LECTURE 11: SYMPLECTIC TORIC MANIFOLDS

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1. Symplectic toric manifolds

¶ Orbit of torus actions.
Recall that in lecture 9 we showed \( \ker(d\mu_m) = (T_m(G \cdot m))^\omega_m \).

**Proposition 1.1.** Let \((M, \omega, \mathbb{T}^k, \mu)\) be a compact connected Hamiltonian \(\mathbb{T}^k\)-space, then for any \(m \in M\), then orbit \(\mathbb{T}^k \cdot m\) is an isotropic submanifold of \(M\).

**Proof.** The moment map \(\mu\) is \(\mathbb{T}^k\)-invariant, so on the orbit \(\mathbb{T}^k \cdot m\), \(\mu\) takes a constant value \(\xi \in \mathfrak{t}^*\). It follows that the differential \(d\mu_m : T_m M \to T_\xi \mathfrak{t}^* \simeq \mathfrak{t}^*\) maps the subspace \(T_m(\mathbb{T}^k \cdot m)\) to 0. In other words,

\[
T_m(\mathbb{T}^k \cdot m) \subset \ker(d\mu_m) = (T_m(\mathbb{T}^k \cdot m))^\omega_m.
\]

So \(\mathbb{T}^k \cdot m\) is an isotropic submanifold of \(M\).

¶ Effective torus actions.

**Definition 1.2.** An action of a Lie group \(G\) on a smooth manifold \(M\) is called effective (or faithful) if each group element \(g \neq e\) moves at least one point \(m \in M\), i.e.

\[
\bigcap_{m \in M} G_m = \{e\}.
\]

(Equivalently, if the group homomorphism \(\tau : G \to \text{Diff}(M)\) is injective.)

**Remark.** If a group action \(\tau\) of \(G\) on \(M\) is not effective, then \(\ker(\tau)\) is a normal subgroup of \(G\), and the action \(\tau\) induces a smooth action of \(G/\ker(\tau)\) on \(M\) which is effective.

A remarkable fact on effective \(\mathbb{T}^k\)-action is
Theorem 1.3. Suppose $\mathbb{T}^k$ acts on $M$ effectively. Then the set of points where the action is free,
\[
\widetilde{M} = \{ m \in M \mid G_m = \{ e \} \},
\]
is an open and dense subset in $M$.


An important consequence is

Corollary 1.4. Let $(M, \omega, \mathbb{T}^k, \mu)$ be a compact connected Hamiltonian $\mathbb{T}^k$-space, if the $\mathbb{T}^k$-action is effective, then $\dim M \geq 2k$.

Proof. Pick any point $m$ in $M$ where the $\mathbb{T}^k$-action is free, i.e. $(\mathbb{T}^k)_m = \{ e \}$. Then the orbit $\mathbb{T}^k \cdot m$ is diffeomorphic to $\mathbb{T}^k/(\mathbb{T}^k)_m = \mathbb{T}^k$, and thus has dimension $k$. But we have just seen that $\mathbb{T}^k \cdot m$ is an isotropic submanifold of $M$. So
\[
k = \dim(\mathbb{T}^k \cdot m) \leq \frac{1}{2} \dim M.
\]
\[\square\]

\section*{Symplectic Toric manifolds.}

Definition 1.5. A compact connected symplectic manifold $(M, \omega)$ of dimension $2n$ is called a symplectic toric manifold if it is equipped with an effective Hamiltonian $\mathbb{T}^n$-action.

Example. $\mathbb{C}^n$ admits an effective Hamiltonian $\mathbb{T}^n$ action,
\[
(t_1, \cdots, t_n) \cdot (z_1, \cdots, z_n) = (t_1z_1, \cdots, z_nt_n),
\]
and thus is a symplectic toric manifold.

Example. $\mathbb{CP}^n$ admits an effective Hamiltonian $\mathbb{T}^n$ action,
\[
(t_1, \cdots, t_n) \cdot [z_0 : z_1 : \cdots : z_n] = [z_0 : t_1z_1 : \cdots : t_nz_n]
\]
and thus is a symplectic toric manifold. The image of the moment map is the simplex in $\mathbb{R}^n$ with $n + 1$ vertices $\frac{1}{2}e_i$ and $(0, \cdots, 0)$, where $e_i = (0, \cdots, 1, \cdots, 0)$.

Example. The products of toric manifolds is still toric.

Remark. A symplectic toric manifold is a special complete integrable system because for any $X, Y \in t$,
\[
\{ \mu^X, \mu^Y \}(m) = \omega_m(X_M(m), Y_M(m)) = 0.
\]
Delzant polytopes.

According to the Atiyah-Guillemin-Sternberg convexity theorem, the image of the moment map is always a convex polytope in \( \mathbb{R}^n \). The moment polytope of \( \mathbb{C}P^1 \), \( \mathbb{C}P^2 \) and \( \mathbb{C}P^1 \times \mathbb{C}P^1 \) are

\[
\begin{align*}
S^2 & \quad \mathbb{C}P^2 & \quad \mathbb{C}P^1 \times \mathbb{C}P^1
\end{align*}
\]

Definition 1.6. A polytope \( \Delta \in \mathbb{R}^n \) is called a Delzant polytope if

1. (simplicity) there are \( n \) edges meeting at every vertex \( p \).
2. (rationality) the edges meeting at \( p \) are of the form \( p + tu_i \), with \( u_i \in \mathbb{Z}^n \).
3. (smoothness) at each \( p \), \( u_1, \ldots, u_n \) form a \( \mathbb{Z} \)-basis of \( \mathbb{Z}^n \).

Obviously the previous examples are Delzant polytopes. More examples of Delzant polytopes

\[
\begin{align*}
(0,1) & \quad (1,1) \\
(0,0) & \quad (4,0)
\end{align*}
\]

The following polytopes are not Delzant:

\[
\begin{align*}
(0,2) & \quad (2,2) \\
(0,0) & \quad (3,0)
\end{align*}
\]

\[
\begin{align*}
(0,0,1) & \quad (0,1,0) \\
(0,0,0) & \quad (1,0,0)
\end{align*}
\]

Remark. Suppose a Delzant polytope has \( d \) faces. Let \( v_i \), \( 1 \leq i \leq d \), be the primitive outward-pointing normal vectors to the faces of \( \Delta \), then \( \Delta \) can be described via a set of inequalities

\[
\langle x, v_i \rangle \leq \lambda_i, \quad i = 1, \ldots, d
\]

Moment polytopes are Delzant.

Now we are ready to prove

Theorem 1.7. For any symplectic toric manifold \( (M, \omega) \), its moment polytope \( \Delta \) is a Delzant polytope.
Proof. Let $m \in M$ be a fixed point of the Hamiltonian torus action, then $p = \mu(m)$ is a vertex of the moment polytope. We have seen from the proof of the Atiyah-Guillemin-Sternberg convexity theorem that the moment polytope near $p$ is

$$\{ p + \sum_{i=1}^{n} s_i w_i \mid s_i \geq 0 \}$$

where $w_1, \ldots, w_n$ are the weights of the linearized isotropic action of the torus on $T_m M$. Thus $\Delta$ satisfies the conditions (1) and (2).

Suppose $\Delta$ does not satisfy the condition (3). Let $W$ be the $\mathbb{Z}$-matrix whose row vectors are the vectors $w_i$’s. Then $W$ is not invertible as a $\mathbb{Z}$-matrix. We take a vector $\tau \notin \mathbb{Z}^n$ such that $W \tau \in \mathbb{Z}^n$. (If $W$ is not invertible, we can take $\tau$ be any non-integer vector in the kernel of $W$. If $W$ is invertible as an $\mathbb{R}$-matrix but not invertible as a $\mathbb{Z}$-matrix, then $W^{-1}$ can not map all $\mathbb{Z}$-vectors to $\mathbb{Z}$ vectors ). So we have

$$\langle w_i, \tau \rangle \in \mathbb{Z}$$

for all $i$.

Recall that in a neighborhood of $m$, there exists coordinate system $(z_1, \cdots, z_n)$ so that the action of $\mathbb{T}^n$ is given by

$$\exp(X) \cdot (z_1, \cdots, z_n) = (e^{2\pi i \langle w_1, X \rangle} z_1, \cdots, e^{2\pi i \langle w_n, X \rangle} z_n).$$

So $\exp(\tau)$ acts trivially on a neighborhood of $m$, but $\exp(\tau)$ is not the identity element in $\mathbb{T}^n$. This contradicts with the fact that in a dense open subset of $M$ the action is free. So $\Delta$ satisfies (3). \qed

2. Delzant’s theorem

¶ Statement of main theorem.

The main result is the following classification for symplectic toric manifold, which says that symplectic toric manifolds are characterized by their moment polytopes:

**Theorem 2.1** (Delzant, 1990). There is a one-to-one correspondence between symplectic toric manifolds (up to $\mathbb{T}^n$ equivariant symplectomorphisms) and Delzant polytopes. More precisely,

1. The moment polytope of a toric manifold is a Delzant polytope.
2. Every Delzant polytope is the moment polytope of a symplectic toric manifold.
3. Two toric manifolds with the same moment polytope are equivariantly symplectomorphic.

The proof is divided into several steps:

Step 1: $M$ toric $\Rightarrow \mu(M)$ Delzant. (Done as theorem 1.7.)

Step 2: $\Delta$ Delzant $\Rightarrow$ construct compact connected symplectic manifold $M_\Delta$. 
Step 3: Check $M_\Delta$ is toric and $\mu(M_\Delta) = \Delta$.

Step 4: $\Delta(M_1) = \Delta(M_2) \iff M_1 \simeq M_2$.

Construction of $M_\Delta$ from Delzant polytope $\Delta$.

Now let $\Delta$ be a Delzant polytope in $\mathbb{R}^n$. Suppose $\Delta$ has $d$ facets, then by the algebraic description one can find primitive outward pointing vectors $v_1, \ldots, v_d$ so that

$$\Delta = \{ x \in (\mathbb{R}^n)^* \mid \langle x, v_i \rangle \leq \lambda_i, i = 1, \ldots, d \}.$$  

By translation we may assume $0 \in \Delta$, and thus $\lambda_i \geq 0$ for all $i$. We shall construct $M_\Delta$ as the symplectic quotient of $\mathbb{R}^d$ by a Hamiltonian action of a torus $N$ of dimension $d - n$.

▶ Step 2.a The $(d - n)$-torus $N$.

Let $e_1, \ldots, e_d$ be the standard basis of $\mathbb{R}^d$. Define linear map

$$\pi : \mathbb{R}^d \to \mathbb{R}^n, \quad e_i \mapsto v_i.$$  

Then since $\Delta$ is Delzant, $\pi$ is onto and maps $\mathbb{Z}^d$ onto $\mathbb{Z}^n$. So we get an induced surjective Lie group homomorphism

$$\pi : \mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d \to \mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n.$$  

Let $N = \ker(\pi)$. It is a $(d - n)$-subtorus of $\mathbb{T}^d$.

Note that from the exact sequence of Lie group homomorphisms

$$0 \to N \xrightarrow{i} \mathbb{T}^d \xrightarrow{\pi} \mathbb{T}^n \to 0$$

one gets an exact sequence of Lie algebras

$$0 \to \mathfrak{n} \xrightarrow{i} \mathbb{R}^d \xrightarrow{\pi} \mathbb{R}^n \to 0,$$

and thus an exact sequence of dual Lie algebras

$$0 \to (\mathbb{R}^n)^* \xrightarrow{\pi^*} (\mathbb{R}^d)^* \xrightarrow{i^*} \mathfrak{n}^* \to 0.$$

▶ Step 2.b The Hamiltonian $N$-action on $\mathbb{C}^d$.

The standard $\mathbb{T}^d$-action on $\mathbb{C}^d$ is given by

$$(e^{i\theta_1}, \ldots, e^{i\theta_d}) \cdot (z_1, \ldots, z_d) = (e^{i\theta_1}z_1, \ldots, e^{i\theta_d}z_d).$$

The action is Hamiltonian with moment map

$$\phi : \mathbb{C}^d \to (\mathbb{R}^d)^*, \quad \phi(z_1, \ldots, z_d) = -\frac{1}{2}(\lvert z_1 \rvert^2, \ldots, \lvert z_d \rvert^2) + c.$$  

We choose $c = \lambda = (\lambda_1, \ldots, \lambda_d)$.

Since $N$ is a sub-torus of $\mathbb{T}^d$, the induced $N$-action on $\mathbb{C}^d$ is Hamiltonian with moment map $i^* \circ \phi : \mathbb{C}^d \to \mathfrak{n}^*$.

▶ Step 2.c The zero level set $Z = (i^* \circ \phi)^{-1}(0)$ is compact.

Let $\Delta' = \pi^*(\Delta)$. Then $\Delta$ is compact. We claim
Claim I: $\text{Im}(\pi^*) \cap \text{Im}(\phi) = \Delta'$.

According to this claim,

$$Z = (i^* \circ \phi)^{-1}(0) = \phi^{-1}(\ker(i^*)) = \phi^{-1}(\text{Im}(\pi^*)) = \phi^{-1}(\Delta').$$

So $Z$ is compact since the map $\phi$ proper.

Proof of claim I: Obviously $\Delta' = \pi^*(\Delta) \subset \text{Im}(\pi^*)$. By the definition $\phi$, $\text{Im}(\phi)$ consists of those points $y$ with $\langle y, e_i \rangle \leq \lambda_i$. For any $x \in \Delta$,

$$\langle \pi^*(x), e_i \rangle = \langle x, v_i \rangle \leq \lambda_i.$$ 

So $\pi^*(\Delta) \subset \text{Im}(\phi)$, and thus $\Delta' \subset \text{Im}(\pi^*) \cap \text{Im}(\phi)$.

Conversely suppose $y = \pi^*(z) = \phi(w)$. Then

$$\langle z, v_i \rangle = \langle \pi^*(z), e_i \rangle \leq \lambda_i.$$ 

In other words, $z \in \Delta$. It follows $\text{Im}(\pi^*) \cap \text{Im}(\phi) \subset \Delta'$. □

\[ \text{Step 2.d} \] $N$ acts freely on $Z$.

For $z \in \mathbb{Z}^d$, let $I_z = \{ i \mid z_i = 0 \}$. Then

$$(\mathbb{T}^d)_z = \{ t \in \mathbb{T}^d \mid t_i = 1 \text{ for } i \notin I_z \}.$$ 

Claim II: The restriction map of $\pi$, $\pi : (\mathbb{T}^d)_z \to \mathbb{T}^n$, is injective.

This implies

$$N_z = N \cap (\mathbb{T}^d)_z = i(N) \cap (\mathbb{T}^d)_z = \ker(\pi) \cap (\mathbb{T}^d)_z = \ker(\pi|_{(\mathbb{T}^d)_z}) = \{1\}.$$ 

So the action is free.

Proof of claim II. Suppose $z \in Z = (i^* \circ \phi)^{-1}(0)$, i.e.

$$\phi(z) \in \ker(i^*) = \text{Im}(\pi^*).$$

Then $\phi(z) \in \text{Im}(\pi^*) \cap \text{Im}(\phi) = \Delta'$. So one can find $x \in \Delta$ so that $\phi(z) = \pi^*(x)$. Thus

$$i \in I_z \iff z_i = 0 \iff \langle \phi(z), e_i \rangle = \lambda_i \iff \langle \pi^*(x), e_i \rangle = \lambda_i \iff \langle x, v_i \rangle = \lambda_i.$$ 

In other words, $x$ is a point in the intersection of facets whose normal vectors are $v_i$. As a consequence, we see that the set of vectors

$$\{ v_i \mid i \in I_z \}$$ 

are linearly independent.

Now let $[t], [s] \in (\mathbb{T}^d)_z$. If $\pi([t]) = \pi([s])$, then

$$\pi(t) - \pi(s) = \sum_{i \in I_z} (t_i - s_i)v_i \in \mathbb{Z}^n.$$ 

It follows that $t_i - s_i \in \mathbb{Z}$ for $i \in I_z$. So $[t] = [s]$. □
In conclusion, we see that $M_{\Delta} = \mathbb{C}^d / N = Z / N$ is a compact symplectic manifold of dimension $2d - 2(d - n) = 2n$.

Remark. Since $M_{\Delta}$ is constructed via $\mathbb{C}^d$ which is Kähler, with more works one can prove that $M_{\Delta}$ is actually a Kähler manifold.

The moment polytope of $M_{\Delta}$ is $\Delta$.

We need to show that $M_{\Delta}$ admits a Hamiltonian $\mathbb{T}^n$ action which is effective. The action is actually very natural:

\begin{itemize}
  \item[\textbf{Step 3.a}] Hamiltonian $\mathbb{T}^n$-action on $M_{\Delta}$.
  Suppose $z$ is a point such that
  \[ \phi(z) = \pi^*(x) \]
  for a vertex $x$ of $\Delta$. Then from the proof of claim I we see that $\dim(\mathbb{T}^d)_z$ equals the number of facets of $\Delta$ that meets at $p$, i.e.
  \[ \dim(\mathbb{T}^d)_z = n. \]
  So by claim I, the map
  \[ \pi : (\mathbb{T}^d)_z \to \mathbb{T}^n \]
  is bijective. By identifying $\mathbb{T}^n$ with $(\mathbb{T}^d)_z$ we get an embedding $\tilde{j} : \mathbb{T}^n \hookrightarrow \mathbb{T}^d$ with
  \[ \pi \circ \tilde{j} = \text{Id}. \]
  So $\mathbb{T}^n$ acts on $\mathbb{C}^d$ in a Hamiltonian way, with moment map $\tilde{j}^* \phi$. Moreover, this $\mathbb{T}^n$-action commutes with the $N$-action we constructed above. Thus by the reduction by stages arguments (presented by Chao'en last time), we get an induced Hamiltonian $\mathbb{T}^n$-action on $M_{\Delta}$, whose moment map $\mu$ satisfies
  \[ \mu \circ \text{pr} = \tilde{j}^* \circ \phi \circ j, \]
  where $\text{pr}$ is the projection from $Z$ to $M_{\Delta}$, and $j$ is the inclusion from $Z$ to $\mathbb{C}^d$.
  \item[\textbf{Step 3.b}] The above $\mathbb{T}^n$-action is effective.
  Since the $\mathbb{T}^d$ action is effective, this induced $\mathbb{T}^n$-action is also effective.
  \item[\textbf{Step 3.c}] The moment polytope of $M_{\Delta}$ is $\Delta$.
\end{itemize}
Equivariantly symplectomorphic toric manifolds.

- **Step 4.a** If \((M_1, \omega_1, T^n, \mu_1)\) and \((M_2, \omega_2, T^n, \mu_2)\) are two equivariantly diffeomorphic toric manifolds, then \(\mu_1(M_1)\) and \(\mu_2(M_2)\) differ by a translation.

  In fact, if we let \(\Phi : M_1 \to M_2\) be the equivariant diffeomorphism. Then \(\Phi^* \mu_2\) is also a moment map for the \(T^n\)-action on \(M_1\), because

  \[
d\langle \Phi^* \mu_2, X \rangle = \Phi^* d\langle \mu_2, X \rangle = \Phi^* \iota_{X_{M_2}} \omega_2 = \iota_{X_{M_1}} \omega_1.
\]

  So there exists a constant \(\xi \in t^*\) so that \(\Phi^* \mu_2 = \mu_1 + \xi\). It follows that

  \[
  \mu_2(M_2) = \Phi^* \mu_2(M_1) = \mu_1(M_1) + \xi.
  \]

Now suppose \((M_1, \omega_1, T^n, \mu_1)\) and \((M_2, \omega_2, T^n, \mu_2)\) are two symplectic toric manifolds with \(\mu_1(M_1) = \mu_2(M_2)\). We would like to prove that there exists an equivariant diffeomorphism that sends \(M_1\) to \(M_2\).

- **Step 4.b** If \(\mu_1(M_1) = \mu_2(M_2)\), one can construct by induction a diffeomorphism that intertwines the torus actions and moment maps.

- **Step 4.c** Show that the cohomology class of \(\omega\) is determined by the moment polytope.

- **Step 4.d** Apply Moser’s trick.

  For more details, c.f. Kai Cieliebak’s notes, P. 40-45.

3. Symplectic cut

Student presentation