LECTURE 10: DYNAMICS OF VECTOR FIELDS

1. Integral Curves

Suppose M is a smooth manifold. Recall that a *smooth curve* in M is a smooth map $\gamma : I \to M$, where I is an interval in $\mathbb{R}$. For any $a \in I$, the tangent vector of γ at the point $\gamma(a)$ is

$$\dot\gamma(a) = \frac{d\gamma}{dt}(a) := d\gamma_a(\frac{d}{dt}),$$

where $\frac{d}{dt}$ is the standard coordinate tangent vector of $\mathbb{R}$.

Definition 1.1. Let X be a smooth vector field on M. We say that a smooth curve $\gamma : I \to M$ is an *integral curve* of X if for any $t \in I$,

$$\dot\gamma(t) = X_{\gamma(t)}.$$

Example. Consider the coordinate vector field $X = \frac{\partial}{\partial x^1}$ on $\mathbb{R}^n$. Then the integral curves of X are the straight lines parallel to the x^1-axis, parametrized as

$$\gamma(t) = (c_1 + t, c_2, \cdots, c_n).$$

To check this, we notice that for any smooth function f on $\mathbb{R}^n$,

$$d\gamma(\frac{d}{dt})f = \frac{d}{dt}(f \circ \gamma) = \nabla f \cdot \frac{d\gamma}{dt} = \frac{\partial f}{\partial x^1}.$$

Example. Consider the vector field $X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$ on $\mathbb{R}^2$. Then if $\gamma(t) = (x(t), y(t))$ is an integral curve of X, we must have for any $f \in C^\infty(\mathbb{R}^2)$,

$$x'(t)\frac{\partial f}{\partial x} + y'(t)\frac{\partial f}{\partial y} = \nabla f \cdot \frac{d\gamma}{dt} = X_{\gamma(t)}f = x(t)\frac{\partial f}{\partial y} - y(t)\frac{\partial f}{\partial x},$$

which is equivalent to the system

$$x'(t) = -y(t), \quad y'(t) = x(t).$$

The solution to this system is

$$x(t) = a\cos t - b\sin t, \quad y(t) = a\sin t + b\cos t.$$

These are circles centered at the origin in the plane parametrized by the angle (with counterclockwise orientation).

Remark. In general, a re-parametrization of an integral curve is no longer an integral curve. However, it is not hard to see that if $\gamma : I \to M$ is an integral curve of X, then

- Let $I_a = \{t \mid t + a \in I\}$ and $\gamma_a(t) := \gamma(t + a)$, then $\gamma_a : I_a \to M$ is an integral curve of X.
- Let $I^a = \{t \mid at \in I\}$ and $\gamma^a(t) := \gamma(at)$, then $\gamma^a : I^a \to M$ is an integral curve for $X^a = aX$.

Remark. Suppose $\varphi : M \to N$ is smooth, and $X \in \Gamma^\infty(TM), Y \in \Gamma^\infty(TN)$ are φ-related. If γ is an integral curve of X, then $\varphi \circ \gamma$ is an integral curve of Y, since

$$d(\varphi \circ \gamma)_a(\frac{d}{dt}) = d\varphi_{\gamma(a)} \circ d\gamma_a(\frac{d}{dt}) = d\varphi_{\gamma(a)} X_{\gamma(a)} = Y_{\varphi\circ\gamma(a)}.$$

Let (φ, U, V) be a local chart on M and let $X = \sum_i X^i \partial_i$ be a smooth vector field. Since $\partial_i(x^j) = \delta_i^j$, we have $X(x^j) = \sum X^i \partial_i(x^j) = X^j$ and thus $X = \sum_i (Xx^i)\partial_i$. Now let $\gamma : I \to M$ be an integral curve of X. Then we get

$$\dot\gamma(t) = d\gamma_t(\frac{d}{dt}) = \sum_i d\gamma_t(\frac{d}{dt})(x^i)\partial_i = \sum_i (x^i \circ \gamma)'(t)\partial_i$$

So the integral curve equation $\dot\gamma(t) = X_{\gamma(t)}$ becomes

$$\sum_i (x^i \circ \gamma)'(t)\partial_i = \sum_i X^i(\gamma(t))\partial_i$$

for all $t \in I$, i.e.

$$(x^i \circ \gamma)'(t) = X^i(\gamma(t)) = X^i \circ \varphi^{-1}(x^1(\gamma(t)), \cdots, x^n(\gamma(t)))$$

for all $t \in I$ and all $1 \le i \le n$. This is a system of first order ODEs on the functions $y^i = x^i \circ \gamma$. Conversely, any solution to this system of ODEs defines an integral curve of the vector field X inside the open set U.

According to the fundamental theorem of ODEs, we conclude

Corollary 1.2. *Suppose X is a smooth vector field on M. Then for any point $q_0 \in M$, there exists a neighborhood U of p_0, an $\varepsilon > 0$ and a smooth map*

$$\Gamma : (-\varepsilon, \varepsilon) \times U \to M$$

so that for any $p \in U$, the curve $\gamma_p : (-\varepsilon, \varepsilon) \to M$ defined by

$$\gamma_p(t) := \Gamma(t, p)$$

is an integral curve of X with $\gamma(0) = p$. Moreover, this integral curve is unique in the sense that $\sigma : I \to M$ is another integral curve of X with $\sigma(0) = p$, then $\sigma(t) = \gamma_p(t)$ for $t \in I \cap (-\varepsilon, \varepsilon)$.

As a consequence of the uniqueness, any integral curve has a *maximal defining interval.* We are interested in those vector fields whose maximal defining interval is $\mathbb{R}$.

Definition 1.3. A vector field X on M is *complete* if for any $p \in M$, there is an integral curve $\gamma : \mathbb{R} \to M$ such that $\gamma(0) = p$.

As in the case of functions, we can define the *support* of a vector field by

$$\text{supp}(X) = \overline{\{p \in M \mid X(p) \neq 0\}}.$$

Theorem 1.4. *If X is a compactly supported vector field on M, then it is complete.*

Proof. Let $C = \mathrm{supp}(X)$. Then any integral curve starting at $q \in M \setminus C$ stays at q. Thus every integral curve starting at $p \in C$ stays in C. It follows that for any $q \in C$, there is an interval $I_q = (-\varepsilon_q, \varepsilon_q)$, a neighborhood U_q of q in C and a smooth map

$$\Gamma : I_q \times U_q \to C$$

such that for all $p \in U_q$,

$$\gamma_p(t) = \Gamma(t, p)$$

is an integral curve of X with $\gamma_p(0) = p$. Since $\cup_q U_q = C$, and C is compact, one can find a finite many points $q_1, \cdots q_N$ in C so that $\{U_{q_1}, \cdots, U_{q_N}\}$ cover C. Let $I = \cap_k I_{q_k} = (-\varepsilon_0, \varepsilon_0)$, then for any $q \in C$, there is an integral curve $\gamma_q : I \to C$. Now suppose the maximal defining interval for $p \in C$ is I_p. I claim that $I_p = \mathbb{R}$. In fact, if $I_p \neq \mathbb{R}$, without loss of generality, we may assume that $\sup I_p = c < \infty$. Then from $q = \gamma_p(c - \frac{\varepsilon_0}{2})$, there is an integral curve $\gamma_q : (-\varepsilon_0, \varepsilon_0) \to M$ of X. By uniqueness, $\gamma_q(t) = \gamma_p(t + c - \frac{\varepsilon_0}{2})$. It follows that the defining interval of γ_p extends to $c + \frac{\varepsilon}{2}$. A contradiction. □

Corollary 1.5. *Any smooth vector field on a compact manifold is complete.*

Proof. The set $\mathrm{Supp}(X)$, as a closed set in the compact manifold, is compact. □

2. Flows generated by vector fields

Now suppose M is a smooth manifold and X is a complete vector field on M. Then for any $p \in M$, there is a unique integral curve $\gamma_p : \mathbb{R} \to M$ such that $\gamma_p(0) = p$. From this one can, for any $t \in \mathbb{R}$, define a map

$$\phi_t : M \to M, \quad p \mapsto \gamma_p(t).$$

Lemma 2.1. *$\phi_t : M \to M$ is bijective with $\phi_t^{-1} = \phi_{-t}$.*

Proof. Notice that for any $p \in M$ and any $t, s \in \mathbb{R}$,

$$\gamma_1(t) = \phi_t \circ \phi_s(p) \quad \text{and} \quad \gamma_2(t) = \phi_{t+s}(p)$$

are both integral curves for X starting at the same point

$$\gamma_1(0) = \phi_s(p) = \gamma_2(0).$$

By uniqueness of integral curves, we have

$$\phi_t \circ \phi_s = \phi_{t+s}.$$

Since $\phi_0 = \mathrm{Id}$, we conclude that $\phi_t^{-1} = \phi_{-t}$, so in particular ϕ_t is bijective. □

Lemma 2.2. *The map*

$$\Phi : \mathbb{R} \times M \to M, \quad (t, p) \mapsto \phi_t(p)$$

is smooth.

Proof. We have seen that the integral curves depends on the initial condition smoothly. In other words, for any $p \in M$, there is a neighborhood U_p of p and an interval $I_p = (-\varepsilon_p, \varepsilon_p) \ni 0$ such that $\Phi|_{I_p \times U_p}$ is smooth. To show that Φ is smooth near a point $(t_0, p) \in \mathbb{R} \times M$ for larger t_0, we notice that $\phi_t(p) = \gamma_p(t)$ is smooth on t, so the set $K = \gamma_p([-\varepsilon_p, t_0 + \varepsilon_p])$ is compact. It follows that one can find finitely many points $p_1, \cdots, p_N$ in K so that the open sets $U_{p_1}, \cdots, U_{p_N}$ cover K. As a result, the set $U = \cup U_{p_k}$ is an open neighborhood of K, and $I = \cap_k I_{p_k} = (-\varepsilon_0, \varepsilon_0)$ is an interval containing 0, such that Φ is smooth in $U \times I$. It follows that if $|t - t_0| < \varepsilon_0$, and if we take N large enough so that $\frac{t_0}{N} < \varepsilon_0$, then

$$\Phi(t,p) = \Phi(t_0 + s, p) = \Phi(t_0/N, \Phi(t_0/N, \cdots, \Phi(t_0/N, \Phi(s,p))))$$

is smooth in both t and p. □

It follows that ϕ_t's are diffeomorphisms for all t. In other words, the family of maps $\{\phi_t\}$ is a family of diffeomorphisms of M. They are called *one-parameter subgroup of diffeomorphisms* since they satisfies the group law $\phi_t \circ \phi_s = \phi_{t+s}$. Notice that the group law can also be rewritten in terms of the map Φ as

$$\Phi(t+s, p) = \Phi(t, \Phi(s,p)).$$

Definition 2.3. We will call $\Phi : \mathbb{R} \times M \to M$, $(t,p) \mapsto \phi_t(p)$ the *flow* of X.

Example. The flow generated by the vector field $X = \frac{\partial}{\partial x^1}$ is the translation

$$\Phi : \mathbb{R} \times \mathbb{R}^n, \quad (t, x^1, x^2, \cdots, x^n) \mapsto (t + x^1, x^2, \cdots, x^n).$$

More generally, the flow generated by a constant vector field $X = \sum c^i \frac{\partial}{\partial x^i}$ is

$$\Phi : \mathbb{R} \times \mathbb{R}^n, \quad (t, x^1, x^2, \cdots, x^n) \mapsto (c^1 t + x^1, \cdots, c^n t + x^n).$$

Example. If we identify $\mathbb{R}^2$ with $\mathbb{C}$, then the flow generated by the vector field

$$X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$

is the counterclockwise rotation

$$\Phi : \mathbb{R} \times \mathbb{C} \to \mathbb{C}, \quad (t,z) \mapsto e^{it} z.$$

Note that this vector field is tangent to circles centered at the origin. We will denote the induced vector fields on such circles by $\frac{d}{d\theta}$.

Remark. If X is not complete, one can also derive a similar theory of *local flow* generated by X. In that case the group law still holds for small t and s.

Remark. Sometimes we will denote $\phi_t = \exp(tX)$ to emphasis the X-dependence. In this case the group law becomes

$$\exp(tX)\exp(sX) = \exp((s+t)X).$$

Note that in general $\exp(tX)\exp(sY) \neq \exp(sY)\exp(tX)$, unless X, Y commutes, i.e. $[X,Y] = 0$.