

LECTURE 12: GEOMETRIC PREQUANTIZATION

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1. THE IDEA OF QUANTIZATION

¶ Classical mechanics modelled on symplectic manifolds.

Recall that in the Hamiltonian formulation of classical mechanics, the phase space of a mechanical system is a symplectic manifold (M, ω) . The symplectic manifold could be a cotangent space T^*X , or more generally an arbitrary symplectic manifold (e.g. a symplectic quotient). Any point in M represents a possible *state* of the system. A Hamiltonian function H is a (smooth) function on M which represents a conserved quantity of the system. The trajectory of the system is an integral curve of the Hamiltonian vector field Ξ_H . In local Darboux coordinates the integral curve is given by the system of Hamiltonian equations

$$\dot{x}(t) = \frac{\partial H}{\partial \xi}, \quad \dot{\xi}(t) = -\frac{\partial H}{\partial x}.$$

A classical observable of the system is just a smooth function a on M . We have seen in lecture 6 that the evolution of a satisfies the equation

$$(1) \quad \dot{a} = \{a, H\},$$

where $\{\cdot, \cdot\}$ is the Poisson bracket on $C^\infty(M)$ induced by the symplectic structure.

¶ Quantum mechanics modeled on Hilbert space.

In the Schrödinger formulation of quantum mechanics, the state space of a quantum mechanical system is a Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$. (Or more precisely, the projectified Hilbert space $\mathbb{P}\mathcal{H}$.) A quantum state is a unit vector in \mathcal{H} . A quantum Hamiltonian is a self-adjoint operator \hat{H} acting on \mathcal{H} , whose eigenvalues represents the quantum energy level of the system, and whose normalized eigenfunctions represents the corresponding quantum states. We shall denote the eigenvalues and

eigenfunctions of \hat{H} by λ_j and ψ_j . The equation describing the evolution of the system is the Schrödinger equation

$$i\hbar \frac{d\psi(t)}{dt} = \hat{H}\psi.$$

A quantum observable is a self-adjoint operator A acting on \mathcal{H} . The expectation value of a quantum observable A in state ψ is given by

$$\langle A \rangle_\psi := \langle A\psi, \psi \rangle.$$

Proposition 1.1. *The evolution of the quantum observable is*

$$\frac{d}{dt} \langle A \rangle_\psi = \frac{1}{i\hbar} \langle [A, \hat{H}] \rangle_\psi.$$

Proof.

$$\frac{d}{dt} \langle A\psi(t), \psi(t) \rangle = \left\langle \frac{1}{i\hbar} A\hat{H}\psi, \psi \right\rangle + \left\langle A\psi, \frac{1}{i\hbar} \hat{H}\psi \right\rangle = \frac{1}{i\hbar} \langle [A, \hat{H}]\psi, \psi \rangle.$$

□

Comparing this with (1) we see that the quantum analogy of the Poisson bracket should be the Lie bracket of operators.

¶ The idea of quantization.

The word “quantization” represents a procedure (a correspondence, a functor ...) that convert a classical Hamiltonian system to its quantum analogue. More precisely, we want a “dictionary”

$$\begin{aligned} (M, \omega) &\rightsquigarrow (\mathcal{H}, \langle \cdot, \cdot \rangle) \\ H &\rightsquigarrow \hat{H} \\ a &\rightsquigarrow A \\ \{\cdot, \cdot\} &\rightsquigarrow [\cdot, \cdot] \end{aligned}$$

At the very beginning of the whole story, Dirac proposed a set of axioms that a quantization procedure should satisfy.

Dirac’s axiom: A quantization procedure assigns self-adjoint operators $Q(a)$ on some Hilbert space \mathcal{H} to classical observable $a \in C^\infty(M)$, so that

- (1) (Linearity) $Q(\lambda a + \mu b) = \lambda Q(a) + \mu Q(b)$.
- (2) (Normalization) $Q(1) = \text{Id}$.
- (3) (Quantum condition) $Q(\{a, b\}) = \frac{1}{i\hbar} [Q(a), Q(b)]$.
- (4) (Minimality) A complete set of (Poisson commuting) functions is quantized to a complete set of (Lie commuting) operators.

[A procedure that only satisfies (1)-(3) is called a *prequantization*.]

Unfortunately it was shown by Groenewold and Van Hove that such a quantization procedure never exist. As a result, mathematicians have developed many different kinds of quantization procedure, each have a different emphasis. For example, the Weyl's quantization works for nice *symbols* on \mathbb{R}^{2n} , or more generally T^*X , with quantum condition replaced by an asymptotic expansion. The deformation quantization concerns more on the affection on the Poisson algebra without indicating a Hilbert space.

In the last two lectures of this course, we will discuss the so-called *geometric quantization*. It was proposed by Kostant and has the advantage that it is coordinate free and works for a very wide class of symplectic manifolds. The Hilbert space quantizing (M, ω) will be a space of sections of a complex line bundle over M .

2. COMPLEX LINE BUNDLES

¶ Complex line bundles via transition functions.

Let M be a smooth manifold, $\mathbb{L} \rightarrow M$ a smooth complex line bundle over M , and $\pi : \mathbb{L} \rightarrow M$ the projection map. Recall that this means

- At each $m \in M$, $\mathbb{L}_m = \pi^{-1}(m)$ is a complex vector space of dimension 1
- there exists an open covering $\{U_i\}$ of M and diffeomorphisms

$$\varphi_i : \pi^{-1}(U_i) \rightarrow U_i \times \mathbb{C}$$

so that for $m \in U_i$, the restriction

$$\varphi_i|_{\mathbb{L}_m} : \mathbb{L}_m \rightarrow \{m\} \times \mathbb{C}$$

is a linear isomorphism.

Consider the transition maps $g_{ij} : U_i \cap U_j \rightarrow \mathbb{C}^* = \text{GL}(\mathbb{C})$ defined by

$$g_{ij}(m) = \varphi_i|_{\mathbb{L}_m} \circ (\varphi_j|_{\mathbb{L}_m})^{-1}.$$

Obviously the transition functions $\{g_{ij}\}$ satisfy the relations

$$g_{ii} = 1, \quad g_{ij}g_{ji} = 1, \quad g_{ij}g_{jk}g_{ki} = 1.$$

Conversely, it is well known that given any open covering $\{U_i\}$ of M and functions $\{g_{ij} : U_i \cap U_j \rightarrow \mathbb{C}^*\}$ satisfying relations above, there exists a complex line bundle \mathbb{L} over M so that $\{g_{ij}\}$ are the transition functions of \mathbb{L} .

So line bundles are completely determined by their transition functions. Using transition functions one can characterize whether two line bundles are isomorphic: two line bundles \mathbb{L} and $\tilde{\mathbb{L}}$ are isomorphic if and only if on a common refinement $\{U_i\}$ of the defining coverings, there exists smooth functions $\lambda_i : U_i \rightarrow \mathbb{C}^*$ such that on $U_i \cap U_j$,

$$\lambda_i g_{ij} \lambda_j^{-1} = \tilde{g}_{ij}.$$

Now suppose \mathbb{L} is an Hermitian line bundle over M . That means, we have an Hermitian inner product $\langle \cdot, \cdot \rangle_m$ on \mathbb{L}_m , which varies smoothly on m and is preserved by $\varphi_i|_{\mathbb{L}_m}$. As a consequence, each transition map becomes $g_{ij} : U_i \cap U_j \rightarrow S^1$.

¶ Connections and curvature forms.

Now suppose \mathbb{L} is a complex line bundle over smooth manifold M . The space of smooth sections of \mathbb{L} is denoted by $\Gamma^\infty(M, \mathbb{L})$. The space of smooth k -forms on M with coefficients in \mathbb{L} is

$$\Omega^k(M, \mathbb{L}) := \Gamma^\infty(M, \wedge^k T^*M \otimes \mathbb{L}).$$

Note that $\Omega^0(M, \mathbb{L}) = \Gamma^\infty(M, \mathbb{L})$. In the case $\mathbb{L} = M \times \mathbb{C}$ is a trivial bundle, one has the identification $\Omega^k(M, \mathbb{L}) \simeq \Omega^k(M)$.

Definition 2.1. A connection ∇ on \mathbb{L} is a linear map

$$\nabla : \Gamma^\infty(M, \mathbb{L}) \rightarrow \Omega^1(M, \mathbb{L})$$

so that for any $f \in C^\infty(M)$ and any $s \in \Gamma^\infty(M, \mathbb{L})$, one has the Leibniz rule

$$\nabla(fs) = df \otimes s + f\nabla s.$$

For any smooth vector field $X \in \text{Vect}(M)$, one can contract X with ∇ to get the *covariant derivative* in the direction of X ,

$$\nabla_X : \Gamma^\infty(M, \mathbb{L}) \rightarrow \Gamma^\infty(M, \mathbb{L}), \quad \nabla_X s := \iota_X \nabla s.$$

A connection ∇ on \mathbb{L} can be extended uniquely to a linear map

$$\nabla : \Omega^k(M, \mathbb{L}) \rightarrow \Omega^{k+1}(M, \mathbb{L})$$

so that for any $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^\bullet(M, \mathbb{L})$,

$$\nabla(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge \nabla \beta.$$

Using this generalized Leibniz rule, it is easy to see that for any $f \in C^\infty(M)$ and any $\beta \in \Omega^\bullet(M, \mathbb{L})$, the map $\nabla^2 : \Omega^\bullet(M, \mathbb{L}) \rightarrow \Omega^{\bullet+2}(M, \mathbb{L})$ satisfies

$$\nabla^2(f\beta) = \nabla(f\nabla\beta + df \wedge \beta) = f\nabla^2\beta + df \wedge \nabla\beta - df \wedge \nabla\beta = f\nabla^2\beta.$$

As a consequence, the map ∇^2 is given by “multiplication by a 2-form”, i.e. there exists a 2-form $\Omega \in \Omega^2(M)$ so that for any $s \in \Gamma^\infty(M, \mathbb{L})$,

$$\nabla^2 s = \Omega s.$$

Definition 2.2. The 2-form Ω is called the *curvature* of ∇ .

Exercise 1. As a map from $\Gamma^\infty(M, \mathbb{L})$ to $\Gamma^\infty(M, \mathbb{L})$,

$$\Omega(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X, Y]}.$$

¶ Differential geometry of complex line bundles.

Now suppose \mathbb{L} be an Hermitian line bundle over M .

Definition 2.3. We say a connection ∇ on \mathbb{L} is *unitary*, or is *compatible* with the Hermitian structure, if for any $s, t \in \Gamma^\infty(M, \mathbb{L})$,

$$d\langle s, t \rangle = \langle \nabla s, t \rangle + \langle s, \nabla t \rangle.$$

Now let $\{e_i\}$ be a unitary frame of the Hermitian line bundle \mathbb{L} . Then there exists locally defined *connection 1-form* θ_i so that

$$\nabla e_i = \theta_i e_i.$$

Note that if \tilde{e}_i is another local frame with $\tilde{e}_i = g(x)e_i$, then

$$\nabla \tilde{e}_i - \nabla(g e_i) = (dg + g\theta_i)e_i = (d \log g + \theta_i)\tilde{e}_i,$$

i.e.

$$\tilde{\theta}_i = d \log g + \theta_i.$$

As a consequence, $\tilde{\theta}_i - \theta_i$ is a closed 1-form.

Moreover, since e_i is unitary,

$$0 = d\langle e_i, e_i \rangle = \theta_i + \bar{\theta}_i,$$

i.e. θ_i is pure imaginary.

Exercise 2. Using connection 1-forms, one can write the curvature 2-form as

$$\Omega = d\theta - \theta \wedge \theta = d\theta.$$

(It is not exact since θ is not globally defined.)

As a consequence, Ω is globally defined, closed, purely imaginary 2-form.

Definition 2.4. The *first Chern class* of the line bundle \mathbb{L} is

$$c_1(\mathbb{L}) := \left[\frac{1}{2\pi i} \Omega \right] \in H_{deRham}^2(M, \mathbb{R}).$$

Remark. According to the famous Chern-Weil theorem, $c_1(\mathbb{L})$ is independent of the choice of the connection and the Hermitian metric on \mathbb{L} , and thus is a topological invariant of \mathbb{L} .

¶ Weil's theorem.

Recall how the de Rham isomorphism sends an element $[a] \in H_{deRham}^2(M, \mathbb{R})$ to an element $[c] \in H_{Cech}^2(M, \mathbb{R})$:

- First take a *good cover* $\{U_i\}$ so that all intersections are contractible.
- For $[a] \in H_{deRham}^2(M)$, one can find a 1-form b_i on U_i so that $a = db_i$ on U_i .
- On the contractible set $U_i \cap U_j$, $db_i = db_j$. So one can find a function c_{ij} on $U_i \cap U_j$ so that $b_i - b_j = dc_{ij}$.

- On $U_i \cap U_j \cap U_k$, $dc_{ij} + dc_{jk} + dc_{ki} = 0$. So the function $c_{ijk} = c_{ij} + c_{jk} + c_{ki}$ is a constant function. They define a Čech cohomology class in $H^2(M, \mathbb{R})$.

Back to $c_1(\mathbb{L})$. On each U_i we have $\Omega = d\theta_i$, and on $U_i \cap U_j$ we have $\theta_i - \theta_j = d \log g_{ij}$. So the de Rham isomorphism sends $c_1(\mathbb{L}) = [\frac{1}{2\pi i} \Omega]$ to $[c] \in H^2(M, \mathbb{R})$ with

$$c_{ijk} = \frac{1}{2\pi i} (\log g_{ij} + \log g_{jk} + \log g_{ki}).$$

Since the g_{ij} 's are the transition functions and satisfy the cocycle condition $g_{ij}g_{jk}g_{ki} = 1$, we get

$$e^{2\pi i c_{ijk}} = 1.$$

This means $c_{ijk} \in \mathbb{Z}$, i.e. $[c] \in H^2(M, \mathbb{Z})$.

In conclusion, we $c_1(\mathbb{L})$ is an *integral* cohomology class:

Proposition 2.5. $c_1(\mathbb{L}) \in H^2(M, \mathbb{Z})$.

Conversely, we have

Theorem 2.6 (Weil). *Let M be a smooth manifold and ω a real, closed 2-form whose cohomology class $[c]$ is integral. Then there is a unique Hermitian line bundle \mathbb{L} over M with unitary connection ∇ so that $c_1(\mathbb{L}) = [c]$.*

Sketch of Proof. Existence: Reverse the arguments above. Since $[c]$ is integral, $e^{2\pi i c_{ijk}} = 1$. As a consequence, $g_{ij}g_{jk}g_{kl} = 1$. So g_{ij} 's are the transition function for some line bundle whose first Chern class is $[c]$.

Uniqueness: Suppose $c_1(\mathbb{L}) = c_1(\tilde{\mathbb{L}})$. Let $h_{ij} = \frac{1}{2\pi i} g_{ij}$ and define \tilde{h}_{ij} similarly. Then the functions $\hat{h}_{ij} = h_{ij} - \tilde{h}_{ij}$ satisfies the relation

$$\hat{h}_{ij} + \hat{h}_{jk} + \hat{h}_{ki} = 0.$$

We take a partition of unity ρ_k and let $\lambda_i = e^{2\pi i \sum \hat{h}_{ki} \rho_k}$. Then

$$\lambda_i g_{ij} \lambda_j^{-1} = e^{2\pi i (h_{ij} + \sum (\hat{h}_{ki} - \hat{h}_{kj}) \rho_k)} = e^{2\pi i (h_{ij} - \tilde{h}_{ij})} = \tilde{g}_{ij}.$$

So as line bundles $\mathbb{L} \simeq \tilde{\mathbb{L}}$. □

3. GEOMETRIC PREQUANTIZATION

Recall that a prequantization is a process that assign to each $a \in C^\infty(M)$ a self-adjoint operator $Q(a)$ on some Hilbert space \mathcal{H} so that the conditions (1), (2) and (3) of Dirac's axiom holds.

Definition 3.1. A symplectic manifold (M, ω) is called *pre-quantizable* if

$$[\frac{\omega}{2\pi}] \in H^2(M, \mathbb{Z}).$$

An Hermitian line bundle (\mathbb{L}, h, ∇) over (M, ω) with $\Omega = \frac{\omega}{i\hbar}$ is called a pre-quantum line bundle.

Now let (M, ω) be a pre-quantizable symplectic manifold, and (\mathbb{L}, h, ∇) a pre-quantum line bundle over M . The Hilbert space \mathcal{H} that we are going to use is $\mathcal{H} = L^2(M, \mathbb{L})$, where the inner product between two sections is given by

$$\langle s_1, s_2 \rangle = \frac{1}{(2\pi\hbar)^n} \int_M h(s_1, s_2) \frac{\omega^n}{n!}.$$

For any $a \in C^\infty(M)$ we let Ξ_a be the Hamiltonian vector field associated to a , and let m_a be the operator “multiplication by a ” on \mathcal{H} .

Definition 3.2. For any $a \in C^\infty(M, \mathbb{R})$ we define

$$Q(a) = -i\hbar\nabla_{\Xi_a} + m_a.$$

Proposition 3.3. *The operator $Q(a)$ is self-adjoint on \mathcal{H} .*

Proof. It is enough to check that the operator $i\nabla_{\Xi_a}$ is self-adjoint:

$$\begin{aligned} \langle i\nabla_{\Xi_a} s_1, s_2 \rangle &= \int_M h(i\nabla_{\Xi_a} s_1, s_2) \frac{\omega^n}{n!} \\ &= i \int_M \Xi_a(h(s_1, s_2)) \frac{\omega^n}{n!} + \int_M h(s_1, i\nabla_{\Xi_a} s_2) \frac{\omega^n}{n!} \\ &= i \int_M \Xi_a(h(s_1, s_2)) \frac{\omega^n}{n!} + \langle s_1, i\nabla_{\Xi_a} s_2 \rangle. \end{aligned}$$

Recall from PSet 2 that for any smooth functions a and b on M , $\int_M \{a, b\} \omega^n = 0$. So

$$\int_M \Xi_a(h(s_1, s_2)) \omega^n = 0.$$

The conclusion follows. \square

Now we are ready to prove

Theorem 3.4 (Kostant-Souriau). *The assignment $a \rightsquigarrow Q(a)$ is a prequantization, i.e. satisfies conditions (1), (2) and (3) in Dirac’s axioms.*

Proof. The conditions (1) and (2) are obvious.

To prove (3), we calculate

$$\begin{aligned} \frac{1}{i\hbar} [Q(a), Q(b)] &= \frac{1}{i\hbar} (Q(a)Q(b) - Q(b)Q(a)) \\ &= \frac{1}{i\hbar} [(-i\hbar\nabla_{\Xi_a} + m_a)(-i\hbar\nabla_{\Xi_b} + m_b) - (-i\hbar\nabla_{\Xi_b} + m_b)(-i\hbar\nabla_{\Xi_a} + m_a)] \\ &= \frac{1}{i\hbar} [(-i\hbar)^2 (\nabla_{\Xi_a} \nabla_{\Xi_b} - \nabla_{\Xi_b} \nabla_{\Xi_a}) - i\hbar (\nabla_{\Xi_a} m_b + m_a \nabla_{\Xi_b} - \nabla_{\Xi_b} m_a - m_b \nabla_{\Xi_a})] \\ &= i\hbar [\nabla_{\Xi_a}, \nabla_{\Xi_b}] - (db(\Xi_a) - da(\Xi_b)) \\ &= i\hbar [\nabla_{\Xi_a}, \nabla_{\Xi_b}] + 2\{a, b\}. \end{aligned}$$

On the other hand, since $\Omega(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}$ we get

$$\begin{aligned} [\nabla_{\Xi_a}, \nabla_{\Xi_b}] &= \Omega(\Xi_a, \Xi_b) + \nabla_{[\Xi_a, \Xi_b]} \\ &= \frac{1}{i\hbar} \omega(\Xi_a, \Xi_b) + \nabla_{[\Xi_a, \Xi_b]} \\ &= -\frac{1}{i\hbar} \{a, b\} + \nabla_{\Xi_{\{a, b\}}}, \end{aligned}$$

so the conclusion follows. □

Remark. The problem for this geometric prequantization is that the Hilbert space $\mathcal{H} = L^2(X, \mathbb{L})$ is too large and thus is very far from satisfying the minimality condition (4) of Dirac's axiom. In fact, one can regard $L^2(M, \mathbb{L})$ as a "twisted version" of $L^2(M)$. Comparing to the Weyl's quantization which quantize (T^*X, ω_{can}) to $L^2(X)$, we see that in some sense one should try to "cut $L^2(M, \mathbb{L})$ into a half".