LECTURE 10: DYNAMICS OF VECTOR FIELDS

1. Integral Curves

Suppose $M$ is a smooth manifold. Recall that a smooth curve in $M$ is a smooth map $\gamma : I \to M$, where $I$ is an interval in $\mathbb{R}$. For any $a \in I$, the tangent vector of $\gamma$ at the point $\gamma(a)$ is

$$\dot{\gamma}(a) = \frac{d\gamma}{dt}(a) := d\gamma_a(\frac{d}{dt}),$$

where $\frac{d}{dt}$ is the standard coordinate tangent vector of $\mathbb{R}$.

**Definition 1.1.** Let $X$ be a smooth vector field on $M$. We say that a smooth curve $\gamma : I \to M$ is an integral curve of $X$ if for any $t \in I$,

$$\dot{\gamma}(t) = X_{\gamma(t)}.$$

**Example.** Consider the coordinate vector field $X = \frac{\partial}{\partial x^1}$ on $\mathbb{R}^n$. Then the integral curves of $X$ are the straight lines parallel to the $x^1$-axis, parametrized as $\gamma(t) = (c_1 + t, c_2, \ldots, c_n)$.

To check this, we notice that for any smooth function $f$ on $\mathbb{R}^n$,

$$d\gamma \left( \frac{d}{dt} \right) f = \frac{d}{dt} (f \circ \gamma) = \nabla f \cdot \frac{d\gamma}{dt} = \frac{\partial f}{\partial x^1}.$$

**Example.** Consider the vector field $X = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}$ on $\mathbb{R}^2$. Then if $\gamma(t) = (x(t), y(t))$ is an integral curve of $X$, we must have for any $f \in C^\infty(\mathbb{R}^2)$,

$$x'(t) \frac{\partial f}{\partial x} + y'(t) \frac{\partial f}{\partial y} = \nabla f \cdot \frac{d\gamma}{dt} = X_{\gamma(t)} f = x(t) \frac{\partial f}{\partial y} - y(t) \frac{\partial f}{\partial x},$$

which is equivalent to the system

$$x'(t) = -y(t), \quad y'(t) = x(t).$$

The solution to this system is

$$x(t) = a \cos t - b \sin t, \quad y(t) = a \sin t + b \cos t.$$ 

These are circles centered at the origin in the plane parametrized by the angle (with counterclockwise orientation).

**Remark.** In general, a re-parametrization of an integral curve is no longer an integral curve. However, it is not hard to see that if $\gamma : I \to M$ is an integral curve of $X$, then

- Let $I_a = \{ t \mid t + a \in I \}$ and $\gamma_a(t) := \gamma(t + a)$, then $\gamma_a : I_a \to M$ is an integral curve of $X$.
- Let $I^a = \{ t \mid at \in I \}$ and $\gamma^a(t) := \gamma(at)$, then $\gamma^a : I^a \to M$ is an integral curve for $X^a = aX$. 

Remark. Suppose \( \varphi: M \to N \) is smooth, and \( X \in \Gamma^\infty(TM), Y \in \Gamma^\infty(TN) \) are \( \varphi \)-related. If \( \gamma \) is an integral curve of \( X \), then \( \varphi \circ \gamma \) is an integral curve of \( Y \), since
\[
d(\varphi \circ \gamma) = d\varphi(a) \circ d\gamma(a) = d\varphi(a)X_{\varphi(a)} = Y_{\varphi(a)}.
\]

Let \( (\varphi, U, V) \) be a local chart on \( M \) and let \( X = \sum_i X^i \partial_i \) be a smooth vector field. Since \( \partial_i(x^j) = \delta^j_i \), we have \( X(x^i) = \sum_i X^i \partial_i(x^j) = X^j \) and thus \( X = \sum_i (X^i \partial_i) \). Now let \( \gamma: I \to M \) be an integral curve of \( X \). Then we get
\[
\dot{\gamma}(t) = d\gamma_t \left( \frac{d}{dt} \right) = \sum_i d\gamma_t(x^i) \partial_i = \sum_i (x^i \circ \gamma)'(t) \partial_i
\]
So the integral curve equation \( \dot{\gamma}(t) = X_{\gamma(t)} \) becomes
\[
\sum_i (x^i \circ \gamma)'(t) \partial_i = \sum_i X^i(\gamma(t)) \partial_i
\]
for all \( t \in I \), i.e.
\[
(x^i \circ \gamma)'(t) = X^i(\gamma(t)) = X^i \circ \varphi^{-1}(x^1(\gamma(t)), \ldots, x^n(\gamma(t)))
\]
for all \( t \in I \) and all \( 1 \leq i \leq n \). This is a system of first order ODEs on the functions \( y^i = x^i \circ \gamma \). Conversely, any solution to this system of ODEs defines an integral curve of the vector field \( X \) inside the open set \( U \).

According to the fundamental theorem of ODEs, we conclude

**Corollary 1.2.** Suppose \( X \) is a smooth vector field on \( M \). Then for any point \( p_0 \in M \), there exists a neighborhood \( U \) of \( p_0 \), an \( \varepsilon > 0 \) and a smooth map
\[
\Gamma: (-\varepsilon, \varepsilon) \times U \to M
\]
so that for any \( p \in U \), the curve \( \gamma_p: (-\varepsilon, \varepsilon) \to M \) defined by
\[
\gamma_p(t) := \Gamma(t, p)
\]
is an integral curve of \( X \) with \( \gamma(0) = p \). Moreover, this integral curve is unique in the sense that \( \sigma: I \to M \) is another integral curve of \( X \) with \( \sigma(0) = p \), then \( \sigma(t) = \gamma_p(t) \) for \( t \in I \cap (-\varepsilon, \varepsilon) \).

As a consequence of the uniqueness, any integral curve has a **maximal defining interval**. We are interested in those vector fields whose maximal defining interval is \( \mathbb{R} \).

**Definition 1.3.** A vector field \( X \) on \( M \) is **complete** if for any \( p \in M \), there is an integral curve \( \gamma: \mathbb{R} \to M \) such that \( \gamma(0) = p \).

As in the case of functions, we can define the **support** of a vector field by
\[
\text{supp}(X) = \{ p \in M \mid X(p) \neq 0 \}.
\]

**Theorem 1.4.** If \( X \) is a compactly supported vector field on \( M \), then it is complete.
Proof. Let $C = \text{supp}(X)$. Then any integral curve starting at $q \in M \setminus C$ stays at $q$. Thus every integral curve starting at $p \in C$ stays in $C$. It follows that for any $q \in C$, there is an interval $I_q = (-\varepsilon_q, \varepsilon_q)$, a neighborhood $U_q$ of $q$ in $C$ and a smooth map

$$\Gamma : I_q \times U_q \to C$$

such that for all $p \in U_q$,

$$\gamma_p(t) = \Gamma(t, p)$$

is an integral curve of $X$ with $\gamma_p(0) = p$. Since $\bigcup_q U_q = C$, and $C$ is compact, one can find a finite many points $q_1, \ldots, q_N$ in $C$ so that $\{U_{q_1}, \ldots, U_{q_N}\}$ cover $C$. Let $I = \bigcap_k I_{q_k} = (-\varepsilon_0, \varepsilon_0)$, then for any $q \in C$, there is an integral curve $\gamma_q : I \to C$. Now suppose the maximal defining interval for $p \in C$ is $I_p$. I claim that $I_p = \mathbb{R}$. In fact, if $I_p \neq \mathbb{R}$, without loss of generality, we may assume that $\sup I_p = c < \infty$. Then from $q = \gamma_p(c - \frac{\varepsilon_0}{2})$, there is an integral curve $\gamma_q : (-\varepsilon_0, \varepsilon_0) \to M$ of $X$. By uniqueness, $\gamma_q(t) = \gamma_p(t + c - \frac{\varepsilon_0}{2})$. It follows that the defining interval of $\gamma_p$ extends to $c + \frac{\varepsilon_0}{2}$. A contradiction. □

Corollary 1.5. Any smooth vector field on a compact manifold is complete.

Proof. The set Supp$(X)$, as a closed set in the compact manifold, is compact. □

2. Flows generated by vector fields

Now suppose $M$ is a smooth manifold and $X$ is a complete vector field on $M$. Then for any $p \in M$, there is a unique integral curve $\gamma_p : \mathbb{R} \to M$ such that $\gamma_p(0) = p$. From this one can, for any $t \in \mathbb{R}$, define a map

$$\phi_t : M \to M, \quad p \mapsto \gamma_p(t).$$

Lemma 2.1. $\phi_t : M \to M$ is bijective with $\phi_t^{-1} = \phi_{-t}$.

Proof. Notice that for any $p \in M$ and any $t, s \in \mathbb{R}$,

$$\gamma_1(t) = \phi_t \circ \phi_s(p) \quad \text{and} \quad \gamma_2(t) = \phi_{t+s}(p)$$

are both integral curves for $X$ starting at the same point

$$\gamma_1(0) = \phi_s(p) = \gamma_2(0).$$

By uniqueness of integral curves, we have

$$\phi_t \circ \phi_s = \phi_{t+s}.$$

Since $\phi_0 = \text{Id}$, we conclude that $\phi_t^{-1} = \phi_{-t}$, so in particular $\phi_t$ is bijective. □

Lemma 2.2. The map

$$\Phi : \mathbb{R} \times M \to M, \quad (t, p) \mapsto \phi_t(p)$$

is smooth.
Proof. We have seen that the integral curves depend on the initial condition smoothly. In other words, for any \( p \in M \), there is a neighborhood \( U_p \) of \( p \) and an interval \( I_p = (-\varepsilon_p, \varepsilon_p) \) such that \( \Phi|_{I_p \times U_p} \) is smooth. To show that \( \Phi \) is smooth near a point \( (t_0, p) \in \mathbb{R} \times M \) for larger \( t_0 \), we notice that \( \phi_t(p) = \gamma_p(t) \) is smooth on \( t \), so the set \( K = \gamma_p([-\varepsilon_p, t_0 + \varepsilon_p]) \) is compact. It follows that one can find finitely many points \( p_1, \ldots, p_N \) in \( K \) so that the open sets \( U_{p_1}, \ldots, U_{p_N} \) cover \( K \). As a result, the set \( U_0 = \bigcup U_{p_k} \) is an open neighborhood of \( K \), and \( I_0 = \bigcap I_{p_k} = (-\varepsilon_0, \varepsilon_0) \) is an interval containing 0, such that \( \Phi \) is smooth in \( U_0 \times I_0 \). It follows that if \( |t - t_0| < \varepsilon_0 \), and if we take \( N \) large enough so that \( \frac{t_0}{N} < \varepsilon_0 \), then
\[
\Phi(t, p) = \Phi(t_0 + s, p) = \Phi(t_0/N, \Phi(t_0/N, \cdots, \Phi(t_0/N, \Phi(s, p))))
\]
is smooth in both \( t \) and \( p \).

It follows that \( \phi_t \)'s are diffeomorphisms for all \( t \). In other words, the family of maps \( \{\phi_t\} \) is a family of diffeomorphisms of \( M \). They are called one-parameter subgroup of diffeomorphisms since they satisfies the group law \( \phi_t \circ \phi_s = \phi_{t+s} \). Notice that the group law can also be rewritten in terms of the map \( \Phi \) as
\[
\Phi(t + s, p) = \Phi(t, \Phi(s, p)).
\]

Definition 2.3. We will call \( \Phi : \mathbb{R} \times M \to M \), \( (t, p) \mapsto \phi_t(p) \) the flow of \( X \).

Example. The flow generated by the vector field \( X = \frac{\partial}{\partial x} \) is the translation
\[
\Phi : \mathbb{R} \times \mathbb{R}^n, \quad (t, x^1, x^2, \cdots, x^n) \mapsto (t + x^1, x^2, \cdots, x^n).
\]
More generally, the flow generated by a constant vector field \( X = \sum c^i \frac{\partial}{\partial x^i} \) is
\[
\Phi : \mathbb{R} \times \mathbb{R}^n, \quad (t, x^1, x^2, \cdots, x^n) \mapsto (c^1 t + x^1, \cdots, c^n t + x^n).
\]

Example. If we identify \( \mathbb{R}^2 \) with \( \mathbb{C} \), then the flow generated by the vector field
\[
X = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}
\]
is the counterclockwise rotation
\[
\Phi : \mathbb{R} \times \mathbb{C} \to \mathbb{C}, \quad (t, z) \mapsto e^{it} z.
\]
Note that this vector field is tangent to circles centered at the origin. We will denote the induced vector fields on such circles by \( \frac{d}{d\theta} \).

Remark. If \( X \) is not complete, one can also derive a similar theory of local flow generated by \( X \). In that case the group law still holds for small \( t \) and \( s \).

Remark. Sometimes we will denote \( \phi_t = \exp(tX) \) to emphasis the \( X \)-dependence. In this case the group law becomes
\[
\exp(tX) \exp(sX) = \exp((s + t)X).
\]
Note that in general \( \exp(tX) \exp(sY) \neq \exp(sY) \exp(tX) \), unless \( X, Y \) commutes, i.e. \( [X, Y] = 0 \).