LECTURE 5: SUBMERSIONS, IMMERSEIONS AND EMBEDDINGS

1. Properties of the Differentials

Recall that the tangent space of a smooth manifold $M$ at $p$ is the space of all derivatives at $p$, i.e. all linear maps $X_p : C^\infty(M) \to \mathbb{R}$ so that the Leibnitz rule holds:

$$X_p(fg) = g(p)X_p(f) + f(p)X_p(g).$$

The differential (also known as the tangent map) of a smooth map $f : M \to N$ at $p \in M$ is defined to be the linear map $df_p : T_pM \to T_{f(p)}N$ such that

$$df_p(X_p)(g) = X_p(g \circ f)$$

for all $X_p \in T_pM$ and $g \in C^\infty(N)$.

**Remark.** Two interesting special cases:

- If $\gamma : (-\varepsilon, \varepsilon) \to M$ is a curve such that $\gamma(0) = p$, then $d\gamma_0$ maps the unit tangent vector $\frac{d}{dt}$ at $0 \in \mathbb{R}$ to the tangent vector $\dot{\gamma}(0) = d\gamma_0(\frac{d}{dt})$ of $\gamma$ at $p \in M$.
- If $f : M \to \mathbb{R}$ is a smooth function, we can identify $T^*_pM$ with $\mathbb{R}$ by identifying $a \frac{d}{dt}$ with $a$ (which is merely the “derivative ↔ vector” correspondence). Then for any $X_p \in T_pM$, $df_p(X_p) \in \mathbb{R}$. Note that the map $df_p : T_pM \to \mathbb{R}$ is linear. In other words, $df_p \in T^*_pM$, the dual space of $T_pM$. We will call $df_p$ a cotangent vector or a 1-form at $p$. Note that by taking $g = \text{Id} \in C^\infty(\mathbb{R})$, we get

$$X_p(f) = df_p(X_p).$$

For the differential, we still have the chain rule for differentials:

**Theorem 1.1** (Chain rule). Suppose $f : M \to N$ and $g : N \to P$ are smooth maps, then $d(g \circ f)_p = dg_{f(p)} \circ df_p$.

**Proof.** For any $X_p \in T_pM$ and $h \in C^\infty(P)$,

$$d(g \circ f)_p(X_p)(h) = X_p(h \circ g \circ f) = df_p(X_p)(h \circ g) = dg_{f(p)}(df_p(X_p))(h).$$

So the theorem follows. \qed

Obviously the differential of the identity map is the identity map between tangent spaces. By repeating the proof of theorem 1.2 in lecture 2 we get

**Corollary 1.2.** If $f : M \to N$ is a diffeomorphism, then $df_p : T_pM \to T_{f(p)}N$ is an isomorphism.

In particular, we have

**Corollary 1.3.** If $\dim M = n$, then $T_pM$ is an $n$-dimensional linear space.
Proof. Let \( \{ \varphi, U, V \} \) be a chart near \( p \). Then \( \varphi : U \to V \) is a diffeomorphism. It follows that \( \dim T_pM = \dim T_qU = \dim T_{f(p)}V = n \).

In particular, we see that the tangent vectors \( \partial_i := d\varphi^{-1}(\frac{\partial}{\partial x^i}) \) form a basis of \( T_pM \). In coordinates, one has the following explicit formula for \( \partial_i \):

\[
\partial_i : C^\infty(M) \to \mathbb{R}, \quad \partial_i(f) = \frac{\partial f \circ \varphi^{-1}}{\partial x^i}(\varphi(p)).
\]

We will abuse the notation and think of \( x^i \) as a function on \( U \) (which really should be \( x^i \circ \varphi \)). Then one can check that \( \{ dx^1_p, \ldots, dx^n_p \} \) is the dual basis of \( \{ \partial_1, \ldots, \partial_n \} \), and for any \( f \in C^\infty(M) \),

\[
df_p = (\partial_1 f) dx^1_p + \cdots + (\partial_n f) dx^n_p.
\]

As in lecture 2, we have the following inverse function theorem:

**Theorem 1.4** (Inverse Mapping Theorem). Suppose \( M \) and \( N \) are both smooth manifolds of dimension \( n \), and \( f : M \to N \) a smooth map. Let \( p \in M \), and \( q = f(p) \in N \). If \( df_p : T_pM \to T_qN \) is an isomorphism, then \( f \) is a local diffeomorphism, i.e. it maps a neighborhood \( U_1 \) of \( p \) diffeomorphically to a neighborhood \( X_1 \) of \( q \).

**Proof.** Take a chart \( \{ \varphi, U, V \} \) near \( p \) and a chart \( \{ \psi, X, Y \} \) near \( f(p) \) so that \( f(U) = X \). Since \( \varphi : U \to V \) and \( \psi : X \to Y \) are diffeomorphisms,

\[
d(d\varphi \circ f \circ \varphi^{-1})(p) = d\psi_q \circ df_p \circ d\varphi^{-1}_{\varphi(p)} : T_{\varphi(p)}V = \mathbb{R}^n \to T_{\psi(q)}Y = \mathbb{R}^n
\]

is a linear isomorphism. It follows from the inverse function theorem in lecture 2 that there exist neighborhoods \( V_1 \) of \( \varphi(p) \) and \( Y_1 \) of \( \psi(q) \) so that \( \psi \circ f \circ \varphi^{-1} \) is a diffeomorphism from \( V_1 \) to \( Y_1 \). Take \( U_1 = \varphi^{-1}(V_1) \) and \( X_1 = \psi^{-1}(Y_1) \). Then

\[
f = \psi^{-1} \circ (\psi \circ f \circ \varphi^{-1}) \circ \varphi
\]

is a diffeomorphism from \( U_1 \) to \( X_1 \). \( \square \)

Again we cannot conclude global diffeomorphism even if \( df_p \) is an isomorphism at any point, since \( f \) might not be invertible. In fact, now we have a much simpler example compared to the example we constructed in lecture 2: Consider \( f : S^1 \to S^1 \) given by \( f(e^{i\theta}) = e^{2i\theta} \). Then it is only a local diffeomorphism.

2. **Submersions, Immersions and Embeddings**

It is natural to ask: what if \( df_p \) is not an isomorphism? Of course the simplest cases are the full-rank cases.

**Definition 2.1.** Let \( f : M \to N \) be a smooth map.

1. \( f \) is a submersion at \( p \) if \( df_p : T_pM \to T_{f(p)}N \) is surjective.
2. \( f \) is an immersion at \( p \) if \( df_p : T_pM \to T_{f(p)}N \) is injective.

We say \( f \) is a submersion/immersion if it is a submersion/immersion at each point.

Obviously
• If $f$ is a submersion, then $\dim M \geq \dim N$.
• If $f$ is an immersion, then $\dim M \leq \dim N$.
• If $f$ is a submersion/immersion at $p$, then it is a submersion/immersion near $p$.

**Example.** If $m \geq n$, then
$$\pi : \mathbb{R}^m \to \mathbb{R}^n, \quad (x^1, \ldots, x^m) \mapsto (x^1, \ldots, x^n)$$
is a submersion.

**Example.** If $m \leq n$, then
$$\iota : \mathbb{R}^m \to \mathbb{R}^n, \quad (x^1, \ldots, x^m) \mapsto (x^1, \ldots, x^m, 0, \ldots, 0)$$
is an immersion.

It turns out that any submersion/immersion locally looks like the above two maps.

**Theorem 2.2** (Canonical Submersion Theorem). Let $f : M \to N$ be a submersion at $p \in M$, then $m = \dim M \geq n = \dim N$, and there exists charts $(\varphi_1, U_1, V_1)$ around $p$ and $(\psi_1, X_1, Y_1)$ around $q = f(p)$ such that
$$\psi_1 \circ f \circ \varphi_1^{-1} = \pi|_{V_1}.$$**Theorem 2.3** (Canonical Immersion Theorem). Let $f : M \to N$ be an immersion at $p \in M$, then $m = \dim M \leq n = \dim N$, and there exists charts $(\varphi_1, U_1, V_1)$ around $p$ and $(\psi_1, X_1, Y_1)$ around $q = f(p)$ such that
$$\psi_1 \circ f \circ \varphi_1^{-1} = \iota|_{V_1}.$$

**Proof of the Canonical Submersion Theorem.** Take a chart $\{\varphi, U, V\}$ near $p$ and a chart $\{\psi, X, Y\}$ near $f(p)$ so that $f(U) \subset X$. Since $f$ is a submersion,
$$d(\psi \circ f \circ \varphi^{-1})_{\varphi(p)} = d\psi_1 \circ df_p \circ d\varphi^{-1}_{\varphi(p)} : T\varphi(p)V = \mathbb{R}^m \to T\psi(q)Y = \mathbb{R}^n$$
is surjective. Denote $F = \psi \circ f \circ \varphi^{-1}$. Then the Jacobian matrix $\left(\frac{\partial F^i}{\partial x^j}\right)$ is an $m \times n$ matrix of rank $n$ at $\varphi(p)$. By reordering the coordinates if necessary, we may assume the sub-matrix
$$\left(\frac{\partial F^i}{\partial x^j}\right), \quad 1 \leq i \leq n, 1 \leq j \leq n$$
is nonsingular at $\varphi(p)$. Note that this re-ordering procedure can be done by modifying $(\psi, X, Y)$ to another chart $(\psi_1, X_1, Y_1)$, and thus we really have $F = \psi_1 \circ f \circ \varphi^{-1}$. Define
$$G : V \to \mathbb{R}^m, \quad (x^1, \ldots, x^m) \mapsto (F^1, \ldots, F^n, x^{n+1}, \ldots, x^m).$$
Then obviously $dG_{\varphi(p)}$ is nonsingular. By the inverse function theorem, there is a neighborhood $V_0$ of $\varphi(p)$ so that $G$ is a diffeomorphism from $V_0$ to $G(V_0)$. Let $H$ be the inverse of $G$ on $G(V_0)$. Note that $F = \pi \circ G$. Let $U_1 = \varphi^{-1}(V_0)$, $V_1 = G(V_0)$, and $\varphi_1 = G \circ \varphi$. Then $(\varphi_1, U_1, V_1)$ is a chart near $p$, and
$$\psi_1 \circ f \circ \varphi_1^{-1} = \psi_1 \circ f \circ (\varphi^{-1} \circ H) = F \circ H = \pi \circ G \circ H = \pi.$$
Proof of the Canonical Immersion Theorem. Similarly we assume the sub-matrix
\[ \left( \frac{\partial F^i}{\partial x^j} \right), \quad 1 \leq i \leq m, 1 \leq j \leq m \]
is nonsingular at \( \varphi(p) \). Define
\[ G : U \times \mathbb{R}^{n-m} \to \mathbb{R}^n, \quad (x^1, \ldots, x^m, y^1, \ldots, y^{n-m}) \mapsto F(x) + (0, \ldots, 0, y^1, \ldots, y^{n-m}). \]
Then \( dG \) is nonsingular at \( (\varphi(p), 0, \ldots, 0) \). So \( G \) is a local diffeomorphism with a smooth local inverse \( H \), and
\[ (H \circ \psi) \circ f \circ \varphi_1^{-1} = H \circ F = H \circ G \circ i = i. \]
\( \square \)

Remark. More generally one has the following constant rank theorem:

**Theorem 2.4** (Constant Rank Theorem). Let \( f : M \to N \) be a smooth map so that \( \text{rank}(df) = r \) near \( p \). Then there exists charts \( (\varphi_1, U_1, V_1) \) around \( p \) and \( (\psi_1, X_1, Y_1) \) near \( f(p) \) such that that
\[ \psi_1 \circ f \circ \varphi_1^{-1}(x^1, \ldots, x^m) = (x^1, \ldots, x^r, 0, \ldots, 0). \]
The proof will be left as an exercise.

Smooth maps are very useful in constructing nice subsets in smooth manifolds. For example, we are interested in

- If \( f \) is an immersion, what can we say about the image \( f(M) \)?
- If \( f \) is a submersion, what can we say about level sets \( f^{-1}(q) \)'s?

Let’s briefly discuss the first case. Let \( f : M \to N \) be an immersion. Then locally \( f(M) \) is a very nice subset, as clarified by the canonical immersion theorem. However, globally \( f(M) \) could be a “bad subset” of \( N \).

*Example.* The following two graphs are the images of two immersions of \( \mathbb{R} \) into \( \mathbb{R}^2 \). For the first one, the immersion is not injective. For the second one, the immersion is injective, while the image still have different topology than \( \mathbb{R} \).

*Example.* A more complicated example: consider \( f : \mathbb{R} \to S^1 \times S^1 \) defined by
\[ f(t) = (e^{it}, e^{i\sqrt{2}t}). \]
Then \( f \) is an immersion, and the image \( f(\mathbb{R}) \) is a dense curve in the torus \( S^1 \times S^1 \).

We are more interested in nice immersions \( f : M \to N \) where the image \( f(M) \), with the subspace topology inherited from \( N \), has the same topology as \( M \).

**Definition 2.5.** Let \( M, N \) be smooth manifolds, and \( f : M \to N \) an immersion. \( f \) is called an *embedding* if it is a homeomorphism onto its image \( f(M) \), where the topology on \( f(M) \) is the subspace topology as a subset of \( N \).