LECTURE 14: LIE GROUP ACTIONS

1. Smooth actions

Let $M$ be a smooth manifold, $\text{Diff}(M)$ the group of diffeomorphisms on $M$.

**Definition 1.1.** An action of a Lie group $G$ on $M$ is a homomorphism of groups $\tau : G \to \text{Diff}(M)$. In other words, for any $g \in G$, $\tau(g)$ is a diffeomorphism from $M$ to $M$ such that

$$\tau(g_1g_2) = \tau(g_1) \circ \tau(g_2).$$

The action $\tau$ of $G$ on $M$ is smooth if the evaluation map $ev : G \times M \to M, \quad (g,m) \mapsto \tau(g)(m)$ is smooth. We will denote $\tau(g)(m)$ by $g \cdot m$.

**Remark.** What we defined above is the left action. One can also define a right action $\hat{\tau} : G \to \text{Diff}(M)$, i.e. such that $\hat{\tau}(g_1g_2) = \hat{\tau}(g_2) \circ \hat{\tau}(g_1)$.

Any left action $\tau$ can be converted to a right action $\hat{\tau}$ by requiring $\hat{\tau}_g(m) = \tau(g^{-1})m$.

**Example.** $S^1$ acts on $\mathbb{R}^2$ by counterclockwise rotation.

**Example.** Any linear group in $\text{GL}(n, \mathbb{R})$ acts on $\mathbb{R}^n$ as linear transformations.

**Example.** If $X$ is a complete vector field on $M$, then

$$\rho : \mathbb{R} \to \text{Diff}(M), \quad t \mapsto \rho_t = \exp(tX)$$

is a smooth action of $\mathbb{R}$ on $M$. Conversely, every smooth action of $\mathbb{R}$ on $M$ is given by this way: one just take $X(p)$ to be $X(p) = \dot{\gamma}_p(t)$, where $\gamma_p(t) := \rho_t(p)$.

**Example.** Any Lie group $G$ acts on itself by many ways, e.g. by left multiplication, by right multiplication and by conjugation. For example, the conjugation action of $G$ on $G$ is given by

$$g \in G \leadsto c(g) : G \to G, x \mapsto gxg^{-1}.$$ 

More generally, any Lie subgroup $H$ of $G$ can act on $G$ by left multiplication, right multiplication and conjugation.

**Example.** Any Lie group $G$ acts on its Lie algebra $\mathfrak{g} = T_eG$ by the adjoint action:

$$g \in G \leadsto \text{Ad}_g = (dc(g))_e : \mathfrak{g} \to \mathfrak{g}.$$ 

For example, one can show that the adjoint action of $\text{GL}(n, \mathbb{R})$ on $\mathfrak{gl}(n, \mathbb{R})$ is given by $A \leadsto \text{Ad}_A : \mathfrak{gl}(n, \mathbb{R}) \to \mathfrak{gl}(n, \mathbb{R}), X \mapsto AXA^{-1}$. Similarly one can define the coadjoint action of $G$ on $\mathfrak{g}^\ast$. 

Now suppose Lie group $G$ acts smoothly on $M$.

**Definition 1.2.** For any $X \in \mathfrak{g}$, the induced vector field $X_M$ on $M$ associated to $X$ is

$$X_M(m) = \left. \frac{d}{dt} \right|_{t=0} \exp(tX) \cdot m.$$

**Lemma 1.3.** For each $m \in M$ and $X \in \mathfrak{g}$,

$$X_M(m) = (ev_m)_e(X),$$

where $ev_m$ is the restricted evaluation map

$$ev_m : G \to M, g \mapsto g \cdot m.$$

**Proof.** For any smooth function $f$ we have

$$(ev_m)_e(X)f = X_e(f \circ ev_m) = \left. \frac{d}{dt} \right|_{t=0} f(\exp(tX) \cdot m) = X_M(m)(f).$$

$\square$

**Remark.** So from any smooth Lie group action of $G$ on $M$ we get a map

$$d\tau : \mathfrak{g} \to \Gamma^\infty(M), \quad X \mapsto X_M.$$

This can be viewed as the differential of the map $\tau : G \to \text{Diff}(M)$. One can prove that $d\tau$ is a Lie algebra anti-homomorphism. It is called the *infinitesimal action* of $\mathfrak{g}$ on $M$. (There is a natural way to regard $\Gamma^\infty(M)$, an infinitely dimensional Lie algebra, as the “Lie algebra” of $\text{Diff}(M)$.)

**Lemma 1.4.** The integral curve of $X_M$ starting at $m \in M$ is

$$\gamma_m(t) = \exp(tX) \cdot m.$$

**Proof.** By definition, we have $\gamma_m(0) = m$, and

$$\dot{\gamma}_m(t) = \left. \frac{d}{dt} \right|_{t=0} \exp(tX \cdot m) = \left. \frac{d}{ds} \right|_{s=0} (\exp sX \circ \exp tX \cdot m) = X_M(\gamma_m(t)).$$

$\square$

As a consequence, we see that the induced vector field is “natural”:

**Corollary 1.5.** Let $\tau : G \to \text{Diff}(M)$ be a smooth action. Then for any $X \in \mathfrak{g}$,

$$\tau(\exp tX) = \exp(tX_M),$$

where $\rho_t = \exp(tX_M)$ is the flow on $M$ generated by $X_M$. 
2. Orbits and the Quotient Space

**Definition 2.1.** Let $\tau : G \to \text{Diff}(M)$ be a smooth action.

1. The *orbit* of $G$ through $m \in M$ is
   \[ G \cdot m = \{ g \cdot m \mid g \in G \} \subset M. \]
2. The *stabilizer* (also called the *isotropic subgroup*) of $m \in M$ is the subgroup
   \[ G_m = \{ g \in G \mid g \cdot m = m \} < G. \]

**Proposition 2.2.** Let $\tau : G \to \text{Diff}(M)$ be a smooth action, $m \in M$. Then

1. The orbit $G \cdot m$ is an immersed submanifold whose tangent space at $m$ is
   \[ T_m(G \cdot m) = \{ X_M(m) \mid X \in \mathfrak{g} \}. \]
2. The stabilizer $G_m$ is a Lie subgroup of $G$, with Lie algebra
   \[ \mathfrak{g}_m = \{ X \in \mathfrak{g} \mid X_M(m) = 0 \}. \]

**Proof.**

1. By definition,
   \[ \text{ev}_m \circ L_g = \tau_g \circ \text{ev}_m. \]
   Taking derivative at $h \in G$, we get
   \[ (d\text{ev}_m)_h \circ (dL_g)_h = (d\tau_g)_{h \cdot m} \circ (d\text{ev}_m)_h. \]
   Since $(dL_g)_h$ and $(d\tau_g)_{h \cdot m}$ are bijective, the rank of $(d\text{ev}_m)_{gh}$ equals the rank of $(d\text{ev}_m)_h$ for any $g$ and $h$. It follows that the map $\text{ev}_m$ is of constant rank. By the constant rank theorem, its image, $\text{ev}_m(G) = G \cdot m$, is an immersed submanifold of $M$.

   The tangent space of $G \cdot m$ at $m$ is the image under $d\text{ev}_m$ of $T_eG = \mathfrak{g}$. But for any $X \in \mathfrak{g}$, we have $(d\text{ev}_m)_e(X) = X_M(m)$. So the conclusion follows.

2. Consider the map
   \[ \text{ev}_m : G \to M, g \mapsto g \cdot m, \]
   then $G_m = \text{ev}_m^{-1}(m)$, so it is a closed set in $G$. It is a subgroup since $\tau$ is a group homomorphism. It follows from Cartan’s theorem that $G_m$ is a Lie subgroup of $G$.

   We have seen that the Lie algebra of the subgroup $G_m$ is
   \[ \mathfrak{g}_m = \{ X \in \mathfrak{g} \mid \exp(tX) \in G_m, \forall t \in \mathbb{R} \}. \]
   It follows that $\exp(tX) \cdot m = m$ for $X \in \mathfrak{g}_m$. Taking derivative at $t = 0$, we get
   \[ \mathfrak{g}_m \subset \{ X \in \mathfrak{g} \mid X_M(m) = 0 \}. \]
   Conversely, if $X_M(m) = 0$, then $\gamma(t) \equiv m$, $t \in \mathbb{R}$, is an integral curve of the vector field $X_M$ passing $m$. It follows that $\exp(tX) \cdot m = \gamma(t) = m$, i.e. $\exp(tX) \in G_m$ for all $t \in \mathbb{R}$. So $X \in \mathfrak{g}_m$. \qed
Obviously if $m$ and $m'$ lie in the same orbit, then $G \cdot m = G \cdot m'$. We will denote the set of orbits by $M/G$. For example, if the action is transitive, i.e. there is only one orbit $M = G \cdot m$, then $M/G$ contains only one point. In general, $M/G$ contains many points. We will equip with $M/G$ the quotient topology. This topology might be very bad in general, e.g. non-Hausdorff.

**Example.** Consider the natural action of $\mathbb{R}_{>0}$ on $\mathbb{R}$ by multiplications, then there are three orbits, $\{+,0,-\}$. The open sets with respect to the quotient topology are $\{+,\} \setminus \{-\}$, $\{+,-\}$, $\{+,0\}$ and the empty set $\emptyset$. So the quotient is not Hausdorff.

Finally we list some conceptions to guarantee that each $G \cdot m$ is an embedded submanifold, and to guarantee that $M/G$ is a smooth manifold.

**Definition 2.3.** Let Lie group $G$ acts smoothly on $M$.

1. The $G$-action is **proper** if the action map
   \[ \alpha : G \times M \to M \times M, \quad (g,m) \mapsto (g \cdot m, m) \]
   is proper, i.e. the pre-image of any compact set is compact.

2. The $G$-action is **free** if $G_m = \{e\}$ for all $m \in M$.

**Remarks.**

1. If the $G$-action on $M$ is proper, then the evaluation map
   \[ ev_m : G \to M, g \mapsto g \cdot m \]
   is proper. In particular, for each $m \in M$, $G_m$ is compact.

2. One can prove that if $G$ is compact, then any smooth $G$-action is proper.

**Example.** Let $H \subset G$ be a closed subgroup. Then the left action of $H$ on $G$, \[ \tau_h : G \to G, \quad g \mapsto hg \]

is free. It is also proper since for any compact subset $K \subset G \times G$, the preimage $\alpha^{-1}(K)$ is contained in the image of $K$ under the smooth map

\[ f : G \times G \to G \times G, \quad (g_1,g_2) \mapsto (g_1g_2^{-1},g_2), \]

which has to be compact. The action is not transitive unless $H = G$.

The major theorem that we will not have time to prove is

**Theorem 2.4.** Suppose $G$ acts on $M$ smoothly, then

1. If the action is proper, then
   a. Each orbit $G \cdot m$ is an embedded closed submanifold of $M$.
   b. The orbit space $M/G$ is Hausdorff.

2. If the action is proper and free, then
   a. The orbit space $M/G$ is a smooth manifold.
   b. The quotient map $\pi : M \to M/G$ is a submersion.
(3) If the action is transitive, then for each \( m \in M \), the map
\[
F : G/G_m \to M, \quad gG_m \mapsto g \cdot m
\]
is a diffeomorphism.

In particular, we see that if the \( G \)-action on \( M \) is transitive, then for any \( m \in M \),
\[
M \simeq G/G_m.
\]
Such a manifold is called a homogeneous space. Obviously any homogeneous space is of the form \( G/H \) for some Lie group \( G \) and some closed Lie subgroup \( H \).

**Example.** According to Gram-Schmidt, the natural action of \( O(n) \) on \( S^{n-1} \) is transitive. It follows that \( S^{n-1} \) is a homogeneous space. Moreover, if we choose \( m \) to be the “north pole” of \( S^{n-1} \), then one can check that the isotropy group \( G_m \) is \( O(n-1) \). It follows
\[
S^{n-1} \simeq O(n)/O(n-1).
\]

**Example.** Let \( k < n \). Consider
\[
Gr_k(n) = \{ k \text{ dimensional linear subspaces of } \mathbb{R}^n \}.
\]
Then \( O(n) \) acts transitively on \( Gr_k(n) \), and the isotropy group of the standard \( \mathbb{R}^k \) inside \( \mathbb{R}^n \) is \( O(k) \times O(k-1) \). It follows
\[
Gr_k(n) \simeq O(n)/(O(k) \times O(n-k))
\]
The manifold \( Gr_k(n) \) is called a Grassmannian manifold. Note that \( Gr_1(n) = \mathbb{RP}^{n-1} \) is the real projective space.