1. Transversality

Last time, we showed that the pre-image of any regular value is a smooth submanifold. More generally, we can prove that the pre-image of submanifolds “in correct positions” are submanifolds:

**Theorem 1.1.** Let \( f : M \to N \) be a smooth map, and let \( S \subset N \) be an embedded submanifold so that for any \( p \in f^{-1}(S), \)
\[
\text{Im}(df_p) + T_{f(p)}S = T_{f(p)}N.
\]
Then \( f^{-1}(S) \) is an embedded submanifold in \( M \) whose codimension equals \( \text{codim}(S) \), and
\[
T_p(f^{-1}(S)) = df_p^{-1}(T_{f(p)}S).
\]

**Definition 1.2.** We will say \( f \) intersect \( S \) transversally, and denote by \( f \triangleleft S \).

**Lemma 1.3.** Suppose \( 0 \) is the regular value of a smooth map \( g : N \to \mathbb{R}^k \), and \( S = g^{-1}(0) \). Then \( f \triangleleft S \) if and only if \( 0 \) is a regular value of \( g \circ f \).

**Proof.** Take any \( p \in f^{-1}(S) \), then \( T_{f(p)}S = \ker(dg_{f(p)}) \) and \( dg_{f(p)}(T_{f(p)}N) = T_0\mathbb{R}^k \). So
\[
f \triangleleft S \iff df_p(T_pM) + T_{f(p)}S = T_{f(p)}N, \ \forall p \in f^{-1}(S)
\]
\[
\iff d(g \circ f)_p(T_pM) = T_0\mathbb{R}^k, \ \forall p \in (g \circ f)^{-1}(0)
\]
\[
\iff 0 \text{ is a regular value of } g \circ f,
\]
where the second “\( \iff \)” follows from the following linear algebra fact: Any linear map \( L : V \to W \) induces an injective linear map \( L : V/\ker(L) \to W \). So if \( V_1 \subset V \), and \( L(V_1) = L(V) \), then \( V = V_1 + \ker(L) \).

**Proof of theorem 1.1.** In PSet 2 you will prove that for any smooth manifold \( S \) on \( N \), there is a smooth map \( g : N \to \mathbb{R}^k \), where \( k \) is the codimension of \( S \), so that \( 0 \) is a regular value of \( g \), and \( g^{-1}(0) = S \). Consider the composition \( g \circ f \). Then \( f^{-1}(S) = (g \circ f)^{-1}(0) \). Moreover, by the previous lemma, \( 0 \) is a regular value of \( g \circ f \). So according to theorem 2.2 in lecture 6, \( f^{-1}(S) \) is an embedded submanifold of \( M \) whose codimension is \( k \), which is also the codimension of \( S \) in \( N \). Finally
\[
T_p(f^{-1}(S)) = (dg_{f(p)} \circ df_p)^{-1}(0) = df_p^{-1}(dg_{f(p)}^{-1}(0)) = df_p^{-1}(T_{f(p)}S).
\]
This completes the proof.

As a special case, if \( S_1, S_2 \) are submanifolds of \( M \), \( \iota : S_1 \hookrightarrow M \) is the canonical embedding, and \( \iota \triangleleft S_2 \), then for any \( p \in S_1 \cap S_2 \), \( T_pS_1 + T_pS_2 = T_pM \).
Definition 1.4. We say $S_1$ and $S_2$ intersect transversally if for any $p \in S_1 \cap S_2$, 
$$T_p S_1 + T_p S_2 = T_p M.$$ 
In this case we write $S_1 \pitchfork S_2$.

So if $S_1 \pitchfork S_2$, then $\iota \pitchfork S_2$, and $S_1 \cap S_2 = \iota^{-1}(S_2)$, where $\iota : S_1 \hookrightarrow M$ is the inclusion. Moreover, the dimension of $S_1 \cap S_2$ equals $\dim S_1 - (\dim M - \dim S_2) = \dim S_1 + \dim S_2 - \dim M$.

Corollary 1.5. If $S_1$ and $S_2$ intersect transversally, then $S_1 \cap S_2$ is a smooth submanifold whose dimension equals $\dim S_1 + \dim S_2 - \dim M$, and for any $p \in S_1 \cap S_2$, 
$$T_p (S_1 \cap S_2) = T_p S_1 \cap T_p S_2.$$ 

We will need the following theorem, which (together with Sard’s theorem below) roughly says that if a family of maps, when viewed as one map, is transverse to a submanifold, then most elements in the family are transverse to the submanifold.

Theorem 1.6. Let $F : S \times M \to N$ be a smooth map, $X \subset N$ be a smooth submanifold, and $F \pitchfork X$. For each $s \in S$, let $f_s : M \to N$ be the map $f_s(p) = F(s,p)$. Then for every regular value $s \in S$ of the projection map 
$$\pi : F^{-1}(X) \to S, \quad \pi(s,p) = s,$$ 
one has $f_s \pitchfork X$.

Proof. Let $s$ be any regular value of $\pi$. For any $p \in f_s^{-1}(X)$, we need to show 
$$\text{Im}(df_s)_p + T_q X = T_q N,$$ 
where $q = f_s(p)$. Since $F \pitchfork X$, for any $Y_q \in T_q N$, there exists $(Z_s, Z_p) \in T_{(s,p)}(S \times M)$ and $Z_q \in T_q X$ such that 
$$Y_q = (dF)_{(s,p)}(Z_s, Z_p) + Z_q.$$ 
But since $s$ is a regular value of $\pi$, for $Z_s \in T_s S$, there exists $Z'_p \in T_p M$ so that $(Z_s, Z'_p) \in T_{(s,p)} F^{-1}(X)$. It follows 
$$Y_q = (dF)_{(s,p)}(0, Z_p - Z'_p) + (dF)_{(s,p)}(Z_s, Z'_p) + Z_q.$$ 
The conclusion follows since $(dF)_{(s,p)}(0, Z_p - Z'_p) = (df_s)_p(Z_p - Z'_p) \in \text{Im}(df_s)_p$, and 
$$(dF)_{(s,p)}(Z_s, Z'_p) \in dF_{(s,p)}(T_{(s,p)} F^{-1}(X)) \subset T_q X.$$ 

2. Morse functions

We start with

Definition 2.1. Let $f : M \to N$ be a smooth map.

1. We say $p \in M$ is a regular point of $f$ if $df_p : T_p M \to T_{f(p)} N$ is surjective.
2. We say $q \in N$ is a regular value of $f$ if any $p \in f^{-1}(q)$ is a regular point.
3. We say $p \in M$ is a critical point of $f$ if it is not a regular point.
4. We say $q \in N$ is a critical value of $f$ if it is not a regular value.
**Remark.** By definition, any \( q \in \mathbb{N} \setminus \text{Im}(f) \) is automatically a regular value.

**Remark.** Critical values are exactly the image of critical points, but the pre-image of critical values may contain regular points. We will denote the set of all critical points of \( f \) by \( \text{Crit}(f) \).

Next time we will prove that the set of critical values is very small: \(^1\)

**Theorem 2.2** (Sard’s theorem). The set of all critical values has measure zero in \( \mathbb{N} \).

For the rest of today’s lecture, we will focus on the case where \( f : M \to \mathbb{R} \) is a smooth function. Then \( p \in M \) is a critical point if and only if \( df_p : T_pM \to \mathbb{R} \) is not surjective. Since \( df_p \) is linear, we must have \( df_p = 0 \). In other words,

\[ p \text{ is a critical point of a smooth function } f \iff df_p = 0. \]

**Example.** Let \( M \) be compact, and \( p \) is a maximal or a minimal value point of \( f \). For any \( X_p \in T_pM \), we take a curve \( \gamma \) passing \( p \) so that \( X_p = \dot{\gamma}(0) \). Then \( t = 0 \) is extremal value point for the function \( f \circ \gamma(t) \). As a consequences, we see

\[ 0 = \left. \frac{d}{dt} \right|_{t=0} f(\gamma(t)) = df_p \left( \left. \frac{d}{dt} \right|_{t=0} \gamma(t) \right) = df_p(X_p). \]

It follows \( df_p = 0 \), i.e. \( p \) is a critical point.

Now suppose \( p \) is a critical point of \( f \). We would like to know: is \( p \) a local extremal value point? As in the Euclidean case, we need to introduce the conception of Hessian. We shall give several different ways to define the Hessian.

We start with a local description. Recall that for any local chart \((\varphi, U, V)\) of \( M \), we have \( df = (\partial_1 f)dx^1 + \cdots + (\partial_n f)dx^n \), where

\[ \partial_i f(p) = \frac{\partial f \circ \varphi^{-1}}{\partial x^i}(\varphi(p)) \]

is a smooth function on \( U \), and

\[ p \text{ is a critical point of a smooth function } f \iff \partial_i f(p) = 0, \ 1 \leq i \leq n. \]

Note that although \( \partial_i f \)'s depends on the choice of local charts, whether \( \partial_i f(p) = 0 \) for all \( i \), i.e. whether \( p \) is a critical point, is independent of the choice of local charts.

Now let \( p \) be a critical point. For any local chart \((\varphi, U, V)\) we will call the matrix

\[ (\partial_i \partial_j f(p)) \]

the **Hessian matrix** of \( f \) at \( p \) (with respect to the given charts). Note that this matrix depends on the choice of local charts. However, one can prove that, at critical points, “whether this matrix is non-singular or not” is independent of the choice of local charts.

**Definition 2.3.** Let \( M \) be a smooth manifold and \( f : M \to \mathbb{R} \) a smooth function.

1. A critical point \( p \in M \) of \( f \) is **non-degenerate** if the Hessian is non-singular.
2. We say \( f \) is a **Morse function** if all its critical points are non-degenerate.

\(^1\)Of course we don’t have a natural measure on \( M \) yet. However, we will explain next time that the notion of “measure zero” is well-defined on manifolds.
Note that the Hessian matrix is symmetric, so it defines a bilinear form on $T_pM$ (with respect to the canonical basis arising from the chart):

$$\text{Hess}_f : T_pM \times T_pM \rightarrow \mathbb{R}, \quad \text{Hess}_f \left( \sum X^i \partial_i, \sum Y^i \partial_i \right) = \sum_{i,j} \partial_i \partial_j f(p) X^i Y^j.$$ 

To see that this bilinear form is in fact independent of the choices of local charts, let’s give an intrinsic ways to define $\text{Hess}_f$. (We will give another “easier” intrinsic definition of $\text{Hess}_f$ later. The advantage of the following point of view is that it gives a geometric interpretation of non-degeneracy. As a consequence, we can use this to deduce the existence of Morse functions.)

We will consider the cotangent bundle $T^*M = \bigcup_p \{p\} \times T^*_pM$.

In PSet 2 you are supposed to prove

- $T^*M$ is a smooth manifold of dimension $2n$.
- The set $\{(p, 0) \mid p \in M\}$, still denoted by $M$ for simplicity, is an $n$ dimensional smooth submanifold of $T^*M$.
- For any $(p, \xi) \in T^*M$, the tangent space $T_{(p,0)}T^*M = T_pM \oplus T^*_pM$.

Now let $f$ be a smooth function on $M$. Then for any $p \in M$, $df_p \in T^*_pM$. So we get a smooth map

$$s_f : M \rightarrow T^*M, \quad p \mapsto s_f(p) = (p, df_p).$$

Note that by definition, the set of critical points are exactly the intersection $M \cap \text{Im}(s_f)$, where we regard $M$ as the “zero submanifold” of $T^*M$. Now pick any critical point $p$ of $f$. For simplicity we denote $\Lambda_f = \text{Im}(s_f)$. Obviously $s_f$ is an embedding, so $\Lambda_f$ is an $n$ dimensional submanifold of $T^*M$. Moreover, the tangent space of $\Lambda_f$ at the critical point $(p, 0)$ is given by

$$T_{(p,0)}\Lambda_f = \text{Im}(ds_f)_p \subset T_{(p,0)}T^*M = T_pM \oplus T^*_pM.$$ 

So any vector in $T_{(p,0)}\Lambda_f$ can be written in the form $(v, \xi)$, where $v \in T_pM$ and $\xi \in T^*_pM$.

**Lemma 2.4.** The projection map $\pi_1 : T_{(p,0)}\Lambda_f \rightarrow T_pM, \quad (v, \xi) \mapsto v$ is bijective.

**Proof.** Since both $T_{(p,0)}\Lambda_f$ and $T_pM$ are $n$-dimensional vector spaces, and since $\pi_1$ is linear, it is enough to prove $\pi_1$ is injective. But this follows from the fact that $\Lambda_f$ is a graph. \hfill $\square$

So if $p$ is a critical point of $f$, then we get a linear map

$$\kappa : T_pM \rightarrow T^*_pM, \quad v \mapsto \pi_2(\pi_1^{-1}(v)),$$

where $\pi_2$ is the second projection $\pi_2 : T_pM \oplus T^*_pM \rightarrow T^*_pM$. The Hessian can be defined by $\kappa$:

$$\text{Hess}_f : T_pM \times T_pM \rightarrow \mathbb{R}, \quad \text{Hess}_f(X_p, Y_p) = \langle \kappa(X_p), Y_p \rangle.$$ 

One should check that with respect to basis that arising from a local chart, the matrix of $\text{Hess}_f$ is exactly the matrix we give earlier.
Now we can give a geometric interpretation of the non-degeneracy condition.

**Proposition 2.5.** A critical point $p$ of $f$ is non-degenerate if and only if $\Lambda_f$ intersect with $M$ transversally at $(p,0)$.

*Proof.* By definition, $p$ is a non-degenerate critical point if and only if $\kappa$ is an isomorphism, i.e. $\ker(\kappa) = 0$. But by definition of $\kappa$,

$$\ker(\kappa) = \{v \in T_p M \mid (v, 0) \in T_{(p,0)} \Lambda_f\} = T_{(p,0)} \Lambda_f \cap T_p M.$$

So by dimension counting, $\ker(\kappa) = 0$ if and only if $T_{(p,0)} \Lambda_f + T_p M = T_{(p,0)} T^* M$, i.e. $\Lambda_f \cap M$ at $(p,0)$. \hfill $\square$

Since both $\Lambda_f$ and $M$ are $n$ dimensional, we see $\text{Crit}(f) = M \cap \Lambda_f$ is zero dimensional if $f$ is a Morse function. In particular, we get

**Corollary 2.6.** If $f$ is a Morse function, then $\text{Crit}(f)$ is a discrete subset in $M$. In particular, if $M$ is compact, then any Morse function has only finitely many critical points.