LECTURE 6: SMOOTH SUBMANIFOLDS

1. Smooth submanifolds as images

Let $M$ be a smooth manifold of dimension $n$, and $k < n$.

**Definition 1.1.** A subset $S \subset M$ is a $k$-dimensional (embedded/regular) smooth submanifold of $M$ if for every $p \in S$, there is a chart $(\varphi, U, V)$ around $p$ in $M$ such that

$$\varphi(U \cap S) = V \cap (\mathbb{R}^k \times \{0\}) = \{x \in \varphi(U) \mid x^{k+1} = \cdots = x^n = 0\}.$$ 

We will call $\text{codim}(S) = n - k$ the codimension of $S$.

**Example.** The sphere $S^n$ is a smooth submanifold of $\mathbb{R}^{n+1}$. Can you construct a local chart of $\mathbb{R}^{n+1}$ near every point of $S^n$ which satisfies the condition in the definition 1.1?

**Example.** For any smooth map $f : M \to N$, the graph

$$\Gamma_f = \{(p, q) \mid q = f(p)\}$$

is a smooth submanifold of $M \times N$.

Recall that an embedding of a smooth manifold $M$ into a smooth manifold $N$ is an immersion $f : M \to N$ (meaning that $df_p$ is injective for all $p \in M$) so that $M$ is homeomorphic to $f(M) \subset N$.

It is not surprising that a smooth submanifold (with the subspace topology) is always a smooth manifold by itself, and the inclusion map from the submanifold to the ambient manifold is always an embedding:

**Theorem 1.2.** Let $S$ be a $k$-dimensional submanifold of $M$. Then with the subspace topology, $S$ admits a unique smooth structure so that

1. $S$ is a smooth manifold of dimension $k$.
2. The inclusion map $\iota : S \hookrightarrow M$ is a smooth embedding.

**Proof.** With the subspace topology, $S$ satisfies the Hausdorff and second-countable conditions. To construct local charts on $S$, we denote

$$\pi : \mathbb{R}^n \to \mathbb{R}^k, \quad (x^1, \ldots, x^n) \mapsto (x^1, \ldots, x^k)$$

$$j : \mathbb{R}^k \hookrightarrow \mathbb{R}^n, \quad (x^1, \ldots, x^k) \mapsto (x^1, \ldots, x^k, 0, \ldots, 0).$$

Fix any chart $(\varphi, U, V)$ of $M$ satisfying definition 1.1. Let $X = U \cap S$, $Y = \pi \circ \varphi(X)$ and $\psi = \pi \circ \varphi$. Then $\psi|_X^{-1} = \varphi^{-1} \circ j$. So $(\psi, X, Y)$ is a chart on $S$. Moreover, charts of this type are compatible, since the transition map

$$\psi_\beta \circ \psi_\alpha^{-1} = \pi \circ \varphi_\beta \circ \varphi_\alpha^{-1} \circ j = \pi \circ \varphi_{\alpha, \beta} \circ j$$

is smooth.
We need to check that the inclusion map \( \iota : S \hookrightarrow M \) is an embedding. Obviously if we endow \( \iota(S) \) with the subspace topology, the map \( \iota : S \rightarrow \iota(S) \) is a homeomorphism. It is an immersion because in each pair of charts as constructed above, \( \iota = \varphi^{-1} \circ \iota \circ \psi \).

We refer to theorem 5.31 in Lee's book (page 114) for the uniqueness of the topology/smooth structures on \( S \) that makes \( \iota \) an embedding. \( \Box \)

It turns out that smooth submanifolds are exactly the image of embeddings:

**Theorem 1.3.** Let \( f : M \rightarrow N \) be an embedding. Then the image \( f(M) \) is a smooth submanifold of \( N \).

**Proof.** Let \( p \in M \) and \( q = f(p) \). Since \( f \) is an immersion, the canonical immersion theorem implies that there exists charts \( (\varphi_1, U_1, V_1) \) near \( p \) and \( (\psi_1, X_1, Y_1) \) near \( q \) such that on \( V_1 \), \( \psi_1 \circ f \circ \varphi_1^{-1} \) is the canonical embedding \( j : \mathbb{R}^m \rightarrow \mathbb{R}^n \) restricted to \( V_1 \), i.e.

\[
\psi_1 \circ f = j \circ \varphi_1
\]
on \( U_1 \). Since \( f \) is a homeomorphism onto its image, \( f(U_1) \) is open in \( f(M) \subset N \). In other words, there exists an open set \( X \subset N \) such that \( f(U_1) = f(M) \cap X \). Replace \( X_1 \) by \( X_1 \cap X \), and \( Y_1 \) by \( \psi_1(X_1 \cap X) \). Then for this new chart \( (\psi_1, X_1, Y_1) \),

\[
\psi_1(X_1 \cap f(M)) = Y_1 \cap \psi_1(f(U_1)) = Y_1 \cap j(\varphi_1(U_1)) = Y_1 \cap (\mathbb{R}^m \times \{0\}).
\]

\( \Box \)

Now let \( S \subset M \) be a submanifold, and \( p \in S \). Since \( \iota : S \hookrightarrow M \) is an embedding, \( d\iota_p : T_pS \rightarrow T_pM \) is injective. We might identify \( T_pS \) as the vector subspace \( d\iota_p(T_pS) \) of \( T_pM \) for every \( p \in S \). In other words, we can identify any vector \( X_p \in T_pS \) with the vector \( \tilde{X}_p = d\iota_p(X_p) \) in \( T_pM \) so that for any \( f \in C^\infty(M) \),

\[
\tilde{X}_p(f) = (d\iota_p(X_p))f = X_p(f \circ \iota) = X_p(f|_S).
\]

A natural question is: which vectors in \( T_pM \) can be regarded as vectors in \( T_pS \)?

**Theorem 1.4.** Suppose \( S \subset M \) is a submanifold, and \( p \in S \). Then

\[
T_pS = \{X_p \in T_pM \mid X_p(f) = 0 \text{ for all } f \in C^\infty(M) \text{ with } f|_S = 0 \}.
\]

**Proof.** Obviously if \( X_p \in T_pS \), then for \( f \in C^\infty(M) \) with \( f|_S = 0 \), \( \tilde{X}_p(f) = 0 \).

Conversely, if \( X_p \in T_pS \) satisfies \( X_p(f) = 0 \) for all \( f \) that vanishes on \( S \), we need to show \( X_p \in T_pS \). Take a coordinate chart \( (\varphi, U, V) \) on \( M \) such that near \( p \), \( S \) is given by \( x^{k+1} = \cdots = x^n = 0 \). Then \( T_pM \) is the span of \( \partial_1, \ldots, \partial_n \), while \( T_pS \) is the subspace spanned by \( \partial_1, \ldots, \partial_k \). In other words, a vector \( X_p = \sum X^i \partial_i \) lies in \( T_pS \) if and only if \( X^i = 0 \) for \( i > k \).

Now let \( h \) be a smooth bump function supported in \( U \) that equals 1 in a neighborhood of \( p \). For any \( j > k \), consider the function \( f_j(x) = h(x)x^j(\varphi(x)) \), extended to be zero on \( M \setminus U \). Then \( f_j|_S = 0 \). So

\[
0 = X_p(f_j) = \sum X^i \frac{\partial(h(\varphi^{-1}(x))x^j)}{\partial x^i}(\varphi(p)) = X^j
\]
for any $j > k$. It follows that $X_p \in T_p S$. \qed

**Remark.** Note the difference between an immersion and an embedding:

- If $f : M \to N$ is an immersion, then by the canonical immersion theorem, any point $p \in M$ has a neighborhood in $M$ whose image is nice in $N$.
- If $f : M \to N$ is an embedding, then by theorem 1.3, any point $q \in f(M)$ has a neighborhood in $f(M)$ that is nice in $N$.

We will call the image of an injective immersion an **immersed submanifold**. Unlike embedded submanifolds, the two topologies of an immersed submanifold $f(M)$, one from the topology of $M$ via the map $f$ and the other from the subspace topology of $N$, might be different, as we have seen from the examples we constructed last time.

**Remark.** More generally, we have

**Theorem 1.5.** If $f : M \to N$ is a smooth map with constant rank $k$ (i.e. $df_p$ is of constant rank $k$ at any point $p \in M$), then the image of $f$ is an immersed submanifold with tangent space the image of the tangent map $df_p$.

### 2. Smooth submanifolds as level sets

Recall that a smooth map $f : M \to N$ is a submersion at $p \in M$ if the differential $df_p : T_p M \to T_{f(p)} N$ is surjective.

**Definition 2.1.** Suppose $f : M \to N$ is a smooth map between smooth manifolds. A point $q \in N$ is called a **regular value** if $f$ is a submersion at each $p \in f^{-1}(q)$.

One can regard the next theorem as a generalization of the implicit function theorem we mentioned in lecture 2.

**Theorem 2.2.** If $q$ is a regular value of a smooth map $f : M \to N$, then $S = f^{-1}(q)$ is a submanifold of $M$ of dimension $\dim M - \dim N$. Moreover, for every $p \in S$, $T_p S$ is the kernel of the map $df_p : T_p M \to T_q N$.

**Proof.** Let $p \in S = f^{-1}(q)$. Then by the canonical submersion theorem, there are charts $(\varphi_1, U_1, V_1)$ centered at $p$ and $(\psi_1, X_1, Y_1)$ centered at $q$ such that $f(U_1) \subset X_1$, and $\pi = \psi_1 \circ f \circ \varphi_1^{-1}$. It follows that $\varphi_1$ maps $U_1 \cap f^{-1}(q)$ onto $V_1 \cap \pi^{-1}(0)$. So $f^{-1}(q)$ is a submanifold of $M$.

Now denote the inclusion by $\iota : S \to M$. Then for any $p \in S$, $f \circ \iota(p) = q$. In other words, $f \circ \iota$ is a constant map on $S$. So $df_p \circ d\iota(p) = 0$, i.e. $df_p = 0$ on the image of $d\iota_p : T_p S \hookrightarrow T_p M$, or in other words, $T_p S \subset \ker(df_p)$. By dimension counting we conclude that $T_p S$ coincides with the kernel of $df_p$. \qed

**Example.** Consider the map

$$f : \mathbb{R}^{n+1} \to \mathbb{R}, \quad (x^1, \ldots, x^{n+1}) \mapsto (x^1)^2 + \cdots + (x^{n+1})^2.$$  

Then $S^n = f^{-1}(1)$. Since the Jacobian

$$Jf(x) = 2(x^1, \ldots, x^{n+1}) \neq 0$$  

we conclude that $S^n$ is a regular value of $f$. Moreover, $S^n$ is a submanifold of $\mathbb{R}^{n+1}$ of dimension $n$. Thus $S^n$ is a smooth submanifold of $\mathbb{R}^{n+1}$.
on $S^n$, we conclude that $S^n$ is an $n$-dimensional submanifold of $\mathbb{R}^{n+1}$. Moreover, for any $a = (a^1, \ldots, a^{n+1}) \in S^n$,

$$T_{a}S^n = \{ v \in \mathbb{R}^{n+1} | v \cdot a = 0 \}.$$  

**Remark.** The theorem above can be generalized to smooth maps of constant ranks.

**Theorem 2.3** (Constant rank level set theorem). Let $M, N$ be smooth manifold, and $f : M \to N$ be a smooth map with constant rank $k$. Then each level set of $f$ is a closed submanifold of codimension $k$ in $M$.

The proof is left as an exercise.