LECTURE 5: SUBMERSIONS, IMMERSIONS AND EMBEDDINGS

1. Properties of the Differentials

Recall that the tangent space of a smooth manifold M at p is the space of all derivatives at p, i.e. all linear maps $X_p : C^\infty(M) \to \mathbb{R}$ so that the Leibnitz rule holds:

$$X_p(fg) = g(p)X_p(f) + f(p)X_p(g).$$

The differential (also known as the *tangent map*) of a smooth map $f : M \to N$ at $p \in M$ is defined to be the linear map $df_p : T_pM \to T_{f(p)}N$ such that

$$df_p(X_p)(g) = X_p(g \circ f)$$

for all $X_p \in T_pM$ and $g \in C^\infty(N)$.

Remark. Two interesting special cases:

- If $\gamma : (-\varepsilon, \varepsilon) \to M$ is a curve such that $\gamma(0) = p$, then $d\gamma_0$ maps *the* unit tangent vector $\frac{d}{dt}$ at $0 \in \mathbb{R}$ to the tangent vector $\dot\gamma(0) = d\gamma_0(\frac{d}{dt})$ of γ at $p \in M$.
- If $f : M \to \mathbb{R}$ is a smooth function, we can identify $T_{f(p)}\mathbb{R}$ with $\mathbb{R}$ by identifying $a\frac{d}{dt}$ with a (which is merely the "derivative $\leftrightarrow$ vector" correspondence). Then for any $X_p \in T_pM$, $df_p(X_p) \in \mathbb{R}$. Note that the map $df_p : T_pM \to \mathbb{R}$ is linear. In other words, $df_p \in T_p^*M$, the dual space of T_pM. We will call df_p a *cotangent vector* or a *1-form* at p. Note that by taking $g = Id \in C^\infty(\mathbb{R})$, we get
$$X_p(f) = df_p(X_p).$$

For the differential, we still have the chain rule for differentials:

Theorem 1.1 (Chain rule)**.** *Suppose $f : M \to N$ and $g : N \to P$ are smooth maps, then $d(g \circ f)_p = dg_{f(p)} \circ df_p$.*

Proof. For any $X_p \in T_pM$ and $h \in C^\infty(P)$,

$$d(g \circ f)_p(X_p)(h) = X_p(h \circ g \circ f) = df_p(X_p)(h \circ g) = dg_{f(p)}(df_p(X_p))(h).$$

So the theorem follows. □

Obviously the differential of the identity map is the identity map between tangent spaces. By repeating the proof of theorem 1.2 in lecture 2 we get

Corollary 1.2. *If $f : M \to N$ is a diffeomorphism, then $df_p : T_pM \to T_{f(p)}N$ is an isomorphism.*

In particular, we have

Corollary 1.3. *If $\dim M = n$, then T_pM is an n-dimensional linear space.*

Proof. Let $\{\varphi, U, V\}$ be a chart near p. Then $\varphi : U \to V$ is a diffeomorphism. It follows that $\dim T_pM = \dim T_pU = \dim T_{f(p)}V = n$. □

In particular, we see that the tangent vectors $\partial_i := d\varphi^{-1}(\frac{\partial}{\partial x^i})$ form a basis of T_pM. In coordinates, one has the following explicit formula for ∂_i:

$$\partial_i : C^\infty(M) \to \mathbb{R}, \quad \partial_i(f) = \frac{\partial f \circ \varphi^{-1}}{\partial x^i}(\varphi(p)).$$

We will abuse the notation and think of x^i as a function on U (which really should be $x^i \circ \varphi$). Then one can check that y $\{dx_p^1, \cdots, dx_p^n\}$ is the dual basis of $\{\partial_1, \cdots, \partial_n\}$, and for any $f \in C^\infty(M)$,

$$df_p = (\partial_1 f)dx_p^1 + \cdots + (\partial_n f)dx_p^n.$$

As in lecture 2, we have the following inverse function theorem:

Theorem 1.4 (Inverse Mapping Theorem)**.** *Suppose M and N are both smooth manifolds of dimension n, and $f : M \to N$ a smooth map. Let $p \in M$, and $q = f(p) \in N$. If $df_p : T_pM \to T_qN$ is an isomorphism, then f is a* local diffeomorphism*, i.e. it maps a neighborhood U_1 of p diffeomorphically to a neighborhood X_1 of q.*

Proof. Take a chart $\{\varphi, U, V\}$ near p and a chart $\{\psi, X, Y\}$ near $f(p)$ so that $f(U) = X$. Since $\varphi : U \to V$ and $\psi : X \to Y$ are diffeomorphisms,

$$d(\psi \circ f \circ \varphi^{-1})_{\varphi(p)} = d\psi_q \circ df_p \circ d\varphi^{-1}_{\varphi(p)} : T_{\varphi(p)}V = \mathbb{R}^n \to T_{\psi(q)}Y = \mathbb{R}^n$$

is a linear isomorphism. It follows from the inverse function theorem in lecture 2 that there exist neighborhoods V_1 of $\varphi(p)$ and Y_1 of $\psi(q)$ so that $\psi \circ f \circ \varphi^{-1}$ is a diffeomorphism from V_1 to Y_1. Take $U_1 = \varphi^{-1}(V_1)$ and $X_1 = \psi^{-1}(Y_1)$. Then

$$f = \psi^{-1} \circ (\psi \circ f \circ \varphi^{-1}) \circ \varphi$$

is a diffeomorphism from U_1 to X_1. □

Again we cannot conclude global diffeomorphism even if df_p is an isomorphism at any point, since f might not be invertible. In fact, now we have a much simpler example compared to the example we constructed in lecture 2: Consider $f : S^1 \to S^1$ given by $f(e^{i\theta}) = e^{2i\theta}$. Then it is only a local diffeomorphism.

2. Submersions, Immersions and Embeddings

It is natural to ask: what if df_p is not an isomorphism? Of course the simplest cases are the full-rank cases.

Definition 2.1. Let $f : M \to N$ be a smooth map.

(1) f is a *submersion* at p if $df_p : T_pM \to T_{f(p)}N$ is surjective.
(2) f is an *immersion* at p if $df_p : T_pM \to T_{f(p)}N$ is injective.

We say f is a submersion/immersion if it is a submersion/immersion at each point.

Obviously

- If f is a submersion, then $\dim M \geq \dim N$.
- If f is an immersion, then $\dim M \leq \dim N$.
- If f is a submersion/immersion at p, then it is a submersion/immersion near p.

Example. If $m \geq n$, then

$$\pi : \mathbb{R}^m \to \mathbb{R}^n, \quad (x^1, \cdots, x^m) \mapsto (x^1, \cdots, x^n)$$

is a submersion

Example. If $m \leq n$, then

$$\iota : \mathbb{R}^m \to \mathbb{R}^n, \quad (x^1, \cdots, x^m) \mapsto (x^1, \cdots, x^m, 0, \cdots, 0)$$

is an immersion.

It turns out that any submersion/immersion locally looks like the above two maps.

Theorem 2.2 (Canonical Submersion Theorem)**.** *Let $f : M \to N$ be a submersion at $p \in M$, then $m = \dim M \geq n = \dim N$, and there exists charts (φ_1, U_1, V_1) around p and (ψ_1, X_1, Y_1) around $q = f(p)$ such that*

$$\psi_1 \circ f \circ \varphi_1^{-1} = \pi|_{V_1}.$$

Theorem 2.3 (Canonical Immersion Theorem)**.** *Let $f : M \to N$ be an immersion at $p \in M$, then $m = \dim M \leq n = \dim N$, and there exists charts (φ_1, U_1, V_1) around p and (ψ_1, X_1, Y_1) around $q = f(p)$ such that*

$$\psi_1 \circ f \circ \varphi_1^{-1} = \iota|_{V_1}.$$

Proof of the Canonical Submersion Theorem. Take a chart $\{\varphi, U, V\}$ near p and a chart $\{\psi, X, Y\}$ near $f(p)$ so that $f(U) \subset X$. Since f is a submersion,

$$d(\psi \circ f \circ \varphi^{-1})_{\varphi(p)} = d\psi_q \circ df_p \circ d\varphi^{-1}_{\varphi(p)} : T_{\varphi(p)}V = \mathbb{R}^m \to T_{\psi(q)}Y = \mathbb{R}^n$$

is surjective. Denote $F = \psi \circ f \circ \varphi^{-1}$. Then the Jacobian matrix $(\frac{\partial F^i}{\partial x^j})$ is an $m \times n$ matrix of rank n at $\varphi(p)$. By reordering the coordinates if necessary, we may assume the sub-matrix

$$(\frac{\partial F^i}{\partial x^j}), \quad 1 \leq i \leq n, 1 \leq j \leq n$$

is nonsingular at $\varphi(p)$. Note that this re-ordering procedure can be done by modifying (ψ, X, Y) to another chart (ψ_1, X_1, Y_1), and thus we really have $F = \psi_1 \circ f \circ \varphi^{-1}$. Define

$$G : V \to \mathbb{R}^m, \quad (x^1, \cdots, x^m) \mapsto (F^1, \cdots, F^n, x^{n+1}, \cdots, x^m).$$

Then obviously $dG_{\varphi(p)}$ is nonsingular. By the inverse function theorem, there is a neighborhood V_0 of $\varphi(p)$ so that G is a diffeomorphism from V_0 to $G(V_0)$. Let H be the inverse of G on $G(V_0)$. Note that $F = \pi \circ G$. Let $U_1 = \varphi^{-1}(V_0)$, $V_1 = G(V_0)$, and $\varphi_1 = G \circ \varphi$. Then (φ_1, U_1, V_1) is a chart near p, and

$$\psi_1 \circ f \circ \varphi_1^{-1} = \psi_1 \circ f \circ (\varphi^{-1} \circ H) = F \circ H = \pi \circ G \circ H = \pi.$$

□

Proof of the Canonical Immersion Theorem. Similarly we assume the sub-matrix

$$(\frac{\partial F^i}{\partial x^j}), \quad 1 \le i \le m, 1 \le j \le m$$

is nonsingular at $\varphi(p)$. Define

$$G : U \times \mathbb{R}^{n-m} \to \mathbb{R}^n, \quad (x^1, \cdots, x^m, y^1, \cdots, y^{n-m}) \mapsto F(x) + (0, \cdots, 0, y_1, \cdots, y_{n-m}).$$

Then dG is nonsingular at $(\varphi(p), 0, \cdots, 0)$. So G is a local diffeomorphism with a smooth local inverse H, and

$$(H \circ \psi) \circ f \circ \varphi_1^{-1} = H \circ F = H \circ G \circ \iota = \iota.$$

□

Remark. More generally one has the following constant rank theorem:

Theorem 2.4 (Constant Rank Theorem)**.** *Let $f : M \to N$ be a smooth map so that* $\operatorname{rank}(df) = r$ *near p. Then there exists charts (φ_1, U_1, V_1) around p and (ψ_1, X_1, Y_1) near $f(p)$ such that that*

$$\psi_1 \circ f \circ \varphi_1^{-1}(x^1, \cdots x^m) = (x^1, \cdots, x^r, 0, \cdots, 0).$$

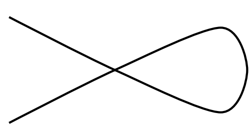
The proof will be left as an exercise.

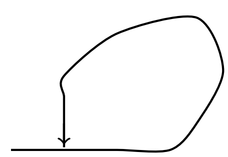
Smooth maps are very useful in contructing nice subsets in smooth manifolds. For example, we are interested in

- If f is an immersion, what can we say about the image $f(M)$?
- If f is a submersion, what can we say about level sets $f^{-1}(q)$'s?

Let's briefly discuss the first case. Let $f : M \to N$ be an immersion. Then lcally $f(M)$ is a very nice subset, as clarified by the canonical immersion theorem. However, globally $f(M)$ could be a "bad subset" of N.

Example. The following two graphs are the images of two immersions of $\mathbb{R}$ into $\mathbb{R}^2$. For the first one, the immersion is not injective. For the second one, the immersion is injective, while the image still have different topology than $\mathbb{R}$.





Example. A more complicated example: consider $f : \mathbb{R} \to \mathbb{S}^1 \times \mathbb{S}^1$ defined by

$$f(t) = (e^{it}, e^{i\sqrt{2}t}).$$

Then f is an immersion, and the image $f(\mathbb{R})$ is a dense curve in the torus $S^1 \times S^1$.

We are more interested in nice immersions $f : M \to N$ where the image $f(M)$, with the subspace topology inherited from N, has the same topology as M.

Definition 2.5. Let M, N be smooth manifolds, and $f : M \to N$ an immersion. f is called an *embedding* if it is a homeomorphism onto its image $f(M)$, where the topology on $f(M)$ is the subspace topology as a subset of N.