LECTURE 9: VECTOR BUNDLES AND VECTOR FIELDS

1. Vector Bundles

Definition 1.1. Let E, M be smooth manifolds, and $\pi : E \to M$ a surjective smooth map. We say (π, E, M) is a *vector bundle of rank k* if for every $p \in M$,

1. $E_p = \pi^{-1}(p)$ is a k dimensional vector space.
2. There exits an open neighborhood U of p and a diffeomorphism $\Phi_U : \pi^{-1}(U) \to U \times \mathbb{R}^k$ so that $\Phi_U(\pi^{-1}(p)) = \{p\} \times \mathbb{R}^k$ which is a linear map.
3. If U, V are two open sets with $p \in U \cap V$, and Φ_U, Φ_V the diffeomorphisms as above, then the transition map
$$g_{UV}(p) = \Phi_U \circ \Phi_V^{-1} : \{p\} \times \mathbb{R}^k \to \{p\} \times \mathbb{R}^k$$
is linear, and smoothly depends on $p \in U \cap V$.

We will call E the *total space*, M the *base*, $\pi^{-1}(p)$ the *fiber* over p, and Φ_U a *local trivialization*. A vector bundle of rank 1 is usually called a *line bundle*. In the case there is no ambiguity about the base, we will denote a vector bundle by E for short.

Example. For any smooth manifold M, $E = M \times \mathbb{R}^k$ is a *trivial bundle* over M.

Example. For any vector bundle (E, M, π) and any open set $U \subset M$, the *restriction bundle* $(\pi^{-1}(U), U, \pi)$ is a vector bundle over U.

Example. Recall that $TM = \cup_p T_pM$, the disjoint union of all tangent spaces, has the structure of a smooth manifold, so that the projection map $\pi : TM \to M$ is a smooth submersion. A local trivialization of TM is given by
$$T\varphi = (\pi, d\varphi) : \pi^{-1}(U) \to U \times \mathbb{R}^n.$$
where $\{\varphi, U, V\}$ is a local chart of M. So TM is a rank n vector bundle over M. We will called TM the *tangent bundle* of M.

Example. Similarly the *cotangent bundle* $T^*M = \cup_p T_p^*M$ is also a rank n vector bundle over M. It is the *dual bundle* of TM.

Example. Let $f : N \to M$ be a smooth map, and (π, E, M) a vector bundle over M. Then one can define a *pull-back bundle* over X by setting the fiber over $x \in X$ to be the fiber of $E_{f(x)}$. More explicitly, let $\Phi_U = (\pi, \Phi_U^2) : \pi^{-1}(U) \to U \times \mathbb{R}^k$ be a local trivialization of E. We choose a coordinate chart X in N so that $f(X) \subset U$. Note that by definition, $(x, \eta) \in (f^*E)_x$ if and only if $\eta \in E_{f(x)}$. Then we define a local trivialization of f^*E over X to be $\Psi_X : \tilde{\pi}^{-1}(X) \to X \times \mathbb{R}^k$, $\Psi_X(x, \eta) = (x, \Phi_U^2(f(x), \eta))$. One can check that f^*E thus defined is a vector bundle over N. [In particular, the restriction of a vector bundle to a submanifold of the base is a vector bundle over the submanifold.]

Definition 1.2. A *(smooth) section* of a vector bundle (π, E, M) is a (smooth) map $s : M \to E$ so that $\pi \circ s = \mathrm{Id}_M$. The set of all sections of E is denoted by $\Gamma(E)$, and the set of all smooth sections of E is denoted by $\Gamma^\infty(E)$.

Remark. Obviously if s_1, s_2 are smooth sections of E, so is $as_1 + bs_2$. So $\Gamma^\infty(E)$ is an (infinitely dimensional) vector space. In fact, one can say more: if s is a smooth section of E and f is a smooth function on M, then fs is a smooth section of E. So $\Gamma^\infty(E)$ is a $C^\infty(M)$-module.

Remark. Many geometrically interesting objects on M are defined as smooth sections of some (vector) bundles over M.

Let E be a vector bundle over M, and U an open set in M.

Definition 1.3. A *local frame* of E over U is an ordered k-tuple $s_1, \cdots, s_k$ of smooth section of E over U so that for each $p \in U$, $s_1(p), \cdots, s_k(p)$ form a basis of E_p.

Obviously if $\Phi_U : \pi^{-1}(U) \to U \times \mathbb{R}^n$ is a local trivialization, and if we let $s_i(p) = \Phi_U^{-1}(p, e_i)$, then $s_1, \cdots, s_k$ form a local frame of E over U. Conversely, if $s_1, \cdots, s_k$ is a local frame of E over U, then for any $p \in U$ and any $v_p \in E_p$, there exists a unique k-tuple of scalars $c_1, \cdots, c_k$ so that $v_p = c_1 s_1(p) + \cdots + c_k s_k(p)$. From this one can define a local trivialization of E over U by setting $\Phi_U(p, v_p) = (p, c_1, \cdots, c_k)$. So the existence of a local frame of E over U is equivalent to the existence of a local trivialization over U.

Example. Let M be a smooth manifold and U be a coordinate patch. Then

- $\partial_1, \cdots, \partial_n$ form a local frame of TM over U.
- $dx^1, \cdots, dx^n$ form a local frame of T^*M over U.

We have the following criterion for the smoothness of sections via local frames:

Theorem 1.4. *A section $s \in \Gamma(E)$ is smooth if and only if for any $p \in M$, there is a neighborhood U of p and a local frame $s_1, \cdots, s_k$ of E over U so that $s = c_1 s_1 + \cdots + c_k s_k$ for some smooth functions $c_1, \cdots, c_k$ defined in U.*

Proof. Let $\Phi_U : \pi^{-1}(U) \to U \times \mathbb{R}^n$ be the local trivialization associated to the frame $s_1, \cdots, s_k$. Then the smoothness of $c_1, \cdots, c_k$ is equivalent to the smoothness of $\Phi_U \circ s|_U$. Since Φ_U is a diffeomorphism, $s|_U$ is a smooth.

Conversely if s is smooth and Φ_U a local trivialization, we let $s_1, \cdots, s_k$ be the frame corresponding to this local trivialization constructed as above. Then the coefficients $c_1, \cdots, c_k$ of $\Phi_U \circ s|_U$ in this basis are smooth, since both s and Φ_U are smooth. □

Note that in general one cannot hope to find global smooth sections $s_1, \cdots, s_k$ defined on the whole manifold M so that $s_1(p), \cdots, s_k(p)$ form a basis of E_p for all $p \in M$. In fact, as we have discussed above, the existence of such a global frame is equivalent to the existence of a global trivialization, i.e. E should be *the same* as $M \times \mathbb{R}^n$. In other words,

Proposition 1.5. *A vector bundle E over M is a trivial bundle if and only if there exists a global frame of E on M.*

2. Smooth vector fields

Definition 2.1. A (smooth) section of TM is called a (smooth) vector field on M.

So by definition, A *vector field* X on M is an assignment that assigns to each point $p \in M$ a tangent vector $X_p \in T_pM$. Locally in a chart $\{\varphi, U, V\}$, any vector field X can be written as

$$X = X^1\partial_1 + \cdots + X^n\partial_n = \sum X^i\partial_i,$$

where X^i's are functions on U. Of course X is smooth if and only if all coefficients X^i's are smooth functions on U. (So one can think of a smooth vector field X as a 1^{st} order differential operator with smooth coefficients.)

Recall that a tangent vector $X_p \in T_pM$ is a linear map $X_p : C^\infty(M) \to \mathbb{R}$ satisfying the Leibnitz law $X_p(fg) = X_p(f)g(p) + f(p)X_p(g)$. So any vector field maps any $f \in C^\infty(M)$ to a function Xf on M defined by $Xf(p) = X_pf$. As a consequence, a smooth vector field X is a map $X : C^\infty(M) \to C^\infty(M)$ that satisfies the Leibnitz law

$$X(fg) = (Xf)g + fXg, \qquad \forall f, g \in C^\infty(M)$$

In what follows we will always assume X to be smooth, unless otherwise stated.

One of the most important conception concerning vector fields is the Lie bracket between two vector fields. Consider two smooth vector fields X and Y on M.

Lemma 2.2. *At each point $p \in M$ the bracket $[X, Y]_p$ defined by*

$$[X, Y]_p(f) = X_p(Yf) - Y_p(Xf)$$

is a tangent vector at p.

Proof. We only need to check the Leibnitz law:

$$\begin{aligned}
[X, Y]_p(fg) =& X_p(Y(fg)) - Y_p(X(fg)) \\
=& X_p((Yf)g + f(Yg)) - Y_p((Xf)g + f(Xg) \\
=& X(Yf)(p)g(p) + Yf(p)Xg(p) + Xf(p)Yg(p) + f(p)X(Yg)(p) \\
& - Y(Xf)(p)g(p) - Xf(p)Yg(p) - Yf(p)Xg(p) - f(p)Y(Xg)(p) \\
=& f(p) \cdot [X, Y]_pg + [X, Y]_pf \cdot g(p).
\end{aligned}$$

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Thus for any vector fields X and Y, the commutator $[X, Y]$ is still a vector field.

Definition 2.3. We call the commutator $[X, Y]$ the Lie bracket of X and Y.

Note that as maps from $C^\infty(M)$ to $C^\infty(M)$, one has

$$[X, Y]f = X(Yf) - Y(Xf).$$

As a consequence of fact that the Lie bracket of two vector fields is again a vector field, we can give a third definition of the Hessian of any smooth function at a critical point p. Recall that p is a critical point of function f if and only if $df_p = 0$. We define the Hessian of f at p to be

$$\mathrm{Hess}_f : T_pM \times T_pM \to \mathbb{R}, \quad (X_p, Y_p) \mapsto X_p(Yf),$$

where Y is any vector field whose value at p is Y_p.

Lemma 2.4. Hess_f *is well-defined, symmetric and bilinear.*

Proof. The linearity in X_p is obvious. For any vector field X whose value at p is X_p, we have

$$X_p(Yf) - Y_p(Xf) = [X,Y]_p f = df_p([X,Y]_p) = 0.$$

This proves symmetry, and also implies well-definedness. ☐

Finally we study relations between vector fields on different manifolds.

Definition 2.5. Suppose $\varphi : M \to N$ is a smooth map, X is a vector field on M and Y is a vector field on N. We say that X and Y are *φ-related* if for any $p \in M$,

$$d\varphi_p(X_p) = Y_{\varphi(p)}.$$

Lemma 2.6. *Suppose $\varphi : M \to N$ is smooth and $X \in \Gamma^\infty(TM), Y \in \Gamma^\infty(TN)$ are φ-related. Then for any $g \in C^\infty(N)$, $X\varphi^* g = \varphi^*(Yg)$.*

Proof. Suppose $q = \varphi(p)$, then

$$\varphi^*(Yg)(p) = (Yg)(q) = Y_q g = d\varphi_p(X_p)g = X_p(g \circ \varphi) = X_p(\varphi^* g) = (X\varphi^* g)(p).$$

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Corollary 2.7. *If X_i are φ-related to Y_i for $i = 1, 2$, then $[X_1, X_2]$ is φ-related to $[Y_1, Y_2]$.*

Proof. For any $g \in C^\infty(N)$,

$$\begin{aligned} d\varphi_p([X_1,X_2]_p)(g) &= X_1(X_2(\varphi^* g))(p) - X_2(X_1(\varphi^* g))(p) \\ &= \varphi^* Y_1(Y_2(g))(p) - \varphi^* Y_2(Y_1(g))(p) \\ &= Y_1(Y_2(g))(\varphi(p)) - Y_2(Y_1(g))(\varphi(p)) \\ &= ([Y_1,Y_2]g)(\varphi(p)) \\ &= [Y_1,Y_2]_{\varphi(p)} g. \end{aligned}$$

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If $f : M \to N$ is a diffeomorphism, and $X \in \Gamma^\infty(TM)$, one can "*push-forward*" X to a smooth vector field $\varphi_* X$ on N by

$$(\varphi_* X)_{\varphi(p)} = d\varphi_p(X_p).$$

Corollary 2.8. *If $\varphi : M \to N$ is a diffeomorphism, $(\varphi_* X)g = (\varphi^{-1})^* X \varphi^* g$.*

Proof. Obviously if $\varphi : M \to N$ is a diffeomorphism, X is φ-related to $\varphi_* X$. ☐