

LECTURE 9: SYMPLECTIC REDUCTION

CONTENTS

1. Reduction	1
2. Reduction at other levels	5
3. Reduction in stages	7

1. REDUCTION

¶ The tangent map of the moment map.

Let G be a compact Lie group action on M . Recall

- The orbit

$$G \cdot m = \{g \cdot m \mid g \in G\}$$

is an embedded submanifold with tangent space at m equals

$$T_m(G \cdot m) = \{X_M(m) \mid X \in \mathfrak{g}\}.$$

- The stabilizer subgroup of each $m \in M$,

$$G_m = \{g \in G \mid g \cdot m = m\},$$

is a Lie subgroup of G whose Lie algebra $\mathfrak{g}_m$ is

$$\mathfrak{g}_m = \{X \in \mathfrak{g} \mid X_M(m) = 0\} \subset \mathfrak{g}.$$

Now suppose (M, ω, G, μ) is a Hamiltonian G -manifold. As before, we let

$$(T_m(G \cdot m))^{\omega_m} = \{v \in T_m M \mid \omega_m(v, w) = 0 \text{ for any } w \in T_m(G \cdot m)\}$$

be the symplectic ortho-complement of $T_m(G \cdot m)$ in $(T_m M, \omega_p)$, and let

$$\mathfrak{g}_m^0 = \{\xi \in \mathfrak{g}^* \mid \langle \xi, X \rangle = 0 \text{ for any } X \in \mathfrak{g}_m\}$$

be the annihilator of $\mathfrak{g}_m$ in $\mathfrak{g}^*$.

Lemma 1.1. *For any $m \in M$,*

- (1) $\ker(d\mu_m) = (T_m(G \cdot m))^{\omega_m}$
- (2) $\text{Im}(d\mu_m) = \mathfrak{g}_m^0$

Proof. For any $v \in T_m M$ and any $X \in \mathfrak{g}$ one has

$$\omega_m(X_M(m), v) = \langle d\mu_m(v), X \rangle.$$

From this one immediately get (1):

$$\ker(d\mu_m) = (T_m(G \cdot m))^{\omega_m},$$

and half of (2):

$$\operatorname{Im}(d\mu_m) \subset \mathfrak{g}_m^0.$$

The other half of (2) follows from dimension counting:

$$\dim \operatorname{Im}(d\mu_m) = \dim T_m M - \dim \ker(d\mu_m) = \dim T_m(G \cdot m) = \dim \operatorname{Im}(d\tau_m),$$

where $d\tau_m$ is the linear map

$$d\tau_m : \mathfrak{g} \rightarrow T_m M, X \mapsto X_M(m),$$

and thus

$$\dim \operatorname{Im}(d\tau_m) = \dim \mathfrak{g} - \dim \ker(d\tau_m) = \dim \mathfrak{g} - \dim \mathfrak{g}_m = \dim \mathfrak{g}_m^0.$$

□

¶ Consequences of freeness.

Recall that an Lie group action is

- *locally free* at m if $\mathfrak{g}_m = \{0\}$.
- *free* if $G_m = \{e\}$ for all m .

Also recall that a regular value of a smooth map $f : M \rightarrow N$ is a point $n \in N$ so that either $f^{-1}(n) = \emptyset$ or df is surjective at each point in $f^{-1}(n)$. According to Sard's theorem, the set of points which are not regular values is a measure zero set. A standard result in manifold theory is: if n is a regular value of $f : M \rightarrow N$, then $f^{-1}(n)$ is a smooth submanifold of M .

Back to the case of Hamiltonian G -manifold (M, ω, G, μ) . According to the equivariance of μ ,

$$\mu(g \cdot m) = \operatorname{Ad}_g^* \circ \mu(m),$$

and use the fact that the co-adjoint action is linear, we conclude

Lemma 1.2. *If $\mu(m) = 0$. then for any $g \in G$, $\mu(g \cdot m) = 0$.*

In other words, the G -actions on M induces a G -action on $\mu^{-1}(0)$. To proceed we want $\mu^{-1}(0)$ to be a smooth manifold.

Proposition 1.3. *The G -action is locally free at each $m \in \mu^{-1}(0)$ if and only if $d\mu_m$ is surjective, i.e. m is a regular point of μ .*

Proof. The action is locally free if and only if $\mathfrak{g}_m = \{0\}$, if and only if $\operatorname{Im}(d\mu_m) = \mathfrak{g}^*$, i.e. $d\mu_m$ is surjective. □

As a consequence,

Corollary 1.4. *If the G -action on $\mu^{-1}(0)$ is free, then 0 is a regular value of μ , and thus $\mu^{-1}(0)$ is a closed submanifold of M whose codimension is $\dim G$.*

Since G is compact, any G -action is proper. So according to Lie theory,

Proposition 1.5. *If the G -action on $\mu^{-1}(0)$ is free, then $\mu^{-1}(0)/G$ is a smooth manifold of dimension $2 \dim M - 2 \dim G$ and the projection map $\pi : \mu^{-1}(0) \rightarrow \mu^{-1}(0)/G$ is a principal G -bundle.*

Remark. If the G -action on $\mu^{-1}(0)$ is not free but locally free (i.e. only assume 0 is a regular value of μ), then by the slice theorem, locally $\mu^{-1}(0)/G$ is just $(G \times_{G_m} D)/G \simeq D/G_m$. In other words, the quotient μ^{-1}/G is an *orbifold* (a topological space that locally looks like $\mathbb{R}^n$ divided by a finite group action, slightly generalizes the conception of manifolds).

¶ The linear reduction.

We would like to construct a symplectic structure on the quotient space $\mu^{-1}(0)/G$. The following theorem appeared in lecture 1:

Theorem 1.6. *Let W be a coisotropic subspace of a symplectic vector space (V, Ω) , then the induced 2-form on W/W^Ω ,*

$$(1) \quad \Omega'([v_1], [v_2]) := \Omega(v_1, v_2),$$

is a symplectic 2-form.

To construct a symplectic structure on $\mu^{-1}(0)/G$, we need to take a closer look at $\mu^{-1}(0)$:

Lemma 1.7. *Suppose G acts freely on $\mu^{-1}(0)$. Then for any $m \in \mu^{-1}(0)$, $T_m(\mu^{-1}(0))$ is a co-isotropic subspace of $T_m M$ whose symplectic ortho-complement is $T_m(G \cdot m)$.*

Proof. Since 0 is a regular value, for each $m \in \mu^{-1}(0)$ we have

$$T_m \mu^{-1}(0) = \ker(d\mu_m).$$

In other words, $T_m \mu^{-1}(0)$ is the symplectic ortho-complement of $T_m(G \cdot m)$. Since G acts on $\mu^{-1}(0)$, $m \in \mu^{-1}(0)$ implies $G \cdot m \in \mu^{-1}(0)$. So $T_m(G \cdot m) \subset T_m \mu^{-1}(0)$. The conclusion follows. $\square$

¶ The Marsden-Weinstein-Meyer theorem.

Now we are ready to prove

Theorem 1.8 (Marsden-Weinstein-Meyer). *Suppose G is a compact Lie group and (M, ω, G, μ) a Hamiltonian G -space. Let $\iota : \mu^{-1}(0) \hookrightarrow M$ be the inclusion map and assume that G acts freely on $\mu^{-1}(0)$. Then*

- (1) *The orbit space $M_{red} := \mu^{-1}(0)/G$ is a smooth manifold.*
- (2) *The projection $\pi : \mu^{-1}(0) \rightarrow M_{red}$ is a principal G -bundle (and in particular π is a submersion).*

(3) *There is a symplectic form ω_{red} on M_{red} such that*

$$\iota^* \omega = \pi^* \omega_{red}.$$

We will call (M_{red}, ω_{red}) the symplectic quotient of (M, ω) . Sometimes we will use the notion $M_{red} = M // G$.

Proof. We have already proved (1) and (2). To prove (3) we shall apply theorem 1.6 and lemma 1.7, to conclude that for each $m \in \mu^{-1}(0)$, one can construct canonically a symplectic structure on $T_m \mu^{-1}(0) / T_m(G \cdot m)$. On the other hand, at each $[m] \in M_{red} = \mu^{-1}(0) / G$, $T_{[m]} M_{red}$ is canonically isomorphic to $T_m \mu^{-1}(0) / T_m(G \cdot m)$. So we get a non-degenerate 2-form on M_{red} . Moreover, by construction (1) one gets

$$\iota^* \omega = \pi^* \omega_{red}.$$

As a consequence,

$$\pi^* d\omega_{red} = d\pi^* \omega_{red} = d\iota^* \omega = \iota^* d\omega = 0.$$

Since π^* is injective, we conclude $d\omega_{red} = 0$, i.e. ω_{red} is a symplectic form on M_{red} . $\square$

¶ Several examples.

Example. Consider the S^1 action on S^2 by rotations with respect to z axis. The action is Hamiltonian with moment map

$$\mu(x, y, z) = z.$$

Obviously S^1 acts freely on the equator $\mu^{-1}(0)$, and the quotient is a single point which is a 0 dimensional manifold.

Example. Consider the diagonal S^1 action on $\mathbb{C}^n$ by

$$e^{i\theta} \cdot (z_1, \dots, z_n) := (e^{i\theta} z_1, \dots, e^{i\theta} z_n).$$

This is a Hamiltonian action with moment maps

$$\mu(z) = \frac{1}{2}(|z_1|^2 + \dots + |z_n|^2) + c,$$

where $c \in \mathbb{R}$ is arbitrary and we identified $\text{Lie}(S^1)^* = \mathbb{R}$. In particular if we take $c = -\frac{1}{2}$, then $\mu^{-1}(0) = S^{2n-1}$ is the unit sphere in $\mathbb{C}^n$, and the symplectic quotient is $\mu^{-1}(0) / S^1 = S^{2n-1} / S^1 \simeq \mathbb{CP}^{n-1}$. The quotient symplectic form is the same (up to a constant factor) as the Fubini-Study symplectic form on $\mathbb{CP}^{n-1}$ that we constructed in problem set 2.

2. REDUCTION AT OTHER LEVELS

¶ Reduction at other level sets - the case of torus.

The Marsden-Weinstein-Meyer theorem tells us how to reduce at the 0 level set of the moment map μ . A natural question is whether we can perform reduction at other level sets of μ . The the case of torus, this is trivial, since if $G = \mathbb{T}^n$, and

$$\mu : M \rightarrow \mathfrak{g}^* \simeq \mathbb{R}^n$$

is a moment map for some Hamiltonian G -action, then for any $\xi \in \mathfrak{g}^*$, the map

$$\mu_\xi = \mu - \xi$$

is a moment map for the same action. So to reduce at the ξ level set of μ is the same as to reduce at the 0 level set of this new moment map μ_ξ .

However, for more general Lie groups the argument above does not work. In fact, for a given moment map μ and an arbitrary $\xi \in \mathfrak{g}^*$, if $m \in M$ is a point with $\mu(m) = \xi$, in general $\mu(g \cdot m) \neq \xi$. In other words, G does not acts on $\mu^{-1}(\xi)$.

Remark. Of course one can modify the torus argument above to general compact Lie groups to get reduction at special level sets: If $\xi \in H^1(\mathfrak{g}, \mathbb{R}) = [\mathfrak{g}, \mathfrak{g}]^0$, then one can reduce at the level set $\mu^{-1}(\xi)$, since in this case $\mu_\xi = \mu - \xi$ is a moment map.

¶ G -action on the pre-image of coadjoint orbits.

Suppose G is a compact Lie group and (M, ω, G, μ) a Hamiltonian G -space. According to the equivariance of μ ,

$$\mu(g \cdot m) = \text{Ad}_g^*(\mu(m)).$$

So for any $m \in M$ and any $g \in G$, $g \cdot m \in \mu^{-1}(\mathcal{O}_{\mu(m)})$, where

$$\mathcal{O}_{\mu(m)} = G \cdot \mu(m) = \text{Ad}_G^*(\mu(m))$$

is the orbit of the coadjoint action $\text{Ad}^* : G \rightarrow \text{Aut}(\mathfrak{g}^*)$ containing $\mu(m)$. As a consequence,

Lemma 2.1. *For each coadjoint orbit $\mathcal{O}$ in $\mathfrak{g}^*$, one gets an induced G -action on the subset $\mu^{-1}(\mathcal{O}) \subset M$.*

The three interesting special cases we studied are:

- (1) Since the coadjoint action is linear, the coadjoint orbit through $0 \in \mathfrak{g}^*$ is $\{0\}$. So we get a G -action on $\mu^{-1}(0)$.
- (2) If G is abelian, i.e. $G = \mathbb{T}^n$ is a torus, then the coadjoint action is the trivial action $\text{Ad}_g^*(\xi) = \xi$. So each $\xi \in \mathfrak{g}^*$ is a coadjoint orbit. As a consequence, G acts on $\mu^{-1}(\xi)$.
- (3) For more general compact connected Lie group G , if $\xi \in [\mathfrak{g}, \mathfrak{g}]^0$, again we have $\text{Ad}_g^*\xi = \xi$ for any $g \in G$. To prove this we write $g = \exp(Y)$ (which

is possible since G is compact, in general we can write g as a product of elements of this form and the same proof below works), then for any $X \in \mathfrak{g}$,

$$\langle \text{Ad}_{\exp Y}^* \xi, X \rangle = \langle \xi, \text{Ad}_{\exp(-Y)} X \rangle = \langle \xi, \exp(-\text{ad} Y) X \rangle = \langle \xi, X \rangle.$$

¶ The shifting trick.

Now let $\mathcal{O}$ be a coadjoint orbit of $\mathfrak{g}$. According to Yuguo's presentation on coadjoint orbits, on $\mathcal{O}$ one can define a G -invariant symplectic form $\omega_{\mathcal{O}}$ (called the *Kostant-Kirillov symplectic structure*) via

$$(\omega_{\mathcal{O}})_\xi(X, Y) := \langle \xi, [X, Y] \rangle.$$

One can prove that the induced G -action on $\mathcal{O}$ is Hamiltonian with moment map $\mu(\xi) = \xi$.

Now consider the product manifold $M \times \mathcal{O}$, equipped with a *twisted* product symplectic form $\omega \times (-\omega_{\mathcal{O}})$. Then the natural product action of G on $(M \times \mathcal{O}, \omega \times (-\omega_{\mathcal{O}}))$ is Hamiltonian with moment map

$$\mu_{\mathcal{O}}(m, \xi) = \mu(m) - \xi.$$

Note that if $\mu_{\mathcal{O}}(m, \xi) = 0$ if and only if $\mu(m) \in \mathcal{O}$. So the natural projection

$$F : \mu_{\mathcal{O}}^{-1}(0) \rightarrow \mu^{-1}(\mathcal{O}), \quad (m, \xi) \mapsto m$$

is bijective, and thus one can identify $\mu^{-1}(0)$ with $\mu_{\mathcal{O}}^{-1}(0)$. As a consequence,

Theorem 2.2. *If the G -action on $\mu^{-1}(\mathcal{O})$ is free, then the quotient $\mu^{-1}(\mathcal{O})/G$ admits a quotient symplectic structure $\omega_{red}^{\mathcal{O}}$.*

The quotient $(\mu^{-1}(\mathcal{O})/G, \omega_{red}^{\mathcal{O}})$ is called the symplectic quotient of (M, ω) with respect to the coadjoint orbit $\mathcal{O}$.

¶ An alternative approach.

Here is an alternative approach to perform symplectic reduction at other levels. Consider the coadjoint action of G on $\mathfrak{g}^*$. Then the stabilizer of $\xi \in \mathfrak{g}^*$ is

$$G_\xi = \{g \in G \mid \text{Ad}_g^* \xi = \xi\}.$$

This is a closed subgroup of G , which acts on the level set $\mu^{-1}(\xi)$. So if this G_ξ -action on $\mu^{-1}(\xi)$ is free, we again get a quotient symplectic manifold $(M_{red}^\xi, \omega_{red}^\xi)$. This is in fact the same as what we get by the shifting trick above. In fact,

Proposition 2.3. *Suppose (M, ω, G, μ) is a Hamiltonian G space, and $\xi \in \mathfrak{g}^*$.*

- (1) *G acts freely on $\mu^{-1}(\mathcal{O}_\xi)$ if and only if G_ξ acts freely on $\mu^{-1}(\alpha)$.*
- (2) *The map $\psi : \mu^{-1}(\xi)/G_\xi \rightarrow \mu^{-1}(\mathcal{O}_\xi)/G$ induced by the inclusion $\mu^{-1}(\xi) \hookrightarrow \mu^{-1}(\mathcal{O}_\alpha)$ is a diffeomorphism.*

Proof.

□

3. REDUCTION IN STAGES

student presentation