LECTURE 3: SMOOTH FUNCTIONS

1. Smooth Functions

Let M be a smooth manifold.

Definition 1.1. We say a function $f: M \to \mathbb{R}$ is *smooth* if for any chart $\{\varphi_{\alpha}, U_{\alpha}, V_{\alpha}\}$ in \mathcal{A} that defines the smooth structure of $M, f \circ \varphi_{\alpha}^{-1}$ is a smooth function on V_{α} .

Example. Each coordinate function $f_i(x^1, \cdots, x^{n+1}) = x^i$ is a smooth function on S^n , since

$$f_i \circ \varphi_{\pm}^{-1}(y^1, \cdots, y^n) = \begin{cases} \frac{2y^i}{1+|y|^2}, & 1 \le i \le n \\ \pm \frac{1-|y|^2}{1+|y|^2}, & i = n+1 \end{cases}$$

are smooth functions on \mathbb{R}^n .

We will denote the set of all smooth functions on M by $C^{\infty}(M)$. Note that this is a (commutative) algebra, i.e. it is a vector space equipped with a (commutative) bilinear "multiplication operation": If f, g are smooth, so are af + bg and fg.

Now suppose $f \in C^{\infty}(M)$. As usual, the *support* of f is by definition the set

$$\operatorname{supp}(f) = \overline{\{p \in M \mid f(p) \neq 0\}}.$$

We say that f is compactly supported, denoted by $f \in C_0^{\infty}(M)$, if the support of f is a compact subset in M. Obviously

- If $f, g \in C_0^{\infty}(M)$, then $af + bg \in C_0^{\infty}(M)$. If $f \in C_0^{\infty}(M)$ and $g \in C^{\infty}(M)$, then $fg \in C_0^{\infty}(M)$.

So $C_0^{\infty}(M)$ is an ideal of $C^{\infty}(M)$. Note that if M is compact, then any smooth function is compactly supported.

Example (Bump function). First suppose $x \in \mathbb{R}$. We know that the function

$$f_1(x) = \begin{cases} e^{-1/x}, & x > 0\\ 0, & x \le 0 \end{cases}$$

is smooth, is strictly positive for all positive x, and vanishes for all negative x. It follows that the function e ()

$$f_2(x) = \frac{f_1(x)}{f_1(x) + f_1(1-x)}$$

is smooth, vanishes for all $x \leq 0$, equals 1 for all $x \geq 1$, and $0 \leq f_2(x) \leq 1$ for all x.

Finally for any $x \in \mathbb{R}^n$, we let

$$f_3(x) = f_2(2 - |x|),$$

then f_3 is a smooth function on \mathbb{R}^n , which vanishes for all $|x| \ge 2$, and 1 for all $|x| \le 1$, and $0 \le f_3(x) \le 1$ for all x.

With the help of these Euclidean bump functions, we can show that on any smooth manifold, there exists many many "bump" functions:

Theorem 1.2. Let M be a smooth manifold, $K \subset M$ is a compact subset, and $U \subset M$ an open subset that contains K. Then there is a "bump" function $\varphi \in C_0^{\infty}(M)$ so that $0 \leq \varphi \leq 1, \ \varphi \equiv 1 \text{ on } K \text{ and } \operatorname{supp}(\varphi) \subset U$.

Proof. For each $q \in K$, there is a chart $\{\varphi_q, U_q, V_q\}$ near q so that $U_q \subset U$ and V_q contains the open ball $B_3(0)$ of radius 3 centered at 0 in \mathbb{R}^n . Let $\widetilde{U}_q = \varphi_q^{-1}(B_1(0))$, and let

$$f_q(p) = \begin{cases} f_3(\varphi_q(p)), & p \in U_q, \\ 0, & p \notin U_q. \end{cases}$$

Then $f_q \in C_0^{\infty}(M)$, $\operatorname{supp}(f_q) \subset U_p$ and $f \equiv 1$ on \widetilde{U}_q . (Why?)

Now the family of open sets $\{\widetilde{U}_q\}_{q\in K}$ is an open cover of K. Since K is compact, there is a finite sub-cover $\{\widetilde{U}_{q_1}, \cdots, \widetilde{U}_{q_N}\}$. Let $\psi = \sum_{i=1}^N f_{q_i}$. Then ψ is a smooth and compactly supported function on M so that $\psi \ge 1$ on K and $\operatorname{supp}(\psi) \subset U$. It follows that the function $\varphi(p) = f_2(\psi(p))$ satisfies all the conditions we required. \Box

As a simple consequence, we see that as a vector space, $C_0^{\infty}(M)$ (and thus $C^{\infty}(M)$) is infinitely dimensional.

2. PARTITION OF UNITY

As we have just seen, for a compact subset $K \subset M$, one can always cover it by finitely many nice neighborhoods on which we can construct nice "local" functions. By adding these (finitely many) local functions, we can get nice global functions on M that behaves nicely on K. It turns out that the same idea applies to the whole manifold M: we can generate an infinite collection of smooth functions on M, and add them to get a global smooth function, provided that near each point, there are only finitely many functions in our collection that are nonzero. More importantly, we can use such a collection of functions to "glue" geometric/analytic objects that can be defined locally using charts.

Definition 2.1. Let M be a smooth manifold, and $\{U_{\alpha}\}$ be an open cover of M. A partition of unity subordinate to the cover $\{U_{\alpha}\}$ is a collection of smooth functions $\{\rho_{\alpha}\}$ so that

- (1) $0 \leq \rho_{\alpha} \leq 1$ for all α .
- (2) $\operatorname{supp}(\rho_{\alpha}) \subset U_{\alpha}$ for all α .
- (3) Each point $p \in M$ has a neighborhood which intersects $\operatorname{supp}(\rho_{\alpha})$ for only finitely many α .¹

¹So in particular, there are only countable ρ_{α} 's whose support are non-empty.

(4) $\sum_{\alpha} \rho_{\alpha}(p) = 1$ for all $p \in M$.

Thanks to the local finiteness condition (3), for each p there is a small neighborhood on which the sum in (4) is in fact a finite sum. So although we might have uncountable many indices α , the sum in (4) is a well-defined smooth function on M.

The main theorem in this lecture is

Theorem 2.2 (Existence of partition of unity). Let M be a smooth manifold, and $\{U_{\alpha}\}$ an open cover of M. Then there exists a partition of unity subordinate to $\{U_{\alpha}\}$.

The proof depends on the following technical lemma whose proof we will postpone for a while:

Lemma 2.3. Let M be any topological manifold. For any open cover $\mathcal{U} = \{U_{\alpha}\}$ of M, one can find two countable family of open covers $\mathcal{V} = \{V_j\}$ and $\mathcal{W} = \{W_j\}$ of M so that

- For each j, \overline{V}_j is compact and $\overline{V}_j \subset W_j$.
- \mathcal{W} is a refinement of \mathcal{U} : For each j, there is an $\alpha = \alpha(j)$ so that $W_j \subset U_\alpha$.
- \mathcal{W} is a locally finite cover: Any $p \in M$ has a neighborhood W such that $W \cap W_i \neq \emptyset$ for only finitely many W_i 's.

Remark. A topological space X is called *paracompact* if every open cover admits a locally finite open refinement.

Proof of theorem 2.2. By theorem 1.2, we can find nonnegative functions $\varphi_j \in C_0^{\infty}(M)$ so that $\varphi_j \equiv 1$ on \overline{V}_j and $\operatorname{supp}(\varphi_j) \subset W_j$. Since \mathcal{W} is a locally finite cover, $\varphi = \sum \varphi_j$ is a well-defined smooth function on M. Since each φ_j is nonnegative, and \mathcal{V} is a cover of M, φ is strictly positive on M. It follows that the functions $\psi_j = \frac{\varphi_j}{\varphi}$ are smooth and satisfy $0 \leq \psi_j \leq 1$ and $\sum_i \psi_j = 1$.

Next let's re-index the family $\{\psi_j\}$ to get the demanded partition of unity. For each j, we fix an index $\alpha(j)$ so that $W_j \subset U_{\alpha(j)}$, and define

$$\rho_{\alpha} = \sum_{\alpha(j)=\alpha} \psi_j.$$

Note that the right hand side is a finite sum near each point, so it does define a smooth function. Clearly the family $\{\rho_{\alpha}\}$ is a partition of unity subordinate to $\{U_{\alpha}\}$.

Remark. As applications of partition of unity theorem, we will

- Define the integral of differential forms in each local chart, and glue them using P.O.U. to get the integral of differential forms on M.
- Construct a metric structure in each local chart, and glue them using P.O.U. to get a global metric structure (the Riemannian metric) on M.
- Construct a connection structure in each local chart, and glue them using P.O.U. to get a global connection structure on M.

As an immediate corollary of partition of unity, we can generalize theorem 1.2 to closed subsets:

Corollary 2.4. Let M be a smooth manifold, $A \subset M$ is a closed subset, and $U \subset M$ an open subset that contains A. Then there is a "bump" function $\varphi \in C^{\infty}(M)$ so that $0 \leq \varphi \leq 1, \ \varphi \equiv 1 \text{ on } A$ and $\operatorname{supp}(\varphi) \subset U$.

Proof. Let $\{\rho_1, \rho_2\}$ be a partition of unity subordinate to the open cover $\{U, M \setminus A\}$. Then $\varphi = \rho_1$ is what we need: $\rho_1 = 1$ on A since $\rho_2 = 0$ on A.

It remains to prove lemma 2.3. First we prove

Lemma 2.5. For any topological manifold M, there exists a countable collection of open sets $\{X_i\}$ so that

- (1) For each j, the closure \overline{X}_j is compact.
- (2) For each $j, \overline{X}_j \subset X_{j+1}$.
- (3) $M = \bigcup_j X_j$.

Proof. Since M is second countable, there is a countable basis of the topology of M. Out of this countable sequence of open sets, we pick those that have compact closure, and denote them by Y_1, Y_2, \cdots . Since M is locally Euclidean, it is easy to see that $\mathcal{Y} = \{Y_j\}$ is an open cover of M.

We let $X_1 = Y_1$. Since \mathcal{Y} is an open cover of \overline{X}_1 which is compact, there exists finitely many open sets Y_{i_1}, \dots, Y_{i_k} so that $\overline{X}_1 \subset Y_{i_1} \cup \dots \cup Y_{i_k}$. Let $X_2 = Y_{i_1} \cup \dots \cup Y_{i_k}$. Obviously \overline{X}_2 is compact. Repeat this procedure again and again, we could get a desired sequence of open sets X_1, X_2, X_3, \dots .

Proof of lemma 2.3. For each $p \in M$, there is an j and an $\alpha(p)$ so that $p \in \overline{X}_{j+1} \setminus X_j$ and $p \in U_{\alpha(p)}$. Since M is locally Euclidean, one can always choose open neighborhoods V_p, W_p of p so that \overline{V}_p is compact and

$$\overline{V}_p \subset W_p \subset U_{\alpha(p)} \cap (X_{j+2} \setminus \overline{X}_{j-1}).$$

Now for each j, since the "stripe" $\overline{X}_{j+1} \setminus X_j$ is compact, one can choose finitely many points $p_1^j, \dots, p_{k_j}^j$ so that $V_{p_1^j}, \dots, V_{p_{k_j}^j}$ is an open cover of $\overline{X}_{j+1} \setminus X_j$. Denote all these $V_{p_k^j}$'s by V_1, V_2, \dots , and the corresponding $W_{p_k^j}$'s by W_1, W_2, \dots . Then $\mathcal{V} = \{V_k\}$ and $\mathcal{W} = \{W_k\}$ are open covers of M that satisfies all the conditions in lemma 2.3. For example, the local finiteness property of \mathcal{W} follows from the fact that there are only finitely many W_k 's (that corresponds to j and j-1 above) intersect $X_{j+1} \setminus \overline{X}_{j-1}$. \Box

The subsets described by lemma 2.5 is called an *exhaustion* of M. More generally, a real-valued continuous function f on M is called an *exhauston function* for M if for any $c \in \mathbb{R}$, the *sublevel set* $f^{-1}((-\infty, c])$ is compact. (So the sublevel sets gives a "continuous exhaustion" of M by compact sets). As an another immediate application of P.O.U, we have

Proposition 2.6. There exists a positive exhausion function on any smooth manifold.

Proof. Let V_j be as constructed in lemma 2.3. Then $\{V_j\}$ is a locally finite open covering of M. Let $\{\rho_j\}$ be a partition of unity subordinate to this covering. Let a_j be any sequence of numbers so that $a_j \ge 1$ and $\lim_{j\to\infty} a_j = +\infty$. We claim that

$$f(p) := \sum a_j \rho_j(p)$$

is a positive exhaustion function.

In fact, f is well-defined and smooth since the covering $\{V_j\}$ is locally finite. It is positive since $a_j \ge 1$ and $\sum \rho_j = 1$. It is an exhaustion function since for any c, we can take N so that $a_j > c$ for all j > N. Then it follows that

$$f^{-1}((-\infty,c]) \subset \bigcup_{j=1}^{N} \overline{V}_j.$$
 (check this!)

Since $\cup_{j=1}^{N} \overline{V}_j$ is compact and $f^{-1}((-\infty, c])$ is closed, $f^{-1}((-\infty, c])$ has to be compact.

We end with two questions:

- **[Easy]** Where do we used the second countable condition in proving P.O.U.?
- [Hard] Where do we used the Hausdorff condition in proving P.O.U.?