In this the usual gradient vector field of f is the vector field ∇f on M so that

$$g(\nabla f, \cdot) = df(\cdot).$$

Using the fact $g(\cdot,\cdot) = \omega(\cdot, J\cdot)$ it is easy to see

Proposition 1.2. *Let M be a symplectic manifold with compatible triple (ω, J, g). Then $\nabla f = J\Xi_f$.*

Proof. We have

$$df = g(\nabla f, \cdot) = \omega(\nabla f, J\cdot) = \omega(-J\nabla f, \cdot).$$

It follows that $-J\nabla f = \Xi_f$, i.e. $\nabla f = J(\Xi_f)$. □

So if one think of an almost complex structure on M as "rotation by 90 degrees couterclockwise", then the usual gradient vector field can be obtained from the symplectic gradient vector fields via "rotation by 90 degrees couterclockwise"!

¶ Symplectic form v.s. Hamiltonian vector field.

Now suppose Ξ_f is the Hamiltonian vector field associated with f. The following properties are easily seen from the definition. One should be aware of the use of the three parts of the definition of a symplectic form.

Proposition 1.3. *Let (M, ω) be a symplectic manifold, and $f \in C^\infty(M)$.*

(1) $\mathcal{L}_{\Xi_f} f = 0$.
(2) $\mathcal{L}_{\Xi_f} \omega = 0$.
(3) *If $\varphi \in \mathrm{Symp}(M, \omega)$, then $\Xi_{\varphi^* f} = \varphi^* \Xi_f$.*

Proof. (1) follows from the skew-symmetry of ω:

$$\mathcal{L}_{\Xi_f} f = \iota_{\Xi_f} df = \iota_{\Xi_f}\iota_{\Xi_f}\omega = 0.$$

(2) follows from the closeness of ω:

$$\mathcal{L}_{\Xi_f}\omega = d\iota_{\Xi_f}\omega = d(df) = 0.$$

(3) follows from the non-degeneracy of ω:

$$\iota_{\Xi_{\varphi^* f}}\omega = d(\varphi^* f) = \varphi^* df = \varphi^*\iota_{\Xi_f}\omega = \iota_{\varphi^*\Xi_f}\varphi^*\omega = \iota_{\varphi^*\Xi_f}\omega.$$

□

2. The Poisson structure

¶ The Poisson bracket.

Applying the identity (2) to Hamiltonian vector fields Ξ_f and Ξ_g, we get

$$[\Xi_f, \Xi_g] = \Xi_{-\omega(\Xi_f,\Xi_g)}. \tag{5}$$

Definition 2.1. For any $f, g \in C^\infty(M)$, we call

$$\{f, g\} = \omega(\Xi_f, \Xi_g)$$

the *Poisson bracket* of f and g.

So by definition,

$$\{f, g\} = \iota_{\Xi_f}\omega(X_g) = df(\Xi_g) = \mathcal{L}_{\Xi_g} f = \Xi_g(f).$$

As a consequence we see

Corollary 2.2. $\{f, g\} = 0$ *if and only if* g *is constant along the integral curves of* Ξ_f*, i.e. the Hamiltonian vector field* Ξ_f *is tangent to the level sets* $g = c$*.*

In particular, the hamiltonian vector field Ξ_f is always tangent to the level sets $f = c$.

The poisson bracket also behaves well under symplectomorphisms:

Proposition 2.3. *If* $\varphi \in \mathrm{Symp}(M, \omega)$*, then* $\{\varphi^* f, \varphi^* g\} = \varphi^*\{f, g\}$*.*

Proof. Using the fact $\Xi_{\varphi^* f} = \varphi^*\Xi_f$ we get

$$\{\varphi^* f, \varphi^* g\} = \omega(\Xi_{\varphi^* f}, \Xi_{\varphi^* g}) = \omega(\varphi^*\Xi_f, \varphi^*\Xi_g) = \varphi^*(\omega(\Xi_f, \Xi_g)) = \varphi^*\{f, g\}.$$

□

¶ The Poisson bracket in local coordinates.

In local Darboux coordinates, (4) gives

$$\{f,g\} = \sum \left(\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial \xi_i} - \frac{\partial f}{\partial \xi_i}\frac{\partial g}{\partial x_i} \right).$$

In particular, the Poisson bracket of the Darboux coordinate functions are simple:

$$\{x_i, x_j\} = \{\xi_i, \xi_j\} = 0, \quad \{x_i, \xi_j\} = \delta_{ij}.$$

Conversely, we have

Proposition 2.4. *A coordinate system $\{x_1, \cdots, x_n, \xi_1, \cdots, \xi_n\}$ on M is a Darboux coordinate system if and only if they satisfies*

$$\{x_i, x_j\} = \{\xi_i, \xi_j\} = 0, \quad \{x_i, \xi_j\} = \delta_{ij}.$$

Proof. We can rewrite these set of equations as

$$dx_i(\Xi_{x_j}) = 0, \quad d\xi_i(\Xi_{\xi_j}) = 0, \quad dx_i(\Xi_{\xi_j}) = -d\xi_i(\Xi_{x_i}) = \delta_{ij}.$$

It follows

$$\Xi_{x_i} = \frac{\partial}{\partial \xi_i}, \quad \Xi_{\xi_i} = -\frac{\partial}{\partial x_i}$$

and thus

$$\omega(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) = \omega(\frac{\partial}{\partial \xi_i}, \frac{\partial}{\partial \xi_j}) = 0, \quad \omega(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial \xi_j}) = \delta_{ij}.$$

□

¶ The Poisson bracket as a Lie bracket.

Using Poisson bracket we can rewrite the equation (5) as

$$\Xi_{\{f,g\}} = -[\Xi_f, \Xi_g]. \tag{6}$$

In particular, if $\{f, g\} = 0$, then $[\Xi_f, \Xi_g] = 0$, thus the Hamiltonian flows of f and g commute.

Theorem 2.5. *Let (M, ω) be a symplectic manifold.*

1. *The Poisson bracket $\{\cdot, \cdot\}$ is a Lie algebra structure on $C^\infty(M)$.*
2. *The map*
$$C^\infty(M) \to \mathrm{Vect}_{Ham}(M, \omega), \quad f \mapsto \Xi_f$$
is a Lie algebra anti-homomorphism.

Proof. (1) Obviously $\{\cdot, \cdot\}$ is bilinear and anti-symmetric. To show it is a Lie algebra structure it remains to check the Jacobi identity:

$$\begin{aligned} 0 &= (\mathcal{L}_{\Xi_f}\omega)(\Xi_g, \Xi_h) = (d\iota_{\Xi_f})(\Xi_g, \Xi_h) \\ &= \Xi_g(\omega(\Xi_f, \Xi_h)) - \Xi_h(\omega(\Xi_f, \Xi_g)) - \omega(\Xi_f, [\Xi_g, \Xi_h]) \\ &= \{\{f, h\}, g\} - \{\{f, g\}, h\} + \omega(\Xi_f, \Xi_{\{g,h\}}) \\ &= \{\{f, h\}, g\} + \{\{g, f\}, h\} + \{\{h, g\}, f\}. \end{aligned}$$

(2) This is just another explanation of (6). □

¶ Poisson manifolds.

Another remarkable fact about the Poisson bracket on $C^\infty(M)$ is that it satisfies the Leibniz law:

Proposition 2.6. *For any $f, g, h \in C^\infty(M)$,*

$$\{fg, h\} = \{f, h\}g + f\{g, h\}. \tag{7}$$

Proof. We have

$$\{fg, h\} = \Xi_h(fg) = \Xi_h(f)g + f\Xi_h(g) = \{f, h\}g + f\{g, h\}.$$

□

Definition 2.7. (1) A *Poisson algebra* $(P, \{\cdot, \cdot\})$ is an associative algebra P with a Lie bracket $\{\cdot, \cdot\}$ such that the Leibniz law (7) holds.
(2) A *Poisson manifold* is a manifold M equipped with a Poisson algebra structure on $C^\infty(M)$.

So any symplectic manifold M is a Poisson manifold. But we have more Poisson manifolds than symplectic manifolds. In fact, any smooth manifold M is a Poisson manifold if we equipped $C^\infty(M)$ with the trivial bracket. But this is of course not interesting.

Example. Let $\mathfrak{g}$ be any Lie algebra of dimension n, and $\mathfrak{g}^*$ its dual. Then one can canonically identify the double dual $(\mathfrak{g}^*)^*$ with $\mathfrak{g}$. We thus get, for any $f \in C^\infty(\mathfrak{g}^*)$ and any $\mu \in \mathfrak{g}^*$, an identification of $d_\mu f$ with an element in $\mathfrak{g}$ via

$$d_\mu f \in T^*_\mu \mathfrak{g}^* = (T_\mu \mathfrak{g}^*)^* = (\mathfrak{g}^*)^* = \mathfrak{g}.$$

Define a bracket structure on $\mathfrak{g}^*$ via

$$\{f, g\}(\mu) = \mu([d_\mu f, d_\mu g]).$$

One can check that this is in fact a Poisson bracket structure.

Let M be a Poisson manifold. According to the Leibniz law, for any $h \in C^\infty(M)$, the map

$$\{\cdot, h\} : C^\infty(M) \to \mathbb{R}$$

is a derivative on $C^\infty(M)$, and thus defines a unique vector field Ξ_h on M so that for any f,

$$\Xi_h(f) = \{f, h\}.$$

As in the symplectic case, one calls Ξ_h the *Hamiltonian vector field* of h.

Poisson manfilds are closely related to symplectic manifolds. One can consult, for example, the book "Lectures on the Geometry of Poisson Manifolds (Progress in Mathematics)" by I. Vaisman.

3. Completely integrable Hamiltonian systems

¶ Hamiltonian system.

In the Hamiltonian fashion of classical machanics, a classical mechanical system is described via a symplectic manifold (M, ω), called the *phase space* of the system. Any point in M is called a *state* of the system. The evolution of the system is then the Hamiltonian flow on M of a Hamiltonian function, and the evolution of a particular state is then given by the integral curve of the Hamiltonian vector field. Any smooth function on M is called a *classical observable*.

Example. Consider a free particle moving under a force field in $\mathbb{R}^n$ (the configuration space). For simplicity we assume the mass of the particle is 1. Denote by $V(x)$ the potential energy function of the force field, and denote by $(x_1(t), \cdot, x_n(t))$ the position vector of the particle at time t. According to the Newton's law, the movement of the particle obey the equation

$$\ddot{x}(t) = -\nabla V.$$

Note that in this system the energy function $E = \frac{1}{2}|\dot{x}(t)|^2 + V(x(t))$ of the particle is a conserved quantity.

To describe this system using the language of Hamiltonian mechanics, we introduce the phase space $\mathbb{R}^{2n}$, with coordinates $x_1, \cdots, x_n, \xi_1, \cdots, \xi_n$, where $(x_1(t), \cdots, x_n(t))$ is still the position vector of the particle in the configuration space, and

$$(\xi_1(t), \cdots, \xi_n(t)) = (\dot{x}_1(t), \cdots, \dot{x}_n(t))$$

is the *momentum vector*. So in particular the energy function becomes a Hamiltonian function on $\mathbb{R}^{2n}$:

$$H(x, \xi) = \frac{1}{2}|\xi|^2 + V(x).$$

As we have seen above, the integral curves of H satisfies the Hamiltonian equation

$$\dot{x}(t) = \frac{\partial H}{\partial \xi_i} = \xi_i(t), \quad \dot{\xi}_(t) = -\frac{\partial H}{\partial x_i} = -\frac{\partial V}{\partial x_i}(t).$$

This system of equations is of course equivalent to the Newtons's equation $\ddot{x} = -\nabla V$.

Definition 3.1. A *Hamiltonian system* is a triple (M, ω, H), where (M, ω) is a symplectic manifold and $H \in C^\infty(M)$ a smooth function (usually called the *Hamiltonian* of the system).

Example. Consider the 1 dimensional Harmonic oscillar. The phase space is $\mathbb{R}^2$ with coordinates x, ξ, where x is the position (the signed distance from the stationary position) and ξ is the momentum. For simplicity we again assume mass=1 and also assume the spring constant =1. The Hamiltonian is the energy function

$$H(x, \xi) = \frac{1}{2}\xi^2 + \frac{1}{2}x^2.$$

The Hamilton's equations becomes

$$\dot{x}(t) = \xi(t), \quad \dot{\xi}(t) = -x(t).$$

Given initial condition $x(0) = x_0, \xi(0) = \xi_0$, one can easily solve the system of equations to get

$$x(t) = \cos(t)x_0 + \sin(t)\xi_0, \quad \xi(t) = \cos(t)\xi_0 - \sin(t)x_0.$$

The Hamilton flow in the phase space is just the counterclockwise rotations on $\mathbb{R}^2$.

From the above example we see that the conserved quantity is important to a Hamiltonian system.

Example. The billiard system in a planar elliptical region. Last time we have seen that the phase space of this system is $C \times (-1, 1)$, where C is the boundary ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

More precisely, a state is a pair (p, t), where p is the position in C where the billiard hit the boundary, and t the the length tangential projection of the unit vector describing the direction that the billiard moves after hitting the boundary. The evolution of the system is a symplectomorphism which maps (p, t) to (p', t'), where (p', t') is the state describing the next hit of the billiard with the boundary. We have seen that the real trajectory of the billiard will always tangent to some confocal ellipse (or confocal hyperbola) whose equation is

$$\frac{x^2}{a^2 - Z^2} + \frac{y^2}{b^2 - Z^2} = 1.$$

Then function Z is invariant under the symplectomorphism and plays the role of the Hamiltonian function for this system.

Now suppose we have a Hamiltonian system (M, ω, H). Let $a \in C^\infty(M)$ be any classcial observable. Let

$$a(t) = a(\exp(t\Xi_H)(p_0))$$

be the evolution of the observable, where p_0 is the initial state of the system.

Proposition 3.2. $\dot{a}(t) = \{a, H\}(\exp(t\Xi_H)(p_0))$.

Proof. Since $\frac{d}{dt}\exp(t\Xi_H)(p) = \Xi_H(\exp(t\Xi_H)(p))$, we have

$$\dot{a}(t) = da(\Xi_H)(\exp(t\Xi_H)(p_0)) = \{a, H\}(\exp(t\Xi_H)(p_0)).$$

□

¶ Completely integrable systems.

Let (M, ω, H) be a Hamiltonain system. Recall that $\{f, H\} = 0$ if and only if f is constant along the integral curves of Ξ_H. Such a function is called an *integral of motion* (or a *first integral*). In general one cannot hope that a Hamiltonian system admits any integral of motion that is *independent* of the Hamiltonian function itself.

Definition 3.3. A set of functions $f_1, \cdots, f_k$ on M are called *independent* if the differentials $df_1, \cdots, df_p$ are linarly independent in an open dense subset of M.

We are interested in Hamiltonian systems with many Poisson commuting integral of motions. Now suppose $f_1 = H, \cdots, f_k$ are independent integral of motions of (M, ω, H) that are Poisson commuting, i.e.

$$\{f_i, f_j\} = 0, \quad \forall 1 \leq i, j, \leq k.$$

It follows that for any i, j,

$$\omega(\Xi_{f_i}, \Xi_{f_j}) = 0.$$

So at a point p where df_i's are linearly independent, the vectors $\Xi_{f_1}(p), \cdots, \Xi_{f_k}(p)$ span an isotropic subspace of T_pM. In particular, we get

Proposition 3.4. *If $f_1, \cdots, f_k$ are independent Poisson commuting integral of motions for (M, ω, H), then $k \leq n$.*

Definition 3.5. A Hamiltonian system (M, ω, H) is called *completely integrable* if it admits $n = \frac{1}{2} \dim M$ independent Poisson commuting integral of motions $f_1 = H, f_2, \cdots, f_n$.

Of course any 2 dimsensional Hamiltonian system (M, ω, H) (with $dH \neq 0$ almost everywhere) is completely integrabel. In particular, the 1 dimensional Harmonic oscillator, the billiard system in the planar ellipse are completely integrable Hamiltonian systems. There exists more complicated examples that we shall discuss later.

Now let (M, ω, H) be a completely integrable Hamiltonian system, and $f_1 = H, f_2, \cdots, f_n$ be independent Poisson commuting integral of motions. Let $c \in \mathbb{R}^n$ be a regular value of the map $F = (f_1, \cdots, f_n) : M \to \mathbb{R}^n$. Then the above argument shows that $F^{-1}(c)$ is a Lagrangian submanifold of M.

¶ Ation-angle coordinates.

Student presentation.