LECTURE 2: SMOOTH MANIFOLDS

1. Review of analysis

Let U be an open set in $\mathbb{R}^n$, and $f : U \to \mathbb{R}$ a continuous function. Recall that f is said to be a *C^k-function*, if all its partial derivatives of order at most k,

$$\partial^\alpha f := \frac{\partial^{|\alpha|} f}{\partial x^\alpha} := \frac{\partial^{|\alpha|} f}{(\partial x^1)^{\alpha_1} \cdots (\partial x^n)^{\alpha_n}}, \quad |\alpha| = \alpha_1 + \cdots + \alpha_n \le k$$

exist and are continuous on U. We say that f is a C^∞ function, or a *smooth function*, if it is of class C^k for all positive integers k. A function f is an *analytic function* (or C^ω) if it is smooth and agrees with its Taylor series in a neighborhood of every point. Note that not all smooth functions are analytic. A standard example is

$$f(x) = \begin{cases} 0, & x \le 0 \\ e^{-\frac{1}{x}}, & x > 0 \end{cases}$$

is a C^∞ function defined on $\mathbb{R}$ but is not analytic at $x = 0$. (Check this!)

Now let U be an open set in $\mathbb{R}^n$ and V be an open set in $\mathbb{R}^m$. Let

$$f = (f_1, \cdots, f_m) : U \to V$$

be a continuous map. We say f is C^∞ (or C^k, or C^ω) if each component f_i, $1 \le i \le m$, is a C^∞ (or C^k, or C^ω) function. [1]

Let f be a smooth map. The *differential* of f, df, assigns to each point $x \in U$ a linear map $df_x : \mathbb{R}^n \to \mathbb{R}^m$ whose matrix (with respect to the canonical basis) is the *Jacobian matrix* of f at x,

$$df_x = \begin{pmatrix} \frac{\partial f_1}{\partial x^1}(x) & \cdots & \frac{\partial f_1}{\partial x^n}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x^1}(x) & \cdots & \frac{\partial f_m}{\partial x^n}(x) \end{pmatrix}.$$

Note that for any $v \in \mathbb{R}^n$, the directional derivative of f at x in the direction v is

$$df_x(v) = \lim_{h \to 0} \frac{f(x + hv) - f(x)}{h}.$$

The well-known **chain rule** asserts that if $f : U \to V$ and $g : V \to W$ are smooth maps, so is the composition map $g \circ f : U \to W$, and

$$d(g \circ f)_x = dg_{f(x)} \circ df_x.$$

Definition 1.1. A smooth map $f : U \to V$ is a *diffeomorphism* if f is one-to-one and onto, and $f^{-1} : V \to U$ is also smooth.

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[1] In this course we will mainly consider C^∞ functions/maps. However, most definitions/theorems can be easily extended to the C^k setting.

Obviously

- If $f : U \to V$ is a diffeomorphism, so is f^{-1}.
- If $f : U \to V$ and $g : V \to W$ are diffeomorphisms, so is $g \circ f$.

As a consequence, we can prove the following smooth "invariance of domain" theorem:

Theorem 1.2. *If $f : U \to V$ is a diffeomorphism, then for each $x \in U$, the differential df_x is a linear isomorphism. In particular, $\dim U = \dim V$.*

Proof. Applying the chain rule to $f^{-1} \circ f = id_U$, and notice that the differential of the identity map $id_U : U \to U$ is the identity transformation $\mathrm{Id} : \mathbb{R}^n \to \mathbb{R}^n$, we get

$$df^{-1}_{f(x)} \circ df_x = \mathrm{Id}_{\mathbb{R}^n}.$$

The same argument applies to $f \circ f^{-1}$, which yields

$$df_x \circ df^{-1}_{f(x)} = \mathrm{Id}_{\mathbb{R}^m}.$$

By basic linear algebra, we conclude that $m = n$ and that df_x is an isomorphism. $\square$

The inverse of the previous theorem is not true. For example, we consider the map

$$f : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2 \setminus \{0\}, \quad (x^1, x^2) \mapsto ((x^1)^2 - (x^2)^2, 2x^1x^2).$$

Then at each point $x \in \mathbb{R}^2 \setminus \{0\}$, df_x is an isomorphism. However, f is not invertible since $f(x) = f(-x)$. (What is the map f if we identify $\mathbb{R}^2$ with $\mathbb{C}$?)

The inverse function theorem is a partial inverse of the previous theorem, which claims that an isomorphism in the linear category implies a *local* diffeomorphism in the smooth category.

Theorem 1.3 (The inverse function theorem)**.** *If $f : U \to V$ is a smooth map, and df_x is an isomorphism, then f is a* local diffeomorphism *near x, i.e. there exists a neighborhood U_x containing x and a neighborhood $V_{f(x)}$ containing $f(x)$ such that*

$$f|_{U_x} : U_x \to V_{f(x)}$$

is a diffeomorphism.

The theorem is a special case of the following

Theorem 1.4 (The implicit function theorem)**.** *Let W be an open set in $\mathbb{R}^n_x \times \mathbb{R}^m_y$, and $F = (F_1, \cdots, F_m) : W \to \mathbb{R}^m$ a smooth map. Let (x_0, y_0) be a point in W so that the $m \times m$ matrix*

$$\begin{pmatrix} \frac{\partial F_1}{\partial y^1}(x_0, y_0) & \cdots & \frac{\partial F_1}{\partial y^m}(x_0, y_0) \\ \vdots & \vdots & \vdots \\ \frac{\partial F_m}{\partial y^m}(x_0, y_0) & \cdots & \frac{\partial F_m}{\partial y^m}(x_0, y_0) \end{pmatrix}.$$

is nonsingular, then there exists a neighborhood $U_0 \times V_0$ of (x_0, y_0) in W and a smooth map $f : U_0 \to V_0$ so that

- *$f(x_0) = y_0$,*
- *If we denote $c = F(x_0, y_0)$, then $F^{-1}(c) \cap (U_0 \times V_0)$ is the graph of f, i.e. $F(x, f(x)) = c$ for all $x \in U_0$.*

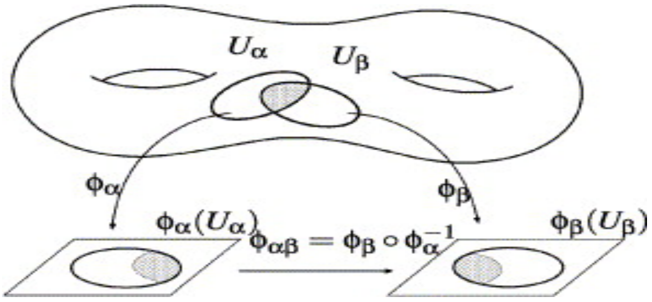
2. Smooth Manifolds

We would like to define smooth structures on topological manifolds so that one can do calculus on it. In particular, we should be able to talk about smoothness of continuous functions on a given smooth manifold M. Since near each point in M, one has a chart $\{\varphi, U, V\}$ which identify the open set U in M with the open set V in $\mathbb{R}^n$, it is natural identify any function f on U with the function $f \circ \varphi^{-1}$ on V, and use the smoothness of $f \circ \varphi^{-1}$ to define the smoothness of f itself. This idea is of course correct. The only issue is that a point on M could sit in many different open charts, and the smoothness of a function at this point should be independent of the choice of chart. In other words, if both φ and ψ are chart maps near a point, we want the maps $f \circ \varphi^{-1}$ and $f \circ \psi^{-1}$ to be simultaneously smooth or non-smooth. This amounts to require $\varphi \circ \psi^{-1}$ to be smooth. With this as our intuition, we define

Definition 2.1. Let M be a topological manifold. We say two charts $\{\varphi_\alpha, U_\alpha, V_\alpha\}$ and $\{\varphi_\beta, U_\beta, V_\beta\}$ of M are *compatible* if the *transition map*

$$\varphi_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

is a diffeomorphism.



Note that both $\varphi_\alpha(U_\alpha \cap U_\beta)$ and $\varphi_\beta(U_\alpha \cap U_\beta)$ are open in $\mathbb{R}^n$, so the smoothness of $\varphi_{\alpha\beta}$ is well understood.

Example. **(Spheres - continued).** Recall that the n-sphere S^n admits two atlas $\{\varphi_+, U_+, \mathbb{R}^n\}$ and $\{\varphi_-, U_-, \mathbb{R}^n\}$, where

$$U_\pm = S^n \setminus \{(0, \cdots, 0, \mp 1)\}$$

and

$$\varphi_\pm(x^1, \cdots, x^n, x^{n+1}) = \frac{1}{1 \pm x^{n+1}}(x^1, \cdots, x^n).$$

It follows that on $\varphi_-(U_+ \cap U_-) = \mathbb{R}^n \setminus \{0\}$,

$$\begin{aligned} \varphi_{-+}(y^1, \cdots, y^n) &= \varphi_+ \circ \varphi_-^{-1}(y^1, \cdots, y^n) \\ &= \varphi_+ \left(\frac{1}{1 + (y^1)^2 + \cdots + (y^n)^2} \left(2y^1, \cdots, 2y^n, -1 + (y^1)^2 + \cdots + (y^n)^2\right) \right) \\ &= \frac{1}{(y^1)^2 + \cdots + (y^n)^2}(y^1, \cdots, y^n), \end{aligned}$$

which is a diffeomorphism from $\mathbb{R}^n \setminus \{0\}$ to itself. So these two charts are compatible.

Example. **(Projective Spaces - continued).** Recall that the n dimensional real projective space $\mathbb{RP}^n$ has $n+1$ charts $\{\varphi_i, U_i, \mathbb{R}^n\}$, $1 \le i \le n+1$, where

$$U_i = \{[x^1 : \cdots : x^{n+1}] \mid x^i \neq 0\}$$

and

$$\varphi_i([x^1 : \cdots : x^{n+1}]) = \left(\frac{x^1}{x^i}, \cdots, \frac{x^{i-1}}{x^i}, \frac{x^{i+1}}{x^i}, \cdots, \frac{x^{n+1}}{x^i}\right).$$

Without loss of generality, let's verify that $\varphi_{1,n+1}$ is a diffeomorphism between

$$\varphi_1(U_1 \cap U_{n+1}) = \{(y^1, \cdots, y^n) \mid y^n \neq 0\} =: V_n$$

and

$$\varphi_{n+1}(U_1 \cap U_{n+1}) = \{(y^1, \cdots, y^n) \mid y^1 \neq 0\} =: V_1.$$

In fact, by definition

$$\begin{aligned}\varphi_{1,n+1}(y^1, \cdots, y^n) &= \varphi_{n+1} \circ \varphi_1^{-1}(y^1, \cdots, y^n) \\ &= \varphi_{n+1}([1 : y^1 : \cdots : y^n]) \\ &= \left(\frac{1}{y^n}, \frac{y^1}{y^n}, \cdots, \frac{y^{n-1}}{y^n}\right)\end{aligned}$$

which is obviously a diffeomorphism from V_n to V_1. Similarly one can show that all transition maps φ_{ij} are diffeomorphisms.

Definition 2.2. (1) An *atlas* $\mathcal{A}$ on M is a collection of charts $\{\varphi_\alpha, U_\alpha, V_\alpha\}$ such that all charts in $\mathcal{A}$ are compatible to each other, and satisfies $\bigcup_\alpha U_\alpha = M$.
(2) Two atlas on M are said to be *equivalent* if their union is still an atlas on M.

Example. We can define two atlas on $\mathbb{R}$ by $\mathcal{A} = \{\varphi_1, \mathbb{R}, \mathbb{R}\}$ and $\mathcal{B} = \{\varphi_2, \mathbb{R}, \mathbb{R}\}$, where $\varphi_1(x) = x$ and $\varphi_2(x) = x^3$. Then they are non-equivalent atlas since

$$\varphi_{21}(x) = \varphi_1 \circ \varphi_2^{-1}(x) = x^{1/3}$$

is not smooth.

Definition 2.3. An n-dimensional *smooth manifold* is an n-dimensional topological manifold M equipped with an equivalence class of atlas. This equivalence class is called its *smooth structure.*

So a smooth manifold is a pair $(M, \mathcal{A})$. In the future we will always omit $\mathcal{A}$ if there is no confusion of the smooth structure.

Remark. Similarly one can define C^k manifolds, real analytic ($=C^\omega$) manifolds and complex manifolds. For example, a complex manifold is a Hausdorff and second countable topological space that locally looks like $\mathbb{C}^n$, so that the transition maps are all bi-holomorphic.

Example. If a topological manifold M can be covered by a single chart, then such a chart automatically determines a smooth structure on M. So in particular, the graph of any continuous function on an open domain of $\mathbb{R}^n$ is a smooth manifold. (However, it is possible that such a graph is not a *smooth submanifold* of $\mathbb{R}^{n+1}$.)

Example. All the examples we studied in lecture 1, including graphs, S^n, $\mathbb{RP}^n$ etc, with the charts we described, are smooth manifolds. In fact, they are also C^ω manifolds.

Remark. Some deep results from differential topology:

- There exists topological manifolds that do not admits smooth structure. The first example was a compact 10-dimensional manifold found by M. Kervaire.
- If a topological manifold admits a C^1 structure, it also admits a C^∞ structure.
- Any manifold M admits a finite atlas consisting of $\dim M + 1$ charts (not necessarily connected).